SPARSE SIGNAL RECONSTRUCTION BASED ON SIGNAL DEPENDENT NON-UNIFORM SAMPLES

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ABSTRACT

The classical approach to A/D conversion has been uniform sampling and we get perfect reconstruction for bandlimited signals by satisfying the Nyquist Sampling Theorem. We propose a non-uniform sampling scheme based on level crossing (LC) time information. We show stable reconstruction of bandpass signals with correct scale factor and hence a unique reconstruction from only the non-uniform time information. For reconstruction from the level crossings we make use of the sparse reconstruction based optimization by constraining the bandpass signal to be sparse in its frequency content. While overdetermined system of equations is resorted to in the literature we use an undetermined approach along with sparse reconstruction formulation. We could get a reconstruction $\mathbf{SNR} > 20\mathbf{dB}$ and perfect support recovery with probability close to 1, in noise-less case and with lower probability in the noisy case. Random picking of LC from different levels over the same limited signal duration and for the same length of information, is seen to be advantageous for reconstruction.

***Index Terms*—** Zero Crossing (ZC), Level Crossing (LC), Sparse Reconstruction, ADCs, Non-uniform Samples (NUS), Compressive Sensing (CS)

1. INTRODUCTION

Natural phenomena, often represented as continuous valued functions, do have a lot of redundancy, owing to the inherent structures generating the signal. There has been lot of work to remove such redundancy for compact transmission or storage and then reconstruct the signal from the compact representation. In some other applications, simply a limited number of samples are available from which the signal itself or its structural parameters need to be estimated. Recent work in compressive sensing (CS) is pushing signal sampling and redundancy removal closer, providing new solutions to signal acquisition as well as showing new applications. We explore in this paper signal reconstruction from signal dependent non-uniform samples, in contrast to signal independent uniform samples, along with sparsity constrained reconstruction.

In contrast to uniform sampling (Nyquist sampling), the signal can be sampled at preset amplitude level crossing (LC). The signal $x(t)$, in this scheme is sampled when it or the output of an operator applied on it, satisfies certain value which triggers the sampling. The sampling time instant sequence is dictated by the signal and is in general non-uniform. Thus, the signal is represented by a sequence of time instants at which it has crossed the preset level instead of signal values at preset times as in Nyquist sampling. This scheme is also referred to as implicit or Lebesgue sampling [1]. Initial work in this regard dates to Logan's theorem [2] which states the sufficient (but not necessary) conditions for reconstructing an octave bandwidth signal from only the zero crossings upto a scale factor. Multiple level based LC sampling (see Fig.1), adaptive level-crossing based sampling [3], computation of LC times of an analog signal refers to some of the work in this regard for using implicit sampling scheme. Also Tsividis in [4] proposes a mixed domain signal and system processing based on input decomposition.

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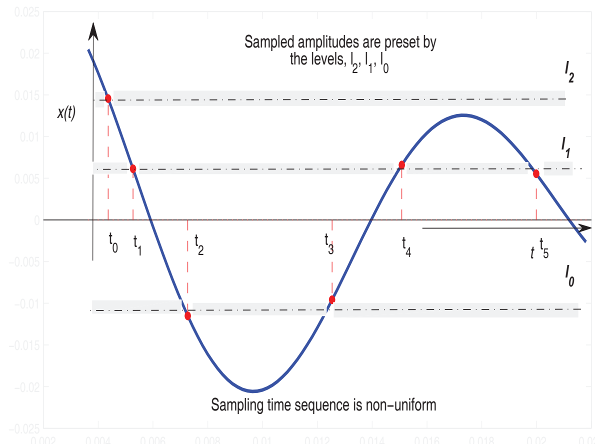

Fig. 1: Level crossing times for signal sampling. For this signal, $x(t)$, sampling instants are $\{(t_0, l_2), (t_1, l_1), (t_2, l_0), (t_3, l_0), (t_4, l_1), (t_5, l_1)\}$.

Implicit sampling in general results in non-uniform time samples. It is well established [5] that a bandlimited signal is uniquely determined from its non-uniform samples (NUS), provided that the average sampling rate exceeds the Nyquist rate. However, NUS based reconstruction based on direct implementation of deterministic functions is computationally impossible because of the need for infinite number of samples. Hence with finite samples we need to approximate the signal as best as possible. The reconstruction approaches in the literature include least-squares and other interpolation techiniques and are reviewed in [6]. The approach of implicit sampling though seems elegant, the reconstruction is not perfect and with a tradeoff of number of samples available along with the issues of robustness to noise. For LC based NUS, the noise can be in the signal itself, or on the LC sampling instants, or in the quantization of these timing values.

