MATRIX PARAMETRIZATION OF COMPACTLY SUPPORTED ORTHONORMAL WAVELETS

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ABSTRACT

We derive a new set of necessary and sufficient conditions for the filter coefficients of the two-scale difference equation to yield an orthogonal wavelet of compact support. The conditions constitute a linear set of equations of an arbitrary decision vector of half the filter size. The vector of the filter coefficients is a differentiable function of the decision vector. The formulation enables the optimization of the filter design under any regular objective function. The proposed parametrization is used to design customized orthonormal wavelets and to reproduce the classical orthogonal wavelets as a solution of a nonlinear optimization problem.

Index Terms— orthogonal wavelets, design, null-space.

1. INTRODUCTION

The construction of compactly supported orthonormal wavelets usually starts from the two-scale difference equation [1] or equivalently the multiresolution analysis [2] to derive conditions on the wavelet filters coefficients. In this work, we derive new conditions for the construction of orthonormal wavelets with compact support. The new parametrization uses a linear algebra approach to derive the equivalent conditions on the filter coefficients of the two-scale difference equation. This results in a system of linear equations whose solution is the wavelet filter coefficients and this parametrization is a differentiable function, almost everywhere, in the design parameters. This parametrization does not restrict the values of the decision variables and offers new flexibility in the design of orthogonal wavelets. In particular, the vector whose entries are the wavelet filter coefficients is shown to be a basis of the null space of a special matrix that is parameterized by the decision variables. This enables subspace wavelet design techniques where desired wavelet features are set as linear constraints and the wavelet design is treated as a standard optimization problem.

Throughout the paper, We use bold-faced capital letters for matrices, and bold-faced small letters for column vectors. $\mathbf{A}'$ denotes the transpose of the matrix $\mathbf{A}$ (all matrices and vectors are assumed real). The notation $\widetilde{\mathbf{a}}$ of a column vector $\mathbf{a} = [a(0), a(1), a(2), ..., a(2k-1)]'$ is:

$$\widetilde{\mathbf{a}} \triangleq [a(2k-1)\ -a(2k-2)\ a(2k-3)\ \ldots\ a(1)\ -a(0)]' \quad (1)$$

2. ORTHOGONAL WAVELETS WITH COMPACT SUPPORT

Consider the two-scale difference equation [3]

$$\phi(x) = \sqrt{2}\sum_{n=0}^{N} h_n \phi(2x-n) \quad (2)$$

The scaling function $\phi(x)$ is related to the mother wavelet $\psi(x)$ by

$$\psi(x) = \sqrt{2}\sum_{n=0}^{N} g_n \phi(2x-n) \quad (3)$$

By choosing

$$g(n) = (-1)^n h(N-1-n) \quad (4)$$

the necessary and sufficient conditions on the lowpass filter $\{h(n)\}$ such that $\psi(x)$ in (3) is an orthonormal wavelet with compact support, are [1], [4],

$$\sum_n h(n)h(n-2i) = \delta(i) \ \text{ for all } i \quad (5)$$

$$\sum_n h(n) = \sqrt{2} \quad (6)$$

$$\sum_n (-1)^n h(n) = 0 \quad (7)$$

We assume without loss of generality that the length of $h(n)$ is $2K+2$ where K is even. The total number of independent conditions on $\{h(n)\}$ is $K+2$.

A single wavelet decomposition stage would involve filtering with $\{h(n)\}$ and $\{g(n)\}$ followed by dyadic subsampling. If the output of the analysis wavelet filters is organized as:

$$\mathbf{y} = [..., x_L(-1), x_H(-1), x_L(0), x_H(0), x_L(1), x_H(1), ...]' \quad (8)$$

where $x_L(n) = (\downarrow 2)(x * h)(n)$ and $x_H(n) = (\downarrow 2)(x * g)(n)$ denote respectively the subsampled approximation and detailed coefficients after a single stage wavelet decomposition; then the wavelet decomposition stage can put in a matrix form [5]

$$\mathbf{y} = \mathbf{H}\mathbf{x} \quad (9)$$

where $\mathbf{H}$ is an infinite-dimensional orthonormal matrix defined as:

$$\mathbf{H} = \begin{pmatrix} \ddots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \ldots & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \ldots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \end{pmatrix} \quad (10)$$

