COMPRESSED SENSING ENHANCED RANDOM EQUIVALENT SAMPLING

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ABSTRACT

The feasibility of compressed sensing (CS) based waveform-reconstruction for data sampled from random equivalent sampling (RES) method is investigated. A novel measurement matrix motivated by the Whittaker-Shannon interpolation formula is proposed for this purpose. Experiments indicate that, for spectrally-sparse signal, the CS reconstructed waveform exhibits significantly higher signal-to-noise ratio (SNR) than that using the traditional time alignment method. A prototype realization of this proposed CS-RES method has been developed using off-the-shelf components. It is able to capture analog waveform at an equivalent sampling rate of 25 GHz while sampled at 100 MHz physically.

Index Terms**—** Random equivalent sampling, non-uniform sampling, sparsity, compressed sensing, signal reconstruction

1. INTRODUCTION

Random equivalent sampling (RES) [1]-[4] is a random sampling method that enables digital sampling of a periodic analog signal using an analog-to-digital converter (ADC) clocked at a frequency much lower than the Nyquist rate.

Compared to uniformly-spaced sampling technique, RES often takes considerably longer sampling time to produce a sufficiently accurate reconstructed waveform. The main cause for such a delay is the uneven random distribution of the relative positions of RES samples within a cycle of the sampling clock. To achieve desired accuracy, one often has to wait patiently until sufficient number of samples to populate the waveform buffer. This long sampling latency may limit potential applications of RES for real time acquisition of high frequency signals. In this work, we propose to use compressed sensing (CS) to reconstruct a spectrally sparse periodic analog waveform to reduce this unwanted delay.

CS [5]-[7] has been proposed as an efficient signal reconstruction method for random samples of spectrally sparse analog signals [8], [9]. Previously, the feasibility of applying CS to sample spectrally sparse periodic signals has been demonstrated [8], [9]. However, both these earlier methods require pre-processing of the analog signal using additional special purpose random sampling circuitry. In particular, a pseudo-random sequence clocked at the Nyquist rate is needed to modulate the analog signal. Implementation of such a pseudo-random sequence generator would be non-trivial and costly.

A key contribution of this work is the successful demonstration of the feasibility and potential advantage of incorporating CS with RES sampling. A novel measurement matrix motivated by the Whittaker-Shannon interpolation formula is proposed to facilitate CS reconstruction. The CS reconstruction algorithm has been incorporated into a prototype of a RES sampling enabled digital oscilloscope [10] developed recently by the first author. This oscilloscope is capable of capturing analog waveform at an equivalent sampling rate of 25 GHz while sampled at 100 MHz physically. It is observed that the CS reconstructed waveform exhibits higher signal-to-noise ratio (SNR) while requiring fewer RES samples (and hence shorter delays). An extended version of this paper will be published in near future [11].

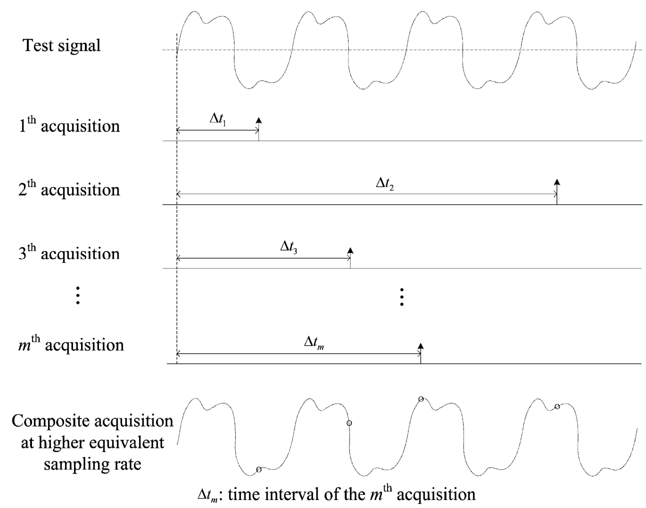

Figure 1. Illustration of random equivalent sampling

2. RANDOM EQUIVALENT SAMPLING

The basic principle of RES sampling is illustrated in Fig. 1. Given a periodic analog signal as shown in the solid line on the top row, a level-trigger circuitry compares this analog waveform against a reference voltage shown as the horizontal dashed line. A trigger pulse will be generated whenever the voltage of the analog signal rises crossing the reference voltage. It is assumed that exactly one trigger pulse will be generated per cycle of the analog signal. If so, the trigger signal would be activated at a rate equal to the fundamental frequency f_0 of the analog signal. The trigger