A general result for the recovery of one dimensional signal from LC instants is still lacking. In this paper, we consider bandpass periodic signals, its sampling based on LC and reconstruction from the NUSs. In many practical settings the signals possess some smoothness and also are bandpass in nature. For the reconstruction, we

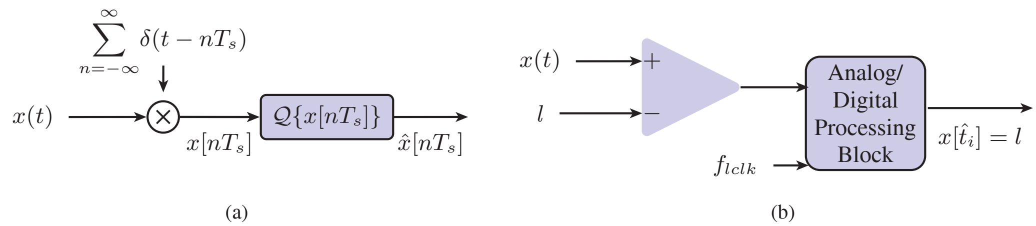

Fig. 2: (a)A Uniform Sampling based ADC. The sampling period is denoted by T_s (b)A generic Non-Uniform Sample based ADC

assume the signal is sparse in the bandpass frequency content. We focus on perfect reconstruction of the signal from less than Nyquist number of samples based on LC based non-uniform samples over a limited finite duration. This is in contrast to sampling by random projections as in CS [7]. We consider LC based sampling, a generalization over ZC. The problem of LC based signal reconstruction has been studied by mainly using interpolation approach. We here explore and compare nonlinear approximations based on solving a non-convex problem of reconstruction from the level crossing information and a sparse reconstruction framework. We deliberately use LCs to enable perfect reconstruction of the signal with accurate scale factor of close to 1. The inclusion of levels other than zero, together with ZCs enables faster and higher probability in perfect reconstruction along with convergence of the algorithms. We examine a random combination among the available LCs for different levels and reconstruction from an undetermined system of equations. Under this formulation we show improvement in signal reconstruction by using multiple levels and find that a random selection amongst LCs giving higher probability of recovery over use of a single level with the same number of level crossing instants. The interplay between reconstruction error and gradual placement of levels farther from zero level is shown.

The paper is organized as follows. The basis for level crossing based sampling is provided in Section 2. Section 3 gives the problem formulation for level crossing based implicit sampling and reconstruction. Section 4 gives the simulation details and the results discussion. We conclude in section 5.

2. LEVEL CROSSINGS BASED SIGNAL SAMPLING

The sampling of a signal based on LCs captures more information than ZCs since LCs include amplitude information also, which is lost in ZCs. The LC based ADCs are asynchronous in nature and provide and compete with synchronous ADCs with the benefits of lower power dissipation, electromagnetic interference reduction and improved Figure of Merit [8, 9]. These can be implemented without a global clock. The sampling rate if any is locally defined by the signal which makes them better suited for non-stationary signals [10] as well as giving more compact representation than Nyquist samples globally. Fig. 2 illustrates uniform sampling vs LCs based nonuniform sampling.

The approaches taken for the reconstruction of the analog signal from the NUSs include linear interpolation[11] based on polynomial kernel [10], sum-of-sincs kernel [12] and piecewise linear interpolation based on splines, prolate spheroidal wave functions[13],iterative methods [14]. In [15] attempt to make use of sparsity based signal reconstruction from ZCs is made. The algorithm proposed in [15] because of depending on ZCs is not able to guarantee on the performance. These reconstruction algorithms need more number of samples so as to solve an overdetermined system of equations and have convergence dependent on choosing of algorithm parameters. We formulate the reconstruction from LCs as a nonlinear optimization problem based on the prior knowledge of the domain in which the signal is sparse. The approach taking the advantage of sparsity is more efficient than a direct pseudo-inverse based solution.

