TWO DIMENSIONAL FREQUENCY ESTIMATION BY INTERPOLATION ON FOURIER COEFFICIENTS

Shanglin Ye^{*} Elias Aboutanios[†]

School of Electrical Engineering and Telecommunications The University of New South Wales Building G17, Sydney, NSW, Australia, 2052 E-mail: *z3368289@student.unsw.edu.au, [†]elias@ieee.org.

ABSTRACT

In this paper, we propose a computationally simple algorithm for the estimation of the frequencies of a random phase two-dimensional (2-D) complex exponential in additive noise by extending the 1-D estimator developed by Aboutanios and Mulgrew. The procedure of the algorithm is based on a twostage scheme consisting of a coarse estimator followed by a fine search stage. The separability of the problem implies that the estimator can be applied in each direction. Theoretical analysis shows, however, that the performance of the algorithm converges to the minimum point of the asymptotic variance after two iterations only if the estimation is applied jointly in the two dimensions. As in the 1-D case, this variance is extremely close to the 2-D Cramer-Rao Lower Bound. The simulation results are presented to verify the analysis.

Index Terms— Digital signal processing, frequency estimation, two-dimensional exponential, interpolation algorithm

1. INTRODUCTION

Frequency estimation, including the two-dimensional (2-D) case, is an important research problem in a wide range of applications, such as the processing of nuclear magnetic resonance (NMR) spectroscopy, radar imaging and biomedical instrumentations. In this paper, the estimation of frequencies of a single 2-D complex exponential in noise is addressed. The signal model can be written as:

$$x(m,n) = Ae^{j\phi + j2\pi(mf_1 + nf_2)} + w(m,n)$$
(1)

where $0 \le m \le M - 1$ and $0 \le n \le N - 1$. For our purposes, the signal amplitude A and random initial phase ϕ are considered to be nuisance parameters. The frequencies f_1 and f_2 are both normalized to [-0.5, 0.5]. The noise terms w(m, n) are the complex additive white Gaussian with zero mean and variance σ^2 . The signal to noise ratio (SNR) is given by $\rho = A^2/\sigma^2$.

The maximum likelihood estimator of the frequencies of a 2-D exponential signal is given by the maximiser of the 2-D periodogram. The Fast Fourier Transform (FFT) algorithm can be used to obtain a sampled version of the periodogram in a computationally efficient manner. However, this approach performs poorly compared to the optimum. To improve the estimation accuracy, high-resolution approaches, such as MEMP [1], 2-D ESPRIT [2] and IMDF [3], have been proposed, but these methods suffer from high computational cost [3]. In [4] the weight phase average (WPA) method is proposed, which is an efficient way for estimating signal 2-D exponential, however, it has a very high breakdown threshold. Therefore, we present in this paper a novel 2-D interpolation algorithm that achieves a good performance while maintaining computational simplicity. To this end, we extend the one-dimensional interpolation algorithm proposed by Aboutanios and Mulgrew (A&M Algorithm) in [5] to the 2-D case.

When assessing a frequency estimation algorithm, it is of prime importance to characterise the best possible performance. The lower bound on the estimation variance is given by the 2-D Cramer-Rao Lower Bound (2-D CRLB). Starting from the matrix form given in [1], we derive the following algebraic formulae of the 2-D CRLB for the signal model (1),

$$CRLB_{f_1} = \frac{6}{(2\pi)^2 \rho M N (M^2 - 1)}$$
(2)

$$CRLB_{f_2} = \frac{6}{(2\pi)^2 \rho NM(N^2 - 1)}.$$
 (3)

The rest of the paper is organized as follows. In section 2, we develop the 2-D interpolation algorithm. This followed by the theoretical analysis in section 3. Simulation results are given in section 4, while relevant conclusions are drawn in section 5.

2. THE 2-D INTERPOLATION ALGORITHM

The 2-D interpolation algorithm is summarised in Table 1. Similarly to the 1-D A&M algorithm, the 2-D estimator comprises two stages, the coarse estimation stage and the fine estimation stage, which is an effective and widely used strategy, [6]. The coarse estimation stage involves a 2-D maximum bin search of the 2-D periodogram to give coarse estimates of both f_1 and f_2 . In the fine estimation stage, two iterative estimators are implemented jointly in an interleaved fashion by interpolating 2-D Fourier coefficients at the edge of the maximum bin in order to refine the estimates.