where the submatrices $\mathbf{U}$, $\mathbf{E}$, and $\mathbf{L}$ (each of size $K \times K$) are block-Toeplitz matrices defined as

$$
\mathbf{U} \triangleq
\begin{pmatrix}
h(2K+1) & h(2K) & h(2K-1) & \dots & h(K+3) & h(K+2) \\
g(2K+1) & g(2K) & g(2K-1) & \dots & g(K+3) & g(K+2) \\
0 & 0 & h(2K+1) & \dots & h(K+5) & h(K+4) \\
0 & 0 & g(2K+1) & \dots & g(K+5) & g(K+4) \\
\vdots & & \ddots & \vdots & \vdots & \\
0 & 0 & 0 & \dots & h(2K+1) & h(2K) \\
0 & 0 & 0 & \dots & g(2K+1) & g(2K)
\end{pmatrix} \quad (11)
$$

$$
\mathbf{E} \triangleq
\begin{pmatrix}
h(K+1) & h(K) & \dots & h(3) & h(2) \\
g(K+1) & g(K) & \dots & g(3) & g(2) \\
h(K+3) & h(K+2) & \dots & h(5) & h(4) \\
g(K+3) & g(K+2) & \dots & g(5) & g(4) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
h(2K-1) & h(2K-2) & \dots & h(K+1) & h(K) \\
g(2K-1) & g(2K-2) & \dots & g(K+1) & g(K)
\end{pmatrix} \quad (12)
$$

$$
\mathbf{L} \triangleq
\begin{pmatrix}
h(1) & h(0) & 0 & \dots & 0 & 0 \\
g(1) & g(0) & 0 & \dots & 0 & 0 \\
h(3) & h(2) & h(1) & \dots & 0 & 0 \\
g(3) & g(2) & g(1) & \dots & 0 & 0 \\
\vdots & & \ddots & \vdots & \vdots & \vdots \\
h(K-1) & h(K-2) & h(K-3) & \dots & h(1) & h(0) \\
g(K-1) & g(K-2) & g(K-3) & \dots & g(1) & g(0)
\end{pmatrix} \quad (13)
$$

It was shown in [6] that if $g(n)$ is computed as in (4), then the Singular Value Decomposition (SVD) of $\mathbf{L}, \mathbf{U}$ and $\mathbf{E}$ are closely related. In particular, it was shown that, the rank of $\mathbf{L}$ and $\mathbf{U}$ is $r \leq K/2$ with equal singular values. Further, all the nonzero singular values are less than one. The SVD of $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{E}$ has the form [6]

$$
\begin{aligned}
\mathbf{L} &= \sum_{i=1}^{r} \delta_i \mathbf{w}_i \mathbf{v}_i' && (14) \\
\mathbf{U} &= -\sum_{i=1}^{r} \delta_i \widetilde{\mathbf{w}}_i \widetilde{\mathbf{v}}_i' && (15) \\
\mathbf{E} &= \sum_{i=1}^{r} s_i \sqrt{1-\delta_i^2}\left(\widetilde{\mathbf{w}}_i \mathbf{v}_i' + \mathbf{w}_i \widetilde{\mathbf{v}}_i'\right) + \sum_{i=2r+1}^{K} \mathbf{w}_i \mathbf{v}_i' && (16)
\end{aligned}
$$

where the singular values $\delta_i \leq 1$ for all i and $s_i \in \{1, -1\}$. Note that, $\{\mathbf{v}_i\}_{i=2r+1}^{K}$ are in the null space of $\mathbf{L} + \mathbf{U}$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR ORTHOGONAL WAVELETS

We need to give some definitions before introducing the main result. Consider a vector $\mathbf{v} = [v_1, v_2, \dots, v_K]'$. Define the square matrix $\mathbf{\Gamma}(\mathbf{v}, \sigma)$ of size $2K+2$ as in (17) (at the top of the following page). The matrix can be written as

$$
\mathbf{\Gamma}(\mathbf{v}, \sigma) = \begin{pmatrix} \widetilde{\boldsymbol{\gamma}}_1 & \boldsymbol{\gamma}_1 & \dots & \widetilde{\boldsymbol{\gamma}}_K & \boldsymbol{\gamma}_K & \widetilde{\mathbf{u}} & \mathbf{u} \end{pmatrix}' \quad (18)
$$