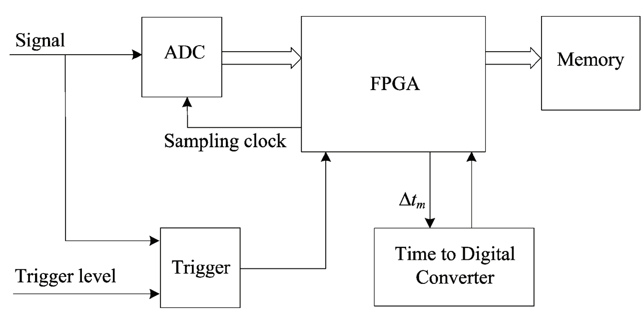

Figure 2. A block diagram of a RES device

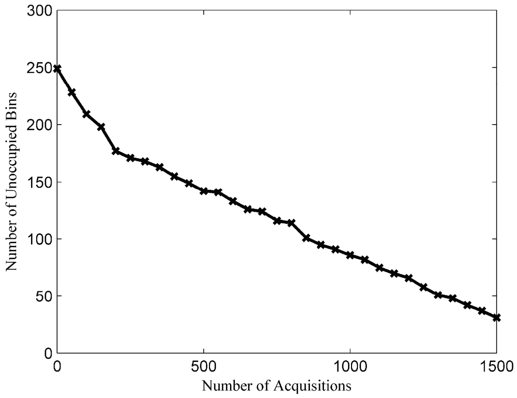

Figure 3. The numbers of the unfilled bins vs the numbers of samples acquired using RES.

pulse provides a fixed reference point to align samples. A block diagram of a RES device is shown in Fig. 2.

The ADC is clocked at a sampling frequency $f_s << f_0$ (sub-Nyquist rate sampling). Let Q be a positive integer, the relation between f_0 and f_s may be explicitly expressed as:

$$T_s = Q \cdot T_0 + \delta \tag{1}$$

where $T_s = 1/f_s$ is the sampling interval, $T_0 = 1/f_0$ is the period of the periodic analog signal, and $0 \le \delta = T_s \bmod T_0 < T_0$. Let us define Δt ($0 \le \Delta t \le T_s$) to be the time difference between the trigger pulse and the immediate sampling clock edge. Therefore, Δt gives the relative position of the sample within one cycle of the sampling clock.

The success of RES hinges upon the evenly distribution of Δt of different samples over the sampling interval $[0, T_s]$. This may be accomplished by injecting a random phase dithering β uniformly over $[0, T_s]$ [11]. The random phase dithering can be achieved by passing the sampling clock through a variable delay circuitry before feeding into the ADC. The variable delay circuitry may be realized with a field programmable gate array (FPGA) where a number of selectable delay paths are implemented. In practical application, the interval T_s is partitioned into equally spaced bins of duration T_e. The value $f_e = 1/T_e$ then becomes the *equivalent sampling frequency* of RES. If $T_e < T_0/2$ (i.e. $f_e > 2f_0$), then the analog waveform may be sampled without aliasing. In other words, the RES should not be applied to sample analog signals whose bandwidth exceeds half of the equivalent sampling frequency.

Due to this uneven distribution of sampling interval, RES would require a rather large number of samples to fill the empty bins and thereby achieve the desired accuracy. This is illustrated in Fig. 3. With T_s = 10 ns and T_e = 40 ps, there are N = 250 bins in the waveform buffer to be filled. After 1500 samples are acquired by the RES circuitry, there are still about 40 bins left empty. On average, one useful sample is obtained for every 7 random samples. Clearly, the inefficiency of the RES sampling method is demonstrated. Below, we propose to use a CS reconstruction method to improve the efficiency and accuracy of the RES sampling when the underlying analog signal has a spectrally sparse frequency spectrum.

3. COMPRESSED SENSING SIGNAL RECONSTRUCTION

Compressed sensing (CS) is an emerging data sampling paradigm that has received much attention [5]- [9], [12]. For a class of signals that exhibit a "spectral sparseness" property, CS method promises perfect reconstruction of the signal using very few measurements. With instrumentation applications, an analog signal is sparse if it is periodic and contains few significant harmonic components. Since RES also requires the underlying analog signal to be periodic, it is natural to consider the feasibility of applying CS to improve the quality of reconstructed signals.