	Table 1. The 2-D Interpolation Algorithm
Let	$X(k,l) = FFT(x)$ and $Y(k,l) = X(k,l) ^2$
Find	$(\hat{m}, \hat{n}) = \arg\max_{k,l} Y(k, l)$
Set	$\hat{\delta}_0 = 0$ and $\hat{\zeta}_0 = 0$
Loop	for <i>i</i> from 1 to Q, do
	$X_p^{(\delta)} = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} x(k,l) e^{-j2\pi(k\frac{\hat{m}+\hat{\delta}_{i-1}+p}{M}+l\frac{\hat{m}+\hat{\zeta}_{i-1}}{N})}$
	$\hat{\delta}_i = \hat{\delta}_{i-1} + h(\hat{\delta}_{i-1})$
	$X_p^{(\zeta)} = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} x(k,l) e^{-j2\pi(k\frac{\hat{m}+\hat{\delta}_i}{M}+l\frac{\hat{m}+\hat{\zeta}_{i-1}+p}{N})}$
	$\hat{\zeta}_i = \hat{\zeta}_{i-1} + g(\hat{\zeta}_{i-1})$
	where
	$h(\hat{\delta}_{i-1}) = \frac{1}{2} \Re \left\{ \frac{X_{0.5}^{(\delta)} + X_{-0.5}^{(\delta)}}{X_{0.5}^{(\delta)} - X_{-0.5}^{(\delta)}} \right\}$
	and
	$g(\hat{\zeta}_{i-1}) = \frac{1}{2} \Re \left\{ \frac{X_{0.5}^{(\zeta)} + X_{-0.5}^{(\zeta)}}{X_{0.5}^{(\zeta)} - X_{-0.5}^{(\zeta)}} \right\}$
Let	$\hat{f}_1 = rac{\hat{m} + \hat{\delta}_Q}{M}$ and $\hat{f}_2 = rac{\hat{n} + \hat{\zeta}_Q}{N}$

Assume that we are operating above the breakdown threshold and (\hat{m}, \hat{n}) is the index of the true maximum bin [5]. The true frequencies can then be written as:

$$f_1 = \frac{\hat{m} + \delta}{M}$$
 and $f_2 = \frac{\hat{n} + \zeta}{N}$ (4)

where δ and ζ are the residuals of f_1 and f_2 in the interval [-0.5, 0.5], respectively.

Let us focus on the frequency f_1 and examine the 2-D DFT coefficients either side of the maximum bin along the f_1 -axis,

$$X_p^{(\delta)} = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} x(k,l) e^{-j2\pi (k\frac{\hat{m}+p}{M} + l\frac{\hat{n}}{N})} + W_p, \quad (5)$$

where $p = \pm 0.5$ and W_p are the Fourier coefficients of the noise. Substituting (1) and (4) into (5) leads to

$$X_p^{(\delta)} = A e^{j\phi} \frac{(1+e^{j2\pi\delta})(1-e^{j2\pi\zeta})}{(1-e^{j2\pi\frac{\delta-p}{M}})(1-e^{j2\pi\frac{\zeta}{N}})} + W_p.$$
(6)

Now for $(\delta - p) \ll M$ and $\zeta \ll N$ we have that $e^{j2\pi\frac{\delta-p}{M}} \approx 1 + j2\pi(\delta - p)/M$ and $e^{j2\pi\frac{\zeta}{N}} \approx 1 + j2\pi\zeta/N$. Ignoring the noise components W_p , (6) becomes

$$X_p^{(\delta)} \approx b_\delta \frac{\delta}{\delta - p},\tag{7}$$

with b_{δ} given by

$$b_{\delta} = -MNAe^{j\phi} \frac{(1+e^{j2\pi\delta})(1-e^{j2\pi\zeta})}{4\pi^2\delta\zeta}$$

The estimation of the frequency residual δ can be obtained from

$$h(\delta) \approx \frac{1}{2} \Re \left\{ \frac{b_{\delta} \frac{\delta}{\delta - 0.5} + b_{\delta} \frac{\delta}{\delta + 0.5}}{b_{\delta} \frac{\delta}{\delta - 0.5} - b_{\delta} \frac{\delta}{\delta + 0.5}} \right\} = \delta.$$
(8)

The real part operation is not necessary here. However, it will result in a real-valued estimation of δ when the complex noise terms W_p are included. Similarly, we can find $g(\zeta) \approx \zeta$ by considering the coefficients $X_p^{(\zeta)}$.

