ON FACTORIZATIONS OF CONJUGATE SYMMETRIC HADAMARD TRANSFORM AND ITS RELATIONSHIP WITH DCT

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ABSTRACT

Complex-valued conjugate symmetric Hadamard transform (C-CSHT) is a variant of complex Hadamard transform and effective for some signal processing and communication applications. Its closed-form factorization of the general N-channel ($N = 2^m$) case was recently proposed, however, there still exist a room to find an effective factorization especially for unified factorization of C-CSHT and its real-valued transform counterpart (R-CSHT). In this paper, we present another simple closed-form factorization of C-CSHT based on that of R-CSHT. The proposed factorization is applicable for both complex- and real-valued CSHTs with one factorization. Furthermore, the relationship with the common block transform, DCT, is revealed.

Index Terms— Hadamard transform, complex Hadamard transform, conjugate symmetric Hadamard transform, DCT, binDCT

1. INTRODUCTION

Hadamard transform (HT) has been widely studied for long years and used for various signal processing and communication applications [1]. It has very low computational complexity since each element of its transformation matrix is 1 or -1: It is often called *antipodal* coefficient [2, 3]. There exists various extended versions of HT, such as lapped HT [2], complex HT [4, 5], etc.

As one of the extensions of HT, complex-valued conjugate symmetric HT (C-CSHT) was proposed by Aung et al. [6, 7]. It is a block transform and each element of the transform is antipodal coefficient or its complex counterpart, i.e., ± 1 and $\pm j$, where $j^2 = -1$. It is a possible extension of HT into the complex-valued transforms, and it is used for signal analysis and synthesis, image watermarking, and so forth. As a real-valued counterpart, there exists the real version of CSHT, which is denoted as R-CSHT hereafter. It has a comparable performance with HT in spite of its less computational complexity.

In the original paper [7], the factorizations of both C-CSHT and R-CSHT were shown for the case $N = 8$, where N is the number of filters. Unfortunately, these complex- and real-valued factorizations were not consistent: One factorization cannot be derived straightforwardly from another. Additionally, no general factorizations about N were presented. Recently, a C-CSHT factorization for the general N was proposed by Bouguezel et al. [8]. However, its implementation is restricted for the C-CSHT version and its R-CSHT counterpart cannot be trivially derived.

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Table 1. Comparison of CSHT factorizations.

	[7]	[8]	Ours
N	8	2^m	2^m
C-CSHT Factorization	✓	✓	✓
R-CSHT Factorization	✓		✓
Consistency b/w C- and R-CSHT			✓

Briefly speaking, the previous methods have the following issues to be solved:

1. General factorization: CSHT should be factorized for the case of the general N and have factorizations for both complex- and real-valued CSHT.
2. Consistency of factorization: C-CSHT and R-CSHT should be factorized based on one approach.

In this paper, we present the most general factorization of CSHT so far. It can be applied for the general N, and both of the complex- and real-valued transforms are derived from one factorization approach. The comparison of CSHT factorization is summarized in Table 1. It should also be mentioned that our factorization is relatively simple compared to the previous one. Finally, thanks to our factorization, the structural relationships between R-CSHT and a special DCT, binDCT [9], are revealed.

1.1. Notation

Matrices are shown as upper-case bold face letters. The $N \times N$ identity and reverse-identity matrices are represented as $\mathbf{I}_N$ and $\mathbf{J}_N$, respectively. The null matrix is $\mathbf{0}_N$. The subscript usually refers to as the size of the matrix unless it is specified. Sometimes the size of matrix is omitted when it is obvious. Matrices $\mathbf{P}_X$ is a permutation matrix which reorders rows of the transformation matrix.

2. CSHT

In this section, first we review the structure of CSHT from its original version, and then the recently proposed factorization is shown. For simplicity, the structure of the permutation matrix just after the core transformation is omitted since the space is limited and it does not affect computational complexity.