In CS, the signal to be reconstructed is denoted by an N dimensional vector $\mathbf{x}$. In the current application, $\mathbf{x}$ would be the unknown analog signal sampled at the equivalent sampling rate f_e. Using RES or other methods, a set of *measurements* of the elements of $\mathbf{x}$ is obtained and denoted by a vector $\mathbf{y}$. In RES, each element of $\mathbf{y}$ is the output of a low-rate ADC. It can be represented as a weighted linear combination of elements in $\mathbf{x}$ that contains evenly spaced samples of the unknown waveform, with the sampling rate $f_e > f_{Nyquist}$ ($f_{Nyquist}$ is the Nyquist rate of the signal to be measured). Let these weights be arranged in a *measurement matrix* $\mathbf{\Phi}$, one may express the relation between $\mathbf{x}$ and $\mathbf{y}$ as follows:

$$\mathbf{y} = \mathbf{\Phi x}. \tag{2}$$

Since the signal is periodic, it has a discrete Fourier spectrum (line spectrum). Such a signal is *spectrally sparse* if only very few Fourier coefficients have significant magnitudes while other are nearly zero. In other words, the energy of the signal is concentrated on few spectral coefficients. In particular, a *K-sparse* signal $\mathbf{x}$ has K significant spectral coefficients where K is the *sparsity level*. If $\mathbf{x}$ is sampled from a *K-sparse* periodic analog signal $x(t)$, it can be approximated by a linear combination of K ($K << N$) discrete Fourier basis functions, i.e.,

$$\mathbf{x} = \sum_{i=1}^{K} \alpha_{n_i} \psi_{n_i} \tag{3}$$

where $\psi_{n_i} \in \mathbf{\Psi}$, $n_i \in \{1, 2, \ldots, N\}$, and $\mathbf{\Psi}$ is the inverse discrete Fourier transform (IDFT) matrix [13]. Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ be the coefficients vector of $\mathbf{x}$ in $\mathbf{\Psi}$, and $\boldsymbol{\alpha}$ is a sparse vector consisting of K non-zero Fourier coefficients. Equation (2) can be rewritten as:

$$\mathbf{y} = \mathbf{\Phi x} = \mathbf{D}\boldsymbol{\alpha} \tag{4}$$

where $\mathbf{D}$ (= $\mathbf{\Phi} \cdot \mathbf{\Psi}$) is the equivalent measurement matrix. Given the observations $\mathbf{y}$, the measurement matrix $\mathbf{\Phi}$, and the transform matrix $\mathbf{\Psi}$, the purpose of CS reconstruction is to find $\boldsymbol{\alpha}$ such that

$$\|\boldsymbol{\alpha}\|_0 \textit{ is minimized subject to } \|\mathbf{y} - \mathbf{D}\boldsymbol{\alpha}\|_2 \le \eta. \tag{5}$$

Here the l_0 norm counts the number of non-zero elements in a vector $\boldsymbol{\alpha}$, and η is a pre-defined acceptable recovery error. Minimizing $\|\boldsymbol{\alpha}\|_0$ is equivalent to minimize the number of non-zero elements of the solution $\boldsymbol{\alpha}^{\#}$ and therefore force the solution to be a sparse vector. Enforcing the constraint $\|\mathbf{y} - \mathbf{D}\boldsymbol{\alpha}\|_2 \le \eta$ would ensure the reconstructed signal $\mathbf{x}^{\#} = \mathbf{\Psi}\boldsymbol{\alpha}^{\#}$ will yield (almost) the same measurements $\mathbf{y}$ if subjected to the same RES sampling process.

3.1 Measurement Matrix

A key contribution of this work is the development of a special kind of measurement matrix. Unlike general CS approaches where random measurement matrices are used, this special measurement matrix for RES sampled signals is motivated by the low-rate ADC sampling mechanism. More specifically, the measurement matrix is formed by the well-known Whittaker-Shannon interpolation