where $\mathbf{u}$ is an all-ones vector, and $\boldsymbol{\gamma}_i$ are vectors of length $2K+2$ defined for $1 \leq i \leq K/2$ as

$$
\boldsymbol{\gamma}_i \triangleq \begin{pmatrix} 0 & 0 & \dots & 0 & \mathbf{a}_i \end{pmatrix} \quad (19)
$$

where $\mathbf{a}_i$ is a vector of size $2i$ defined or $1 \leq i \leq K/2$ as

$$
\begin{aligned}
\mathbf{a}_i &\triangleq \begin{pmatrix} v_K & v_{K-1} & v_{K-2} & \dots & v_{K-2i+1} \end{pmatrix}' \\
\mathbf{a}_{i+K/2} &\triangleq \begin{pmatrix} v_K & v_{K-1} & \dots & v_1 & \sigma v_1 & -\sigma v_2 & \dots & \sigma v_{2i-1} & -\sigma v_{2i} \end{pmatrix}'
\end{aligned} \quad (20)
$$

By direct substitution and straightforward arithmetic we could show that, if $\mathbf{\Gamma}(\mathbf{v}, \sigma)$ in (17) is full-rank, then the columns of

$$
\mathbf{B}_i = \begin{pmatrix}
| & | & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
| & | & & & \\
| & | & \mathbf{0} & \vdots & \mathbf{0} \\
\mathbf{a}_i' & \mathbf{a}_{i-1}' & | & & \mathbf{0} \\
| & | & \mathbf{a}_{i-2}' & & \vdots \\
| & | & | & & \mathbf{a}_1'
\end{pmatrix} \quad (21)
$$

constitute a basis of the null space of

$$
\mathbf{C}_i = \begin{pmatrix}
\widetilde{\mathbf{a}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
& \widetilde{\mathbf{a}}_2 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
& & \widetilde{\mathbf{a}}_3 & \mathbf{0} & \dots & \mathbf{0} \\
& \vdots & & \vdots & & \\
& & & \widetilde{\mathbf{a}}_i & &
\end{pmatrix} \quad (22)
$$

The main result is summarized by the following theorem,

Theorem 1 *The necessary and sufficient condition that a vector* $\mathbf{h} = \begin{pmatrix} h(0) & h(1) & \dots & h(2K+1) \end{pmatrix}'$ *yields an orthogonal wavelet using the two-scale difference equation is that it is the solution of the linear system of equations:*

$$
\mathbf{\Gamma}(\mathbf{v}, \sigma).\mathbf{h} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{2} \end{pmatrix} \quad (23)
$$

where $\mathbf{\Gamma}(\mathbf{v}, \sigma)$*, as defined in (17), is full-rank.*

Proof If $\mathbf{h}$ is a wavelet filter then the SVD of the components of the matrix representation (i.e., $\mathbf{U}, \mathbf{L}$ and $\mathbf{E}$) is as shown in (14), (15), and (16). If $\mathbf{v} = \begin{pmatrix} v(1) & v(2) & \dots & v(K) \end{pmatrix}'$ is a right singular vector of $\mathbf{L}$ with $\delta < 1$ as the corresponding singular value and $\mathbf{w}$ as the corresponding left singular vector, then

$$
\mathbf{L}\widetilde{\mathbf{v}} = \mathbf{0} \quad (24)
$$

which by simple reorganization yields the first K rows of $\mathbf{\Gamma}(\mathbf{v}, \sigma)$. Further we have from (14), and (16)

$$
\begin{aligned}
\mathbf{L}\mathbf{v} &= \delta \mathbf{w} && (25) \\
\mathbf{E}\mathbf{v} &= -\sqrt{1-\delta^2}\,\widetilde{\mathbf{w}} && (26)
\end{aligned}
$$

where we chose the sign in (16) to be negative. then we have

$$
\mathbf{E}\mathbf{v} = -\frac{\sqrt{1-\delta^2}}{\delta}\widetilde{\mathbf{L}\mathbf{v}} \quad (27)
$$

then by setting

$$
\sigma \triangleq \frac{\sqrt{1-\delta^2}}{\delta} \quad (28)
$$

we get the system of equations

$$
\mathbf{E}\mathbf{v} + \sigma\widetilde{\mathbf{L}\mathbf{v}} = 0 \quad (29)
$$

which again yields by simple reorganization the second K rows of $\mathbf{\Gamma}(\mathbf{v}, \sigma)$. Note that the L2-norm of $\mathbf{v}$ needs not to be unity because of the zero in the right hand side. The last two rows in (23) are a direct consequence of (6) and (7).