3. PROBLEM FORMULATION

Consider a real periodic signal of finite energy, $s(t) \in \mathbf{L}^2[0,T]$, satisfying the Dirichlet conditions. Then $s(t)$ has equivalent representation as:

$$s(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \Big(\alpha_k \cos\Big(\frac{2\pi kt}{T}\Big) + \beta_k \sin\Big(\frac{2\pi kt}{T}\Big)\Big). \quad (1)$$

where $\alpha_0,\ \alpha_k,\ \beta_k$ are the Fourier series coefficients. Instead of $s(t)$, we focus on bandpass signal, $x(t)$ with a frequency span of $[\frac{p}{T},\ \frac{q}{T}]$ with $q > p > 0$,and $\{p, q\} \in \mathbb{Z}$. Then $x(t)$ has the representation:

$$x(t) = \sum_{k=p}^{q} \Big(\alpha_k \cos\Big(\frac{2\pi kt}{T}\Big) + \beta_k \sin\Big(\frac{2\pi kt}{T}\Big)\Big). \quad (2)$$

We denote the space of T duration periodic signals band limited to (in rad/sec) $\frac{2\pi p}{T}$ to $\frac{2\pi q}{T}$ by $\mathcal{V}_B$. For $q = 2p$, $x(t)$ is an octave band signal. With the formulation in (2) we have $\mathcal{V}_B$ being spanned by the $2(q-p+1)$ functions $\{e^{-j2\pi k/T},\ p \leq |k| \leq q\}$. The dimension of $\mathcal{V}_B$ is $2(q-p+1)$. For a uniform sampling grid we have with $\mathbf{\Psi}$ being the Fourier dictionary,

$$\mathbf{x} = \mathbf{\Psi}\mathbf{a} = [\mathbf{\Psi}^{\cos}\ \mathbf{\Psi}^{\sin}] \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}_{2(q-p+1)\times 1} \quad (3)$$

In the above equation, $\mathbf{a} = [\boldsymbol{\alpha}^T\ \boldsymbol{\beta}^T]^T$ is the cosine and sine Fourier coefficient vector in order. The goal is to reconstruct $x(t) \in \mathcal{V}_B$ using finite number (M) of NUS taken through LC information. For NUS, with the sampling grid being non-uniform we have the Fourier dictionary defined by the LCs time instants information. For the formulation here, we consider $\mathcal{L}$ to be composed of a single level l. In the simulation we populate the matrices with information of LCs for multiple level. The $x(t)$ is sampled at the LC for level l. The sampled time instants are then represented by $\mathcal{T}^l = \{t_i^l : i \in [1, M]\}$, again with M denoting the number of level crossings over the sampled time duration. The sampling operation is defined using the sampling instants information with the sampling kernel $\mathbf{\Psi}_{\mathcal{T}^l}$ as,

$$\mathbf{x}_l = [l \dots l]_{M\times 1}^T = \mathbf{\Psi}_{\mathcal{T}^l}\mathbf{a} = [\mathbf{\Psi}_{\mathcal{T}^l}^{\cos}\ \mathbf{\Psi}_{\mathcal{T}^l}^{\sin}] \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} \quad (4)$$

For ZCs, $l = 0$, the problem has been approached by solving for the nullspace solution and the reconstruction is difficult, it is only upto a scale factor and the convergence depends on the parameters in the algorithm [15]. For $l \neq 0$ a straight forward approach is to solve the set of M linear equations with $2(q-p+1)$ unknowns. With $N = 2(q-p+1)$ the computation requires inverse and with $N > 2(q-p+1)$ it needs the pseudo-inverse of $\mathbf{\Psi}_{\mathcal{T}^l}$. This is computationally intensive when M is large. We consider the system of linear equations to be under-determined based on not all LCs time instants information is available. This can be formulated by multiplying equation (4) with $\mathbf{\Phi}_{L\times M}$, a rectangular matrix with $L < M$ and the i^{th} row having a 1 at the j^{th} location and all other entries of the row being zeros, which enables picking of the desired LCs. This will pick L of the LCs information and with $L < 2(q-p+1)$ we have,

$$\mathbf{y}_{L\times 1} = \mathbf{\Phi x}_l = \mathbf{\Phi\Psi}_{\mathcal{T}^l}\mathbf{a} \quad (5)$$

The problem of reconstruction is now ill-defined with the linear system of equations being under-determined. We make use of sparsity of $x(t)$ in its bandpass frequency content and obtain an approximate solution.

3.1. Sparse Recovery Using LCs

We assume the signals is sparse in the bandpass frequency content with $\|\mathbf{a}\|_0 \leq K$. This seems to be a reasonable assumption for practical bandpass signals. The assumption of sparsity has been shown to be useful in solving under-determined system of linear equations [16].