2.1. Original CSHT

In the original paper [7], the factorizations of C-CSHT and R-CSHT are separately presented and they are implemented in the case $N = 8$. The C-CSHT is factorized as follows:

$$\mathbf{H}_8 = \mathbf{P}_{H8} \begin{bmatrix} \mathbf{W}_2 & & & \\ & \mathbf{W}_2 & & \\ & & \mathbf{W}_2 & \\ & & & \mathbf{W}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 & & & \\ & \mathbf{C}_2 & & \\ & & \mathbf{I}_2 & \\ & & & \mathbf{I}_2' \end{bmatrix} \times \begin{bmatrix} \mathbf{W}_4 & \\ & \mathbf{W}_4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_4 & \\ & \mathbf{C}_4 \end{bmatrix} \mathbf{W}_8 \tag{1}$$

where

$$\mathbf{W}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{I}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{I}_{N/2} \end{bmatrix}$$
$$\mathbf{C}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{0} \\ \mathbf{0} & j\mathbf{I}_{N/2} \end{bmatrix}, \mathbf{I}_2' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The matrix form of $\mathbf{H}_8$ is shown as

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & j & j & -1 & -1 & -j & -j \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & -1 & -j & j & -1 & 1 & j & -j \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & j & -j & -1 & 1 & -j & j \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & 1 & -j & -j & -1 & -1 & j & j \end{bmatrix}. \tag{2}$$

Depending on the structure of $\mathbf{P}_{H8}$ in (1), the CSHT is called natural- or sequency-ordered CSHT [7, 8].

Furthermore, R-CSHT $\mathbf{S}_N$ can be derived from $\mathbf{H}_N$ as

$$\mathbf{S}_N = \begin{bmatrix} \mathbf{H}_N(0, l) \\ (\mathcal{J}\{\mathbf{H}_N(1, l)\}) - (\mathcal{J}\{\mathbf{H}_N(N-1, l)\})/2 \\ (\mathcal{R}\{\mathbf{H}_N(1, l)\}) + (\mathcal{R}\{\mathbf{H}_N(N-1, l)\})/2 \\ \vdots \\ (\mathcal{R}\{\mathbf{H}_N(N/2-1, l)\}) + (\mathcal{R}\{\mathbf{H}_N(N/2+1, l)\})/2 \\ \mathbf{H}_N(N/2, l) \end{bmatrix} \tag{3}$$

where l is the column index of the matrix and $\mathcal{R}\{\cdot\}$ and $\mathcal{J}\{\cdot\}$ represent real and imaginary parts of the number, respectively. However, this postprocessing system is not effective to implement $\mathbf{S}_N$ itself. Therefore, a factorization of $\mathbf{S}_8$ (eight-channel case) is shown in [7] as follows:

$$\mathbf{S}_8 = \mathbf{P}_{S8} \begin{bmatrix} \mathbf{W}_2 & & & \\ & \mathbf{J}_2 & & \\ & & \mathbf{I}_2'\mathbf{W}_2 & \\ & & & \mathbf{W}_2 \end{bmatrix} \begin{bmatrix} \mathbf{W}_4 & \\ & \hat{\mathbf{J}}_4 \end{bmatrix} \mathbf{W}_8 \tag{4}$$

where $\hat{\mathbf{J}}_4 = \begin{bmatrix} \mathbf{0} & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0} \end{bmatrix}$. Its matrix form is

$$\mathbf{S}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}. \tag{5}$$

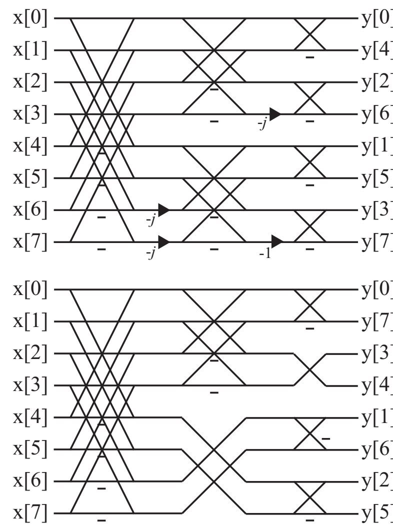

Fig. 1. CSHT factorization in the original paper. Top: C-CSHT. Bottom: R-CSHT.