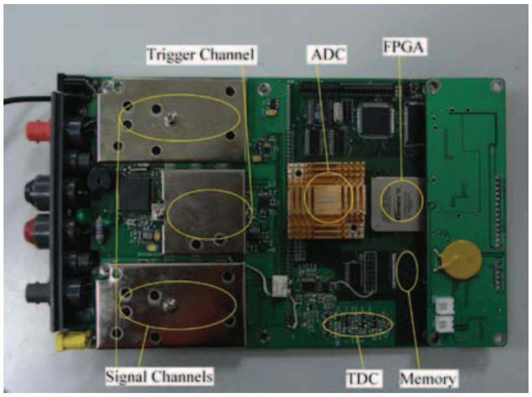

Figure 4. A RES prototype circuit board

formula. According to the Shannon sampling theorem [14], the observation $\boldsymbol{y}$ and the original signal $\boldsymbol{x}$ satisfy the following formula

$$y(\Delta t_m) = \sum_{n=1}^{N} x(nT_e)\,\mathrm{sinc}\left(\frac{\Delta t_m}{T_e} - n\right) \tag{6}$$

where $1 \le m \le M$, $1 \le n \le N$, M is the number of random measurements ($M < N$), Δt_m is time interval for the m^{th} sample as described in Fig. 1, and T_e is the equivalent sampling period. In general, it requires no more than $c{\cdot}K{\cdot}\log(N)$ [9] random measurements (c is a constant, say 5 in practice) to recover the signal with high probability. From (6), the relation between the non-uniformly (randomly) sampled signal $\mathbf{y}$ of size M (i.e. composite acquisitions in Fig. 1) and the uniform equivalent sampling signal $\mathbf{x}$ of size N can be represented by the following matrix-vector representation:

$$\begin{bmatrix} y(\Delta t_1) \\ y(\Delta t_2) \\ \vdots \\ y(\Delta t_M) \end{bmatrix} = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,N} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{M,1} & \phi_{M,2} & \cdots & \phi_{M,N} \end{bmatrix} \begin{bmatrix} x(T_e) \\ x(2T_e) \\ \vdots \\ x(N \cdot T_e) \end{bmatrix}$$

$$\Leftrightarrow \mathbf{y} = \Phi\,\mathbf{x}. \tag{7}$$

In (7), $\mathbf{\Phi}$ is the measurement matrix with $(m,\ n)^{\text{th}}$ element as follows:

$$\phi_{m,n} = \mathrm{sinc}\left(\frac{\Delta t_m}{T_e} - n\right) = \mathrm{sinc}\left(B_m - n\right). \tag{8}$$

Due to the dependence on individually estimated Δt_m, each $\phi_{m,n}$ must be computed after the sample is acquired. However, since $0 \le B_m = \Delta t_m/T_e \le T_s/T_e = N$, the values of $\mathrm{sinc}(B_m - n)$ may be stored in a table and be accessed using a simple look-up table operations. This will significantly reduce the computational complexity.

3.2 OMP Recovery Algorithm

As described in the introduction of Section 3, the dimension of $\mathbf{y}$ is often much lower than that of $\mathbf{x}$ ($M < N$). As such, recovering $\mathbf{x}$ from $\mathbf{y}$ is an ill-posed problem, and there may be many possible solutions of (5). Among them, the CS theory favors the mostly sparse solution that contains minimum number of non-zero elements in $\boldsymbol{\alpha}^{\#}$. However, the l_0 norm regularization problem is NP-hard, and is computationally challenging when N increases. A variety of signal recovery algorithms have been proposed to alleviate this difficulty. To solve the optimization problem of (4), one may either apply convex programming [15] or use a family of greedy pursuit algorithm [16]. For RES signal reconstruction, the large computation cost of convex programming makes it a less appealing approach. In this work, we adopt the orthogonal matching pursuit (OMP) algorithm [17], a variant of the greedy pursuit algorithm for solving the sparse vector $\boldsymbol{\alpha}$.

4. EXPERIMENT

Experiments are performed to compare the performance of the proposed CS-RES method against conventional interpolation based RES method. Two kinds of synthetic analog waveforms are generated and applied to a prototype RES sampling board to yield a list of time-indexed samples. The first kind of analog waveform is a spectrally sparse signal, and the second kind is a spectrally dense signal.

These synthesized analog waveforms are fed into a RES prototype board [13]. A picture of this prototype is shown in Fig. 4. The block diagram of the major component of this prototype is shown in Fig. 2. In this system, f_s = 100 MHz and f_e = 25 GHz. Thus the number of temporal bins $N = f_e/f_s = 250$. The phase of the sampling clock is dithered by passing the clock signal through a variable delay circuitry consisting of 20 different delay paths. For each new sample acquisition run, one of these 20 paths is randomly selected. Thus, β may assume one of 20 different values. After a fixed number of acquisition runs, a sequence of acquired RES samples and corresponding positions within the sampling period are obtained. Then, these RES samples will be used to reconstruct the original analog waveform using both the conventional interpolation method as well as the CS reconstruction method.