3.2. Constrained Optimization Solution

The solution to (5) can be framed as an optimization problem and the system of equations being under-determined different solutions can be obtained depending on the cost function $J(.)$. We use three different approaches well known in solving under-determined problems. The optimization problem for (5) is framed as,

$$\mathcal{P}_J^a : \ \mathbf{min_a} J(\mathbf{a}), \text{ subject to } \mathbf{\Phi\Psi}_{\mathcal{T}^l}\mathbf{a} = \mathbf{y}, \quad (6)$$

The most common choice for $J(.)$ to solve for LCs based on (4) has been the squared Euclidean norm, $\|\mathbf{a}\|_2$. Using this for (6) gives the closed form pseudo-inverse solution,
$\hat{\mathbf{a}} = (\mathbf{\Phi\Psi}_{\mathcal{T}^l})^T(\mathbf{\Phi\Psi}_{\mathcal{T}^l}(\mathbf{\Phi\Psi}_{\mathcal{T}^l})^T)^{-1}\mathbf{y} = (\mathbf{\Phi\Psi}_{\mathcal{T}^l})^{\dagger}\mathbf{y}$.
This is computationally intensive for each LCs set, it does not take into account the sparsity in $\mathbf{a}$ and though closed form is not the best solution. It does not recover the exact support of $\mathbf{a}$ when it is sparse. The other choices are choosing l_p norms which induce sparsity in the solution, and hence we go with $l_p - norm$ as cost function, $\|\mathbf{a}\|_p$ with $0 \leq p \leq 1$. The two well known approaches in this are basis pursuit and orthogonal matching pursuit (OMP) algorithms. The optimization for Basis pursuit is,

$$\mathcal{P}_1^a : \ \mathbf{min_a}\{\lambda\|\mathbf{a}\|_1 + \|\mathbf{\Phi\Psi}_{\mathcal{T}^l}\mathbf{a} - \mathbf{y}\|_2\}, \quad (7)$$

The parameter λ makes the optimization to be tuned between degree of sparsity and accuracy in the solution.

4. SIMULATIONS

For the simulations, a continuous domain signal and its LC instants are obtained approximately by using zero padding in the frequency domain followed by then interpolating linearly in time to find estimate of the LC time instants. We vary the number of levels and their placement. We do not report on adaptation of levels. The LCs level are chosen to be $\eta(\max|x(t)|)$ with $|\eta| \in [0,1)$. In (2) we consider $T = 1$ and $q > (p+1)$. The sparsity factor K is varied from $10\% - 90\%$ of the bandwidth of $x(t)$. The K nonzero location in the sine and the cosine Fourier coefficient vector, $\mathbf{a}$, are selected randomly which also selects the active frequencies in the bandpass spectrum. These active coefficients values are drawn from a normal distribution and the remaining are set to zero. After obtaining the LC instants from the corresponding time domain signal, the LC sampled values are picked up based on the formulation in (5). The L level crossing instant are selected and we solve for the $N = 2(q-p+1)$ unknowns using pseudo-inverse, OMP, and Basis Pursuit. A Monte-Carlo simulation with 100 trials was done and for each trial a reconstruction with $SNR > 20\mathbf{dB}$ was considered perfect recovery. For support recovery error (S_e) [16] we make use of the following with $\hat{\mathbf{a}}$ denoting the recovered coefficient vector,

$$S_e = \frac{\max\{|\mathbf{a}|, |\hat{\mathbf{a}}|\} - |\mathbf{a} \cap \hat{\mathbf{a}}|}{\max\{|\mathbf{a}|, |\hat{\mathbf{a}}|\}}$$

In the simulations we do not assume any apriori information about K except that $\mathbf{a}$ is sparse hence $K < N$.

4.1. Reconstruction from LCs Based on Sparsity

With bandwidth $N/2 = 40$, denoting the cardinality of $\mathbf{a}$, and $K = 20$, the under-determined system of equation is solved using two arbitrary different levels. The reconstruction is done using Basis Pursuit [17]. For the simulations, $\lambda = .000165$ was found to give good reconstruction SNR for (7). The plot in Fig. 3 shows perfect reconstruction with $SNR = 35$ $\mathbf{dB}$. Similar results are obtained with OMP. However, pseudo-inverse solution fails to reconstruct the signal because of inability to recover the sparse support.

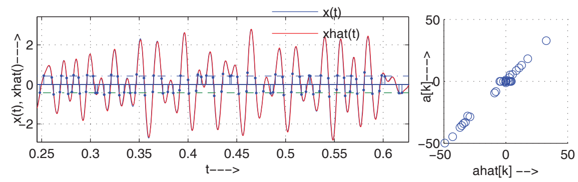

Fig. 3: Reconstruction based on Basis Pursuit with sparsity factor of 25%. (The plot is better visualized in color.) The levels along with the sampling instants are shown in the first plot. The second plot is the scatter plot showing perfect support recovery.