The factorizations of $\mathbf{H}_8$ and $\mathbf{S}_8$ are illustrated in Fig. 1. Unfortunately, the relationships between these two factorizations were not very clear in spite of the fact that one of them is just a counterpart of another.

2.2. Closed-Form Factorization of CSHT

The algorithm used in the original paper [7] is only given for the case $N = 8$. In [8], the closed-form representation of C-CSHT for $N = 2^m$ $\{m \in \mathbb{N}\}$ was presented. Its structure is represented as follows:

$$\mathbf{H}_N = \mathbf{P}_0 \left(\prod_{i=1}^{m-2} (\mathbf{I}_{2^{m-i}} \otimes \mathbf{W}_2 \otimes \mathbf{I}_{2^{i-1}}) \times \left(\begin{bmatrix} \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{2^{m-i-1}-1} \otimes \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2' \end{bmatrix} \end{bmatrix} \otimes \mathbf{I}_{2^{i-1}} \right) \right) \times (\mathbf{I}_2 \otimes \mathbf{W}_2 \otimes \mathbf{I}_{N/4}) \left(\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \otimes \mathbf{I}_{N/4} \right) (\mathbf{W}_2 \otimes \mathbf{I}_{N/2}) \tag{6}$$

where $\otimes$ represents a Kronecker product operator. This representation requires the same number of arithmetic operations as those for the original C-CSHT for $N = 8$, but it presents a more general structure according to N.

3. PROPOSED FACTORIZATION

In this section, we present another closed-form factorization of CSHT. It is expressed as a combination of R-CSHT and a postprocessing with a complex-valued matrix. Similar to the previous factorization [8], we consider the case for $N = 2^m$ $\{m \in \mathbb{N}\}$.

In contrast to the other factorizations, we start from R-CSHT in (5). The factorization of R-CSHT is represented as follows:

$$\mathbf{S}_N = \mathbf{P}_1 \begin{bmatrix} \mathbf{S}_{N/2} & \\ & \bar{\mathbf{W}}_{N/2} \end{bmatrix} \bar{\mathbf{I}}_N \tag{7}$$

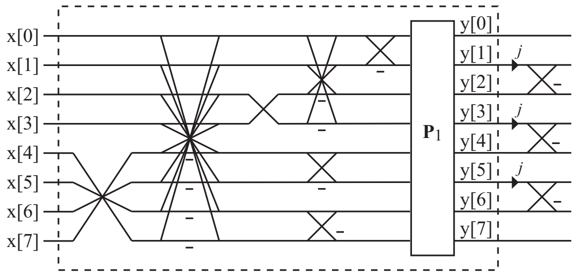

Fig. 2. Eight-point CSHT. The factorization in the dashed box is R-CSHT.

where

$$\bar{\mathbf{W}}_N = \prod_{k=1}^{N/4} \begin{bmatrix} \mathbf{I}_{N/4k} \otimes \mathbf{W}_{2k} & \\ & \mathbf{I}_{N/4k} \otimes \mathbf{W}'_{2k} \end{bmatrix} \tag{8}$$

and

$$\bar{\mathbf{I}}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{J}_{N/2} \\ \mathbf{J}_{N/2} & -\mathbf{I}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{N/2} & \\ & \mathbf{J}_{N/2} \end{bmatrix} \tag{9}$$

$$\mathbf{W}'_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{I}_{N/2} \\ -\mathbf{I}_{N/2} & \mathbf{I}_{N/2} \end{bmatrix}. \tag{10}$$

Moreover, the smallest martix of R-CSHT is of the size 4×4 and defined as

$$\mathbf{S}_4 = \begin{bmatrix} \mathbf{W}_2 & \\ & \mathbf{I}_2 \end{bmatrix} \bar{\mathbf{I}}_4. \tag{11}$$

Obviously, it is a recursive factorization and has a relatively simple implementation.