$\mathbf{\Gamma}(\mathbf{v}, \sigma) \triangleq$

$$
\left(\begin{array}{ccccccc|cc|ccccccc}
v_{K-1} & -v_K & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & v_K & v_{K-1} \\
v_{K-3} & -v_{K-2} & v_{K-1} & -v_K & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & v_K & v_{K-1} & v_{K-2} & v_{K-3} \\
 & & & & & \vdots & & & & & \vdots & & & & & \\
v_1 & -v_2 & v_3 & -v_4 & \dots & v_{K-1} & -v_K & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_K & v_{K-1} & \dots & v_4 & v_3 & v_2 & v_1 \\
\hline
\sigma v_2 & \sigma v_1 & v_1 & -v_2 & \dots & v_{K-3} & -v_{K-2} & v_{K-1} & -v_K & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & v_K & v_{K-1} & v_{K-2} & v_{K-3} & \dots & v_2 & v_1 & -\sigma v_1 & \sigma v_2 \\
\sigma v_4 & \sigma v_3 & \sigma v_2 & \sigma v_1 & \dots & v_{K-3} & -v_{K-2} & v_{K-3} & -v_{K-2} & v_{K-1} & -v_K & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & v_K & v_{K-1} & v_{K-2} & v_{K-3} & v_{K-4} & v_{K-5} & \dots & -\sigma v_1 & \sigma v_2 & -\sigma v_3 & \sigma v_4 \\
 & & & & & \vdots & & & & & \vdots & & & & & \\
\sigma v_K & \sigma v_{K-1} & \sigma v_{K-2} & \sigma v_{K-3} & \dots & \sigma v_2 & \sigma v_1 & v_1 & -v_2 & v_3 & -v_4 & \dots & v_{K-1} & -v_K & 0 & 0 \\
0 & 0 & v_K & v_{K-1} & \dots & v_4 & v_3 & v_2 & v_1 & -\sigma v_1 & \sigma v_2 & \dots & -\sigma v_{K-3} & \sigma v_{k-2} & -\sigma v_{K-1} & \sigma v_K \\
\hline
1 & -1 & 1 & -1 & \dots & 1 & -1 & 1 & -1 & 1 & -1 & \dots & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1
\end{array}\right) \tag{17}
$$

Conversely, if $\mathbf{h}$ is the solution of (23), then (6) and (7) are satisfied. In the following we show that $\mathbf{h}$ also satisfies (5). First, we prove that

$$\sum_n h(n)h(n-2i) = 0 \text{ for } i > 0 \tag{30}$$

The nonzero terms in the left hand side in (30) is the inner product of two vectors $\mathbf{y}_i$ and $\mathbf{z}_i$ of size $2(K+1-i)$, defined as

$$
\begin{aligned}
\mathbf{y}_i &\triangleq \begin{pmatrix} h(2i) & h(2i+1) & \dots & h(2K+1) \end{pmatrix}' \qquad (31) \\
\mathbf{z}_i &\triangleq \begin{pmatrix} h(0) & h(1) & \dots & h(2K+1-2i) \end{pmatrix}' \qquad (32)
\end{aligned}
$$

If $\mathbf{h}$ satisfies (23), then by direct substitution we get,

$$\mathbf{C}_{K-i}\mathbf{z}_i = \mathbf{0} \tag{33}$$

$$\mathbf{B}'_{K-i}\mathbf{y}_i = \mathbf{0} \tag{34}$$

where $\mathbf{B}_i$ and $\mathbf{C}_i$ are as defined in (21) and (22) respectively. Hence, $\mathbf{z}_i$ is in the null space of $\mathbf{C}_{K-i}$. Then we have

$$\mathbf{z}_i = \mathbf{B}_{K-i}\mathbf{s} \tag{35}$$

for some nonzero vector $\mathbf{s}$. Hence, from (34) we get $\langle \mathbf{z}_i, \mathbf{y}_i \rangle = 0$ for any $i > 0$.
Now it remains to prove that $\sum_n |h(n)|^2 = 1$. The last two rows in (23) imply that,

$$\sum_n h(2n) = \sum_n h(2n+1) = \frac{1}{\sqrt{2}} \tag{36}$$

By straightforward arithmetic, we get

$$\left[\sum_n h(2n)\right]^2 + \left[\sum_n h(2n+1)\right]^2 = \sum_n |h(n)|^2 + 2\sum_{i>0}\sum_n h(n)h(n-2i)$$