4.2. Pseudo-inverse (non-iterative) vs OMP

With a fixed sparsity of $\frac{K}{N} = 0.2$, the lower cut off frequency was increased from $p = 20$ to $p = 60$ with the higher cut off being $q = 2p$. The LC set was composed of 3 levels randomly placed. A random selection of LC instants is taken. The reconstruction was carried out by solving for an undetermined system of linear equations with $2(q-p+1)$ unknowns. As seen in Fig.4 OMP takes advantage of sparsity and performs better than direct pseudo-inverse. The advantage is clear in both probability of exact recovery and support recovery of the nonzero values in $\mathbf{a}$. The plot also shows the decrease in probability of recovery as p (hence bandwidth) increases. This means increase in higher frequency content in the signal and its reconstruction is effected by lower resolution in the capturing of the LCs time instants. In a practical realization, the LC time instants will be perturbed. This is analogous to jitter but does not accumulate over time. The sample is taken at an uncertain time instant around the nominal time instant. This can be modeled as an additive noise with the signal. However, the noise depends not only on the distribution of the perturbation, but also on the slope of the measured signal at the LC.

4.3. Level Dependency

We compare the performance with respect to choice of levels for LC as a function of the dynamic range of the signal. The experiment is carried out with a single level, $l = \eta\max|x(t)|$, with $\eta \in (0,1)$. The bandwidth of the signal and the bandpass frequency span is fixed with sparsity made to sweep from $\frac{K}{N} = 0.1$ to 0.9. The plots in Fig.5

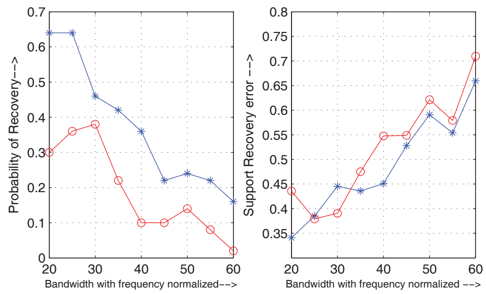

Fig. 4: Effect of bandpass frequency content on reconstruction. $*$ denotes plot for OMP and $\circ$ denotes plot for pseudo-inverse.

show that with a single level the closer l is to 0, higher is the probability of recovery and support recovery. This is an intuitive result as the number of LCs captured is more and we are more likely to capture at least two samples per cycle of the frequencies present in the spectrum. This was carried out with the system of linear equations being not undetermined and hence the performance of both pseudo-inverse and OMP are similar. However, the support recovery is relatively better in OMP than with pseudo-inverse.

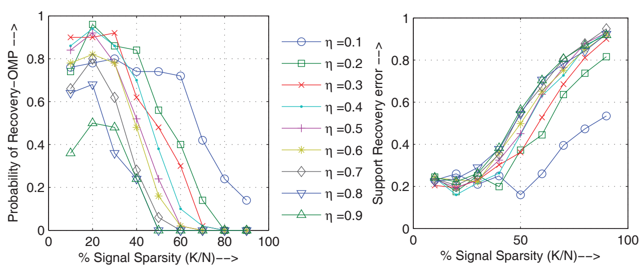

Fig. 5: Reconstruction performance dependency on single level (l) placement, with $l = \eta \max |x(t)|$, with $\eta \in (0, 1)$.

4.4. Random LCs from the preset Levels

Given to choose L LCs time instant information with $L < 2(q-p+1)$ we simulate to see which combination of LCs performs better. The simulation result show that a random selection of LCs amongst the available levels tends to perform better than choosing the time instant information from a single level's LCs. This is interesting as it means in picking up same length information a random picking carries more information and hence all LC time instants are not equally important.

4.5. Sparsity and Reconstruction from random LCs

Here we compare the effect of increasing sparsity and sampling the LC instants randomly by choosing from the obtained LCs for $l = 3$ levels. We do not take all the LC instants corresponding to a single level instead we take a random combinations of LCs instants obtained to make up the new LCs instants set. The plots in Fig.6 show the comparison of Basis Pursuit based recovery and reconstruction with pseudo-inverse. As seen in the plots the performance based on Basis Pursuit is relatively higher because of advantage of sparsity in reconstructing with an restricted number of LCs instants. The advantage decreases with increasing signal sparsity (K/N).