Furthermore, C-CSHT in our factorization can be easily derived from the definition (3):

$$\mathbf{H}_N = \mathbf{P}_2 \begin{bmatrix} 1 & & & & \\ & \tilde{\mathbf{W}}_2 & & & \\ & & \ddots & & \\ & & & \tilde{\mathbf{W}}_2 & \\ & & & & 1 \end{bmatrix} \mathbf{S}_N \tag{12}$$

in which

$$\tilde{\mathbf{W}}_2 = \mathbf{W}'_2 \begin{bmatrix} j & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & j \\ -1 & j \end{bmatrix}. \tag{13}$$

Clearly, our factorization gives both real and complex factorizations of CSHT with one recursive implementation. The structure is shown in Fig. 2 for $N = 8$, and Fig. 3 for $N = 16$ (R-CSHT only).

3.1. Computational Complexity

Even in our factorization, the computational complexity is the same as the other CSHT factorizations. The number of complex multiplications by j in C-CSHT is clearly $N/2 - 1$ since $\tilde{\mathbf{W}}_2$ is the only factor of the complex multiplications in our factorization. The number is strictly the same as those mentioned in [7,8]. Furthermore, the number of additions/subtractions in our factorization is $N \log_2 N$, which is also the same as the previous factorization.

For R-CSHT, the number of butterfly matrices required 18 additions/subtractions for $N = 8$ in the original paper. It is also the same as that for our factorization (see the dashed box in Fig. 2).

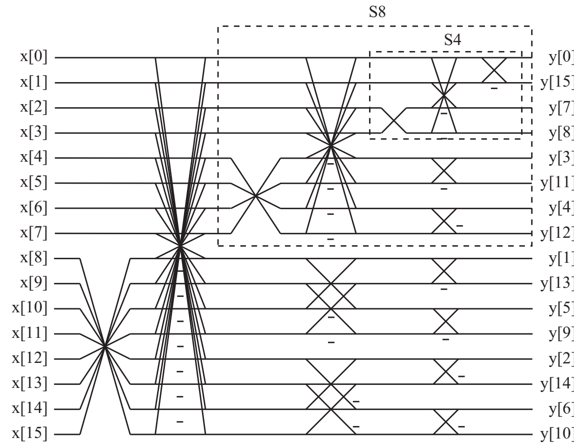

Fig. 3. 16-point R-CSHT based on our factorization where dashed boxes represents four- and eight-point R-CSHTs.

As mentioned in Section 1, the original paper presented factorizations of both C-CSHT and R-CSHT only for the case $N = 8$ and the factorization of R-CSHT cannot be derived from that of C-CSHT straightforwardly (and vice versa). Moreover, the method in [8] is a general C-CSHT factorization in terms of N, however, that of R-CSHT was not shown. Consequently, our factorization gives the consistent and general form of CSHT for both real and complex versions.

4. RELATIONSHIP WITH DCT

In this section, we reveal the relationship between R-CSHT and DCT. The DCT has a lot of efficient factorizations for software/hardware implementations. In this paper, we focus on the one named binDCT [9, 10] which is based on a lifting factorization of the DCT.

The binDCT is known as a computationally effective form of the DCT. It approximates DCT's transformation matrix with the lifting implementation [11]. Originally, it is based on Chen's factorization [12] and Loeffler's one [13]. We consider the former in this paper. It is implemented with butterfly matrices and several lifting steps and gives multiplierless representations of the DCT. Based on the tradeoff between the complexity and the performance, the binDCT has many configurations. Here, we focus on the simplest version of the binDCT: binDCT-C9 in [9]. The factorizations of the general binDCT and binDCT-C9 are illustrated in Fig. 4. It is clear that binDCT-C9 has a quite close structure to that of R-CSHT. In fact, R-CSHT is the same as binDCT-C9 if the rightmost matrix in (9) is removed. Their comparison is illustrated in Fig. 5(a) and (b). The computational complexities of binDCT-C9 and R-CSHT are the same each other since the upper right lifting matrix of binDCT-C9 in Fig. 4 can be represented as one butterfly matrix $\mathbf{W}_2$ as shown in Fig. 5(a). This relationship is effectively utilized for software/hardware implementation of CSHT since both of C- and R-CSHTs can be realized easily if we have the binDCT architecture.