From the previous analysis, the second term of the left hand side is zero and this completes the proof. ■

The structure of $\mathbf{\Gamma}$ allows for an efficient solution of $\mathbf{h}$ that requires little computation. This solution also gives more insight to the design problem using the proposed parametrization. Let $\mathbf{G}$ denote the first $2K+1$ rows of $\mathbf{\Gamma}$, i.e.,

$$\mathbf{G}(\mathbf{v}, \sigma) = \begin{pmatrix} \tilde{\boldsymbol{\gamma}}_1 & \boldsymbol{\gamma}_1 & \dots & \tilde{\boldsymbol{\gamma}}_K & \boldsymbol{\gamma}_K & \tilde{\mathbf{u}} \end{pmatrix}' \tag{37}$$

where the vectors in the right hand side are as defined in (18). The row space of the first $2K$ rows of $\mathbf{G}$ is the union of two disjoint spaces that are composed of the even and odd rows. Denote the matrices that contain the first K odd and even rows of $\mathbf{G}$ by $\mathbf{D}$ and $\mathbf{F}$ respectively. Note that, $\mathbf{D}\mathbf{F}' = \mathbf{0}$. Define the matrix,

$$
\mathbf{Q} \triangleq \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & -1 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 \\
0 & 0 & \dots & 0 & -1 & 0 & 0 \\
0 & 0 & \dots & 1 & 0 & 0 & 0 \\
\vdots & & \vdots & & \vdots & & \\
0 & -1 & \dots & 0 & 0 & 0 & 0 \\
1 & 0 & \dots & 0 & 0 & 0 & 0
\end{pmatrix} \tag{38}
$$

It is straightforward to show that,

$$\mathbf{F} = \mathbf{D}\mathbf{Q} \tag{39}$$

The row spaces of $\mathbf{D}$ and $\mathbf{F}$ are disjoint. Denote $\mathbf{P}_D$ and $\mathbf{P}_F$ as the projection matrices onto these row spaces respectively. Then the projection onto the row space of the first $2K$ rows of $\mathbf{G}$ is

$$
\begin{aligned}
\mathbf{P} &\triangleq \mathbf{P}_D + \mathbf{P}_F \\
&= \mathbf{P}_D - \mathbf{Q}\mathbf{P}_D\mathbf{Q} \qquad (40)
\end{aligned}
$$

where

$$\mathbf{P}_D = \mathbf{D}'(\mathbf{D}\mathbf{D}')^{-1}\mathbf{D} \tag{41}$$

Note that, the matrix to be inverted has only dimension of K. Define the component of the last row of $\mathbf{G}$ that does not lie in the row-space of the first $2K$ rows as:

$$\bar{\mathbf{u}}' \triangleq \tilde{\mathbf{u}}'(\mathbf{I} - \mathbf{P}) \tag{42}$$

then $\mathbf{P}_G$ could be written as:

$$\mathbf{P}_G = \mathbf{P} + \frac{\bar{\mathbf{u}}\bar{\mathbf{u}}'}{\bar{\mathbf{u}}'\bar{\mathbf{u}}} \tag{43}$$

Note that, $\mathbf{h}$ is in the null space of $\mathbf{G}$ and it could be viewed as the component of the last row in $\mathbf{\Gamma}$ which does not belong to the row space of $\mathbf{G}$, i.e.,

$$\mathbf{h} = \frac{\mathbf{P}_G^{\perp}\mathbf{u}}{\|\mathbf{P}_G^{\perp}\mathbf{u}\|} \tag{44}$$

where $\mathbf{P}_G^{\perp} = \mathbf{I} - \mathbf{P}_G$ is the orthogonal projection onto the null space of $\mathbf{G}$, and $\mathbf{u}$ is an all-ones column vector.