3. THEORETICAL ANALYSIS

In this section, we derive the theoretical properties of the algorithm based on the procedure in [5]. We first focus on the derivation of the asymptotic variance and then analyse the iterative implementation of the algorithm. We only consider the analysis procedure for the estimation of f_1 , with the results for f_2 easily obtainable from those for f_1 .

3.1. Asymptotic Variance

Let $\hat{\delta}, \hat{\zeta}, \hat{f}_1, \hat{f}_2$ be the estimated values of their corresponding parameters. Including the noise terms W_p , (8) becomes

$$h \approx \frac{1}{2} \Re \left\{ \frac{2\delta + \frac{\delta^2 - 0.25}{b_{\delta}\delta} (W_{0.5} + W_{-0.5})}{1 + \frac{\delta^2 - 0.25}{b_{\delta}\delta} (W_{0.5} - W_{-0.5})} \right\}.$$
 (9)

Now W_p are of order $O\left(\sqrt{MN\ln(M)\ln(N)}\right)$ whereas b_{δ} is O(MN), resulting in the second term in the denominator of (9) being $O\left(M^{-\frac{1}{2}}N^{-\frac{1}{2}}\sqrt{\ln(M)\ln(N)}\right)$. Therefore,

$$\begin{split} h &\approx \delta + \frac{\delta^2 - 0.25}{2\delta} \, \Re \left\{ \frac{(1 - 2\delta)W_{0.5} + (1 + 2\delta)W_{-0.5}}{b_\delta} \right\} \\ &+ O\left(M^{-1}N^{-1}\ln(M)\ln(N) \right). \end{split}$$

Using the fact that the noise variances $var[\Re\{W_{0.5}\}] = var[\Re\{W_{-0.5}\}] = MN\sigma^2/2$, and

$$|b_{\delta}|^{2} = M^{2} N^{2} A^{2} \frac{\cos^{2}(\pi \delta) \sin^{2}(\pi \zeta)}{(\pi \delta)^{2} (\pi \zeta)^{2}},$$

the asymptotic variance of $\hat{\delta}$ simplifies to

$$\operatorname{var}[\hat{\delta}] = \frac{\pi^2 (\delta^2 - 0.25)^2 (4\delta^2 + 1)}{4M\rho \cos^2(\pi\delta)} \times \frac{(\pi\zeta)^2}{N \sin^2(\pi\zeta)} \quad (10)$$

The asymptotic variance of $\hat{\zeta}$ can be obtained by exchanging δ and ζ as well as M and N,

$$\operatorname{var}[\hat{\zeta}] = \frac{\pi^2 (\zeta^2 - 0.25)^2 (4\zeta^2 + 1)}{4N\rho \cos^2(\pi\zeta)} \times \frac{(\pi\delta)^2}{M \sin^2(\pi\delta)}.$$
 (11)



Fig. 1. Plot of the ratio of the asymptotic variance of \hat{f}_1 to the 2-D ACRLB as a function of δ and ζ .

Given that $\hat{f}_1 = (\hat{m} + \hat{\delta})/M$, the asymptotic variance of \hat{f}_1 becomes

$$\operatorname{var}[\hat{f}_1] = \frac{\pi^2 (\delta^2 - 0.25)^2 (4\delta^2 + 1)}{4M^3 \rho \cos^2(\pi \delta)} \times \frac{(\pi \zeta)^2}{N \sin^2(\pi \zeta)}.$$
 (12)

The ratio of the asymptotic variance to the asymptotic CRLB (ACRLB) is useful in putting the estimation performance in prespective. The 2-D ACRLB expression for f_1 is given by

$$CRLB_{f_1} \approx \frac{6}{(2\pi)^2 \rho M^3 N},\tag{13}$$

and the resulting ratio becomes

$$R_{f_1} = \frac{\pi^4 (\delta^2 - 0.25)^2 (4\delta^2 + 1)}{6\cos^2(\pi\delta)} \times \frac{(\pi\zeta)^2}{\sin^2(\pi\zeta)}.$$
 (14)

This ratio is shown in Fig.1. We see that it is dependent on both δ and ζ , but independent of M, N and the SNR. Also note that when $\delta = \zeta = 0$, $R_{f_1} = R_{f_2} = (\pi^4 (0.25)^2)/6 = 1.0147$, indicating that the 2-D interpolation algorithm has a minimum asymptotic variance that is just slightly above the 2-D CRLB.