It is also mentioned that HT is a special version of the binDCT [9] . HT based on the binDCT factorization is shown in Fig. 5(c). By appending three butterfly matrices and a trivial $\mathbf{J}_2$, we can obtain HT from binDCT-C9. Consequently, these three simple integer transforms are interchangeable each other and binDCT-C9 will present

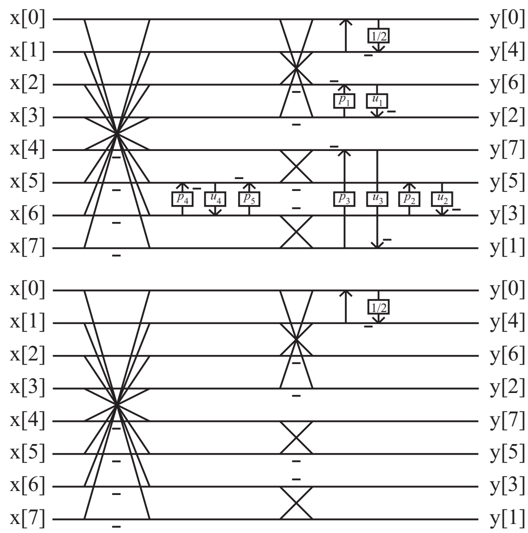

Fig. 4. Structures of binDCT where p_n and u_n are the parameters for lifting steps (scaling parameters are omitted for simplicity). Top: General binDCT structure based on Chen's factorization. Bottom: BinDCT-C9.

the simplest form among them. Additionally, the general structure of CSHT would be derived from the binDCT perspective, which is our ongoing work.

5. CONCLUSIONS

In this paper, we present a general factorization of conjugate symmetric complex Hadamard transform. It is based on the real-valued transform of CSHT, and its complex-valued counterpart can be easily implemented. The structure covers both real and complex versions of CSHT and any $N = 2^m$. Furthermore, the structure comparisons among three simple integer transforms are shown.

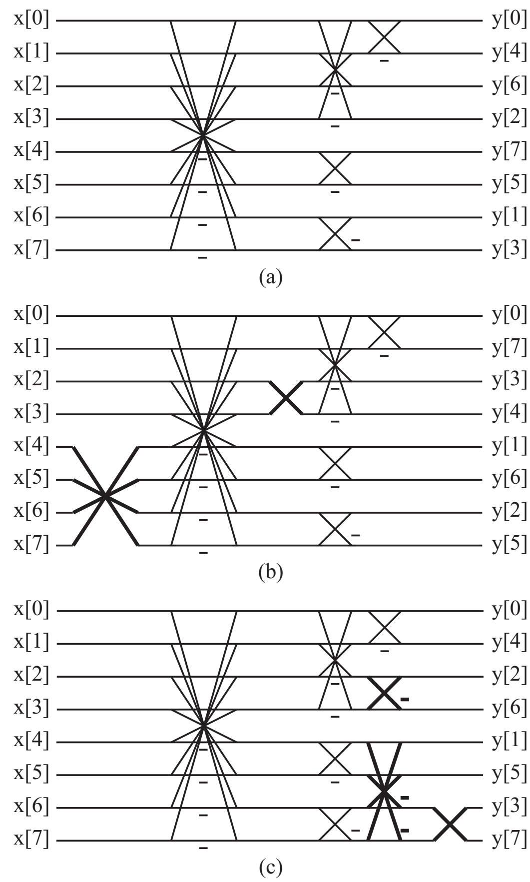

Fig. 5. Comparison of integer-valued transforms. (a) binDCT-C9. (b) R-CSHT. (c) HT. The differences from binDCT-C9 is shown in bold.

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