With the conventional interpolation method, a linear interpolation is applied to the RES sample to fill in sample values of empty bins. Then a low pass digital filter is applied to smooth the resulting waveform. With the CS reconstruction method, the positions of the RES samples are first utilized to compute the measurement matrix using look-up table method. Then the OMP reconstruction is applied to obtain the reconstructed signal $\mathbf{x}$.

SNR defined below is used as a metric to compare the quality of the reconstructed analog waveform:

$$SNR = 20 \cdot \log_{10}\left(\frac{\|\mathbf{x}\|}{\|\mathbf{x} - \mathbf{x}^{\#}\|}\right). \tag{9}$$

Where $\|\cdot\|$ is the Euclidean norm, $\mathbf{x}$ is the signal as measured by a sampling oscilloscope, and $\mathbf{x}^{\#}$ is the reconstructed signal vector.

The first test signal is a spectrally sparse amplitude-modulated (AM) analog waveform

$$f_{\mathrm{AM}}(t) = A\cos(2\pi f_c t) \cdot (m(t) + C) \tag{10}$$

where $m(t)$ is the modulation message signal, f_c = 0.8 GHz is the carrier frequency, A and C are the constants. The AM signal has $2K + 2$ nonzero frequency components ($K = 1$ in this example).
For the traditional time-alignment based reconstruction method, M = 118 RES samples are acquired. For the CS-based reconstruction method, M = 64 RES samples are used. The reconstructed signal using the traditional time alignment method achieve a SNR = 14.56 dB, while the CS reconstruction yield a SNR = 31.51 dB. The original waveform, the time-alignment based reconstructed waveform, and the CS reconstructed waveform are depicted in Fig. 5. Hence, the CS-RES achieves much higher SNR using fewer than half of the RES samples.

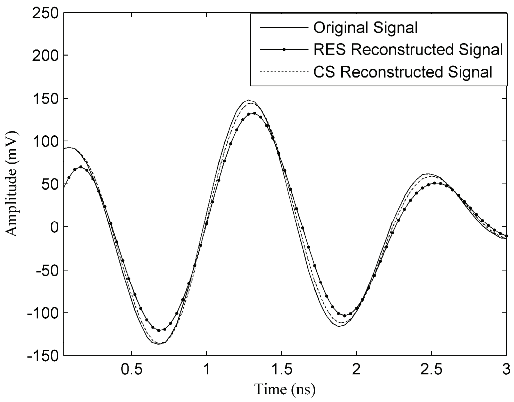

Figure 5. Restored waveform comparison.

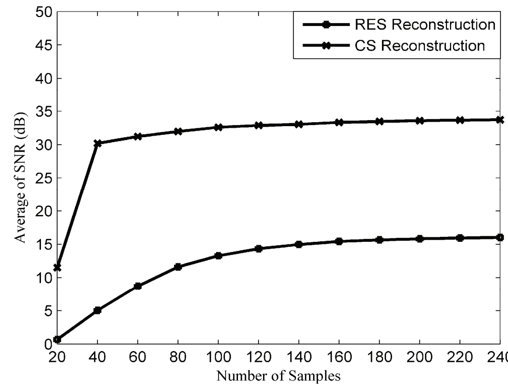

Figure 6. Averaged SNR comparison.

Next, the accuracy of reconstructed analog waveforms at different lengths of the RES sample sequence are compared. For the range of 20 to 240 RES samples in increment of 20 samples, 100 random trials are performed for each specific length. The mean of the SNRs of 100 trials at each length of the random samples is plotted in Fig. 6. It is clear that after the length of random samples increases beyond 40, the CS method yields much higher SNR with very small variance. Thus, for spectrally sparse signal (with very few harmonics), the advantage of the CS method is clearly demonstrated.

5. CONCLUSION

A CS based signal reconstruction method is proposed for reconstructing periodic waveforms sampled using the RES method. Compared to the traditional time-alignment RES reconstruction method, the CS approach yields better performance when the underlying waveform is spectrally sparse in the sense it contains few significant Fourier coefficients. To this aim, a novel measurement matrix structure motivated by the Shannon's sampling theory is proposed. A popular OMP CS reconstruction algorithm is applied to expedite computation. A hardware RES sampling prototype system is used to capture RES samples from synthesized analog waveforms. Using these RES samples as inputs, the CS reconstructed waveforms yield higher fidelity to the original analog waveform for spectrally sparse analog signals.

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