The ratio of $var[\hat{f}_2]$ to the 2-D ACRLB is of course obtained by exchanging δ and ζ ,

$$R_{f_2} = \frac{\pi^4 (\zeta^2 - 0.25)^2 (4\zeta^2 + 1)}{6 \cos^2(\pi\zeta)} \times \frac{(\pi\delta)^2}{\sin^2(\pi\delta)}, \quad (15)$$

and the observations made for f_1 are also applicable to f_2 .

3.2. Iterative Implementation

Now we turn to the analysis of the iterative implementation. As is shown in the algorithm procedure in Section 2, for each iteration, the previous estimation of frequency residuals are removed from the maximum bin, so the estimators are applied to estimate the compensated data. Expanding $h(\delta)$ as a Taylor series about the true residual of f_1 , which we denote as δ_0 , yields

$$h(\delta) = \frac{\sin\left(\frac{2\pi}{M}(\delta_0 - \delta)\right)}{2\sin\left(\frac{\pi}{M}\right)} [1 + O(\lambda)]$$

= $(\delta - \delta_0) h'(\delta_0) [1 + O(\lambda)],$
where $\lambda = M^{-\frac{1}{2}} N^{-\frac{1}{2}} \sqrt{\ln(M) \ln(N)}$ and
 $h'(\delta_0) = -\frac{\pi}{M \sin\left(\frac{\pi}{M}\right)} [1 + O(\lambda)]$
= $-1 + O(\lambda).$

As a result, the estimate $\hat{\delta}$ at the *i*th iteration becomes

$$\begin{aligned}
\delta_i &= \delta_{i-1} + h(\delta_{i-1}) \\
&= \delta_0 + (\hat{\delta}_{i-1} - \delta_0) O(\lambda).
\end{aligned}$$
(16)

Similarly expanding $g(\zeta)$, we can get:

$$\hat{\zeta}_{i} = \hat{\zeta}_{i-1} + g(\hat{\zeta}_{i-1}) = \zeta_{0} + (\hat{\zeta}_{i-1} - \zeta_{0})O(\lambda),$$
(17)

where ζ_0 is the true residual of f_2 . Both (16) and (17) are easily shown to be contractive mappings [5], which ensures by the fixed point theorem that the estimates converge and $\lim_{i\to\infty} \hat{\delta}_i = \delta_0$ and $\lim_{i\to\infty} \hat{\zeta}_i = \zeta_0$. Now the variance expressions in (10) and (11), which we denote as $V_1(\delta, \zeta)$ and $V_2(\zeta, \delta)$ respectively, and their corresponding ratios to the ACRLB, are continuous functions of δ and ζ on $[-0.5, 0.5]^2$. As $h(\delta)$ and $g(\zeta)$ are implemented alternatively in each iteration, the variance of $\hat{\delta}_i$ is $V_1(\delta_0 - \hat{\delta}_{i-1}, \zeta_0 - \hat{\zeta}_{i-1})$, whereas that of $\hat{\zeta}_i$ is $V_2(\zeta_0 - \hat{\zeta}_{i-1}, \delta_0 - \hat{\delta}_i)$. Thus, in the limit, the variances become

$$\operatorname{var}[\hat{\delta}_{\infty}] = \lim_{i \to \infty} V_1(\delta_0 - \hat{\delta}_{i-1}, \zeta_0 - \hat{\zeta}_{i-1}) = V_1(0, 0),$$
(18)

and

$$\operatorname{var}[\hat{\zeta}_{\infty}] = \lim_{i \to \infty} V_2(\zeta_0 - \hat{\zeta}_{i-1}, \delta_0 - \hat{\delta}_i) = V_2(0, 0) = V_1(0, 0).$$
(19)

Thus, the variances of $\hat{\delta}$ and $\hat{\zeta}$ converge to the minimum point.