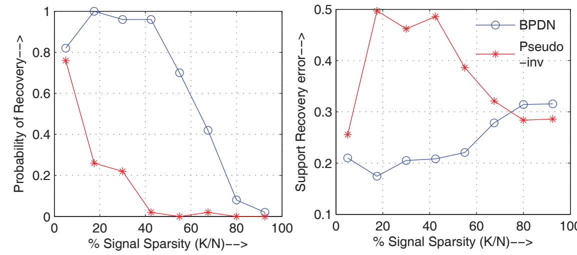

Fig. 6: Effect of sparsity

5. CONCLUSION

It is interesting that sparse reconstruction using either basis pursuit or OMP is quite effective in recovering the signal based on its level crossings information only. Surprisingly a random set of LCs performed better than any single level LCs when the system of equations is underdetermined. With the notion of signal sampling itself based on its local values, there is much promise to explore signal dependent non-uniform samples for many different applications.

6. REFERENCES

[1] K.J. Astrom and B.M. Bernhardsson, "Comparison of riemann and lebesgue sampling for first order stochastic systems," in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002,*, dec. 2002, vol. 2, pp. 2011 – 2016 vol.2.

[2] Jr. B.F. Logan, "Information in the zero crossings of bandpass signals," *AT&T Technical Journal*, Apr 1977.

[3] Suleyman S. Kozat Karen M. Guan and Andrew C. Singer, "Adaptive reference levels in a level-crossing analog-to-digital converter," *EURASIP Journal on Advances in Signal Processing*, 2008.

[4] Y. Tsividis, "Digital signal processing in continuous time: a possibility for avoiding aliasing and reducing quantization error," in *IEEE International Conference on Acoustics, Speech, and Signal Processing, 2004. Proceedings. (ICASSP '04).*, may 2004, vol. 2, pp. ii – 589–92 vol.2.

[5] Kung Yao and J. Thomas, "On some stability and interpolatory properties of nonuniform sampling expansions," *IEEE Transactions on Circuit Theory,*, vol. 14, no. 4, pp. 404 – 408, dec 1967.

[6] F. Marvasti, "Nonuniform sampling: Theory and practice," *Springer*, 2001.

[7] E.J. Candes and M.B. Wakin, "An introduction to compressive sampling," march 2008, vol. 25, pp. 21 –30.

[8] L. Fesquet M. Renaudin E. Allier, G. Sicard, "A new class of asynchronous a/d converters based on time quantization," in *Proc. of ASYNC*. IEEE, 2003, pp. 36–50.

[9] H.V. Sorensen N. Sayine and T. R. Viswanathan., "A level-crossing sampling scheme for a/d conversion," *IEEE Trans. Circuits and Systems-II: Analog and Digital Signal Processing*, Apr 1996.

[10] S. Chandrasekhar and T.V. Sreenivas, "Instantaneous frequency estimation using level-crossing information," in *IEEE International Conference on Acoustics, Speech, and Signal Processing, 2003. Proceedings. (ICASSP '03). 2003*, april 2003, vol. 6, pp. VI – 141–4 vol.6.

[11] E. Margolis and Y.C. Eldar, "Nonuniform sampling of periodic bandlimited signals," july 2008, vol. 56, pp. 2728 –2745.

[12] R. Kumaresan and N. Panchal, "Encoding bandpass signals using zero/level crossings: A model-based approach," *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 18, no. 1, pp. 17 –33, jan. 2010.

[13] Seda Senay, Luis F. Chaparro, and Lutfiye Durak, "Reconstruction of nonuniformly sampled time-limited signals using prolate spheroidal wave functions," *Signal Processing*, vol. 89, no. 12, pp. 2585 – 2595, 2009.

[14] M. MalmirChegini and F. Marvasti, "Performance improvement of level-crossing a/d converters," in *IEEE International Conference on Telecommunications and Malaysia International Conference on Communications, 2007. ICT-MICC 2007.*, may 2007, pp. 438 –441.

[15] P.T. Boufounos and R.G. Baraniuk, "Reconstructing sparse signals from their zero crossings," in *IEEE International Conference on Acoustics, Speech and Signal Processing, 2008. ICASSP 2008.*, 31 2008-april 4 2008, pp. 3361 –3364.

[16] M. Elad, "Sparse and redundant representations," *Springer*, 2010.

[17] "http://sparselab.stanford.edu and CVX, a package for specifying and solving convex programs," .