One advantage of the proposed parametrization in (23) is that we can find closed-form expressions of the wavelet filter coefficients at different orders. For $K = 2$ (i.e., filter order is 6) a closed form is listed in table 1. The design variables in this case are (v_1, v_2, σ). Expression for higher filter order is straightforward using direct arithmetic which is simply the last row of Γ^{-1} in (23) scaled by $\sqrt{2}$.

Table 1. Filter Parameterization for $K = 2$, (where $\alpha \triangleq \sqrt{2}(v_1^2 + v_2^2)(\sigma^2 + 1)$)

Coefficient	Expression
$h(0)$	$v_2(v_1 + v_2 - \sigma v_1 + \sigma v_2)/\alpha$
$h(1)$	$v_1(v_1 + v_2 - \sigma v_1 + \sigma v_2)/\alpha$
$h(2)$	$\sigma(\sigma - 1)/\alpha$
$h(3)$	$\sigma(\sigma + 1)/\alpha$
$h(4)$	$v_1(v_1 - v_2 + \sigma v_1 + \sigma v_2)/\alpha$
$h(5)$	$-v_2(v_1 - v_2 + \sigma v_1 + \sigma v_2)/\alpha$

Note that the only constraint in the choice of the design parameters $(\mathbf{v}, \sigma)$ is that the corresponding matrix $\mathbf{\Gamma}$ is full-rank. There is no restriction on the values of the parameters, and this simplifies search procedures in the design optimization problem.

4. DISCUSSION

The main result of this work is that an orthonormal wavelet basis of compact support could be designed as a function of an almost *arbitrary* vector. If $\mathbf{\Gamma}(\mathbf{v}, \sigma)$ in (17) is full-rank, then each entry of $\mathbf{\Gamma}^{-1}$ is a differentiable function of the entries of $\mathbf{\Gamma}$ almost everywhere because it could be expressed as a ratio of two determinants and the determinant is a continuous function of its entries [7]. Therefore the coefficients $\{h(n)\}_{n=0}^{2K+1}$ of the two scale difference equation in (2) could be expressed as a *differentiable* function of the vector $(\mathbf{v}'\ \sigma)'$ that constitutes the entries of $\mathbf{\Gamma}(\mathbf{v}, \sigma)$ by solving the *linear* system of equations in (23). There is only one loose constraint on our choice: $\mathbf{\Gamma}(\mathbf{v}, \sigma)$ is full-rank. This formulation allows the deployment of standard optimization techniques that requires a regular objective function.

Compared to earlier parametrizations of compactly supported orthonormal wavelets, e.g., [8]-[10], the proposed algorithm provides a single time-domain parametrization that applies to all filter sizes and does not require recursive relations at different orders. Further, for a certain choice of the unknown vector $(\mathbf{v}'\ \sigma)'$ a closed-form solution could be computed using analytic expressions for the cofactors of the elements of the last row in $\mathbf{\Gamma}(\mathbf{v}, \sigma)$. Further, there are no bounds on the values of the decision variables as long as $\mathbf{\Gamma}$ is full-rank.

The proposed parameterization could be used in a wide class of applications where optimized design of orthonormal wavelets is needed. In [11], the proposed parameterization along with the cascade algorithm [4] is used to design orthonormal wavelets and scaling functions of compact support that are matched to a predefined template. The problem is modeled as a standard nonlinear programming problem that is solved using standard techniques. Interestingly, the design of the matched scaling function does not require the use of the iterative cascade algorithm [4]. In [12], the proposed parameterization was used to design a wide class of useful wavelets that has linear constraints. In particular, the classical Daubechies wavelets with maximal number of vanishing moments were reproduced after setting the maximal moments requirement as a linear constraint to the wavelet design problem. Wavelets with close to linear-phase were designed using the same procedure. In [13] the proposed parameterization was used to design orthogonal boundary wavelet filters for a perfect-reconstruction finite-length wavelet transform. These boundary wavelet filters allow for a wider class of boundary linear extension rather than the common periodic extension that results in signal discontinuity.

The proposed parameterization is also applicable to orthogonal filter banks with two extra degrees of freedom by dropping the last two rows of $\mathbf{\Gamma}(\mathbf{v}, \sigma)$ in (17) (that are specific to orthonormal wavelets). This gives an under-determined linear system of equations with a nonzero null space whose solutions are two-channel orthogonal filter banks.

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