When running the estimator iteratively, however, the iterations should be stopped when the residuals become of order lower than or equal to the CRLB, which is $O\left(M^{-\frac{1}{2}}N^{-\frac{1}{2}}\right)$ for both δ and ζ . Following the argument of [5], an examination of the estimation functions $h(\delta)$ and $g(\zeta)$ reveals that only two iterations are needed to reach this condition. At the output of the second iteration, both $\delta_0 - \hat{\delta}_2$ and $\zeta_0 - \hat{\zeta}_2$ are $O\left(M^{-1}N^{-1}\ln(M)\ln(N)\right)$ which, in fact, is $O\left(M^{-\frac{1}{2}}N^{-\frac{1}{2}}\right)$. Therefore, it is sufficient to apply the estimator only for two iterations for the minimum variance to be attained.



Fig. 2. RMSE of the 2-D interpolation algorithm versus SNR (M = N = 32). 10000 Monte Carlo runs were used.

4. SIMULATION RESULTS

The proposed 2-D interpolation algorithm was implemented and the simulation results are reported in this section. Frequencies are randomly selected in all simulations. Fig. 2 shows the estimate root mean squared error (RMSE) versus SNR for M = N = 32. Included for comparison are the 2-D CRLB, the curve for f_1 obtained by the weighed phase average (WPA) method [4] as well as the curve for f_1 obtained by applying the 1-D estimator to each column of the data and averaging the results. As expected, the RMSE of the estimates of both f_1 and f_2 are quite close to the CRLB. It is obvious that the proposed algorithm outperforms WPA by enjoying a 30dB lower breakdown threshold. Also observe that for high SNR, the RMSE of the estimates obtained from the 1-D algorithm is identical to that of the 2-D interpolators. This is expected as the 2-D CRLB for the incoherent case is simply the 1-D CRLB divided by the number of estimates (that is for f_1 , for instance, it is the 1-D CRLB divided by N). However, the threshold of the 1-D case is about 15dB worse than the 2-D case. This results from the SNR gain that the coherent combining gives when the 2-D estimator is used. Since the estimate of f_1 is used in the calculation of \hat{f}_2 in the first iteration, we notice that the RMSE of \hat{f}_2 after the first iteration is slightly better than that of f_1 . Furthermore, the curves after iteration 1 clearly show the estimation bias at high enough SNR, with the bias disappearing at the output of the second iteration. Finally, for the sake of completeness, Fig. 3 shows the performance of the estimator for M = 2N = 64. The lower CRLB, as well as lower estimation variance, of f_1 after the second iteration are evident.

5. CONCLUSION

We have proposed a computationally simple algorithm for estimating the frequencies of a noisy 2-D random phase com-



Fig. 3. RMSE the 2-D interpolation algorithm versus SNR (M = 64, N = 32). 10000 Monte Carlo runs were used.

plex exponential. The algorithm comprises a coarse estimator using the maximum bin search followed by an interpolation stage. In the interpolation stage, two bins either side of the maximum are used to refine the estimates of the frequencies. The theoretical analysis shows that the performance of the iterative algorithm converges after two iterations, with the ratio of the asymptotic variance to the 2-D ACRLB achieving a minimum value of 1.0147. The theoretical results were confirmed by simulations.

6. REFERENCES

- Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil," *IEEE Trans. Signal Process.*, vol. 40, pp. 2267–2280, September 1992.
- [2] S. Rouquette and M. Najim, "Estimation of frequencies and damping factors by two-dimensional ESPRIT type methods," *IEEE Trans. Signal Process.*, vol. 49, pp. 237– 245, January 2001.
- [3] Jun Liu and Xiangqian Liu, "An eigenvector-based approach for multidimensional frequency estimation with improved identifiability," *IEEE Trans. Signal Process.*, vol. 54, pp. 4543–4556, December 2006.
- [4] S. Kay and R. Nekovei, "An efficient two-dimensional frequency estimator," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, pp. 1807–1809, October 1990.
- [5] E. Aboutanios and B. Mulgrew, "Iterative frequency estimation by interpolation on Fourier coefficients," *IEEE Trans. Signal Process.*, vol. 53, pp. 1237–1241, April 2005.
- [6] B. G. Quinn, "Estimating frequency by interpolation using Fourier coefficients," *IEEE Trans. Signal Process.*, vol. 42, pp. 1264–1268, May 1994.