U-INVARIANT SAMPLING AND STABLE RECONSTRUCTION IN ATOMIC SPACES

Volker Pohl and *Holger Boche*.

Chair for Theoretical Information Technology
Technische Universität München, 80290 München, Germany
{volker.pohl, boche}@tum.de

ABSTRACT

Given a U-invariant sampling scheme on an arbitrary Hilbert space $\mathcal{H}$. This paper characterizes atomic subspaces $\mathcal{A}$ of $\mathcal{H}$ such that every signal $x \in \mathcal{A}$ can be reconstructed from its samples acquired with this sampling scheme. If signal recovery is possible a linear filter is derived which reconstructs the signal from the samples.

Index Terms— Atomic spaces, sampling, stationary sequences

1. INTRODUCTION

The Nyquist-Shannon sampling theorem [1] states that every function x in the Paley-Wiener space PW^2_π of square integrable functions which are bandlimited to $[-\pi, \pi]$ can be reconstructed from its uniform samples $\{x(k)\}_{k\in\mathbb{Z}}$ as $x(t) = \sum_k x(k)\, \mathrm{sinc}(t-k)$ where $\mathrm{sinc}(t) := \sin(\pi t)/(\pi t)$. This reconstruction shows in particular that PW^2_π is the closed linear span of integer shifts of the sinc-function, i.e. PW^2_π is an example of a *shift-invariant (SI) space* [2]. In general, an SI space is generated by L functions $\phi^{(l)} \in L^2(\mathbb{R})$ as

$$\mathcal{A}(\phi) = \overline{\mathrm{span}}\{\phi^{(l)}(t - k\,a) \;:\; l = 1, 2, \dots, L;\; k \in \mathbb{Z}\}$$

wherein $a \in \mathbb{R}$ is a certain shift and $\overline{\mathrm{span}}$ stands for the closed linear span. The use of the sinc function as a generator of $\mathcal{A}$, in the Nyquist-Shannon sampling theorem, implies some conceptual and practical problems [3], mainly due the strict band-limits of the sinc function and its slow decay in the time domain. However, by an appropriate choice of the generators $\phi^{(l)}$, these problems can be eliminated such that SI spaces are frequently used as a signal model in sampling problems (see, e.g., [4, 5], and references therein).

Moreover, often the samples are not taken of the signal x itself but of a filtered version of it. This yields to the description of the sampling process as an evaluation of inner products $c_k = \langle x, s_k\rangle$ with a certain set of sampling function $\{s_k\}$ [3]. These sampling functions have often a particular structure. In the classical case, for example, the sampling functions are given by $s_k(t) = s(t - kT)$ where $s \in L^2(\mathbb{R})$ is the impulse response of a linear filter and $T \in \mathbb{R}$ is the sampling period. Thus, similar as in the definition of an SI subspace, the sampling functions are given by time-shifts of a certain generator function. Moreover, a multichannel sampling scheme is characterized by multiple generators, say $s^{(1)}, \dots, s^{(M)}$. Then $s^{(m)}_k(t) = s^{(m)}(t - kT)$ are the sampling functions, and the generalized samples are given by $c^{(m)}_k = \langle x, s^{(m)}_k\rangle$.

Atomic spaces are natural extensions of SI spaces. They are defined [6] by replacing the translation operator $\mathrm{T}_a : \phi(t) \mapsto \phi(t-a)$ in the definition of SI spaces, with an arbitrary unitary operator W on the actual Hilbert space $\mathcal{H}$:

$$\mathcal{A} = \overline{\mathrm{span}}\{\mathrm{W}^k \phi^{(l)} \;:\; l = 1, 2, \dots, L;\; k \in \mathbb{Z}\}\ .$$

Similarly, the sampling functions are often defined by a general unitary operator V on $\mathcal{H}$ as $s^{(m)}_k = \mathrm{V}^k s^{(m)}$ for $m = 1, \dots, M$ and $k \in \mathbb{Z}$. This paper considers the problem of reconstructing $x \in \mathcal{A}$ from the signal samples $c^{(m)}_k = \langle x, s^{(m)}_k\rangle$. This problem was investigated in [6] for the particular case where $\mathrm{W} = \mathrm{V}$ and where the number of generators L is equal to the number M of sampling channels. In the present paper, we allow for $L \neq M$ and we take $\mathrm{W} \neq \mathrm{V}$. However, it is assumed that W and V are similar in the sense that $\mathrm{W} = \mathrm{U}^R$ and $\mathrm{V} = \mathrm{U}^Q$ for some unitary operator U and some non-negative integers Q and R. Then we are able to derive a necessary and sufficient condition on the generators $\{\phi^{(l)}\}$ and $\{s^{(m)}\}$ such that every $x \in \mathcal{A}$ can be perfectly reconstructed from its samples $c^{(m)}_k = \langle x, s^{(m)}_k\rangle$. Moreover the transfer function of a corresponding reconstruction filter is given.

2. SAMPLING IN ATOMIC SPACES

Notations As usual, $L^2(\mathbb{R})$ denotes the Hilbert space of complex square integrable functions on the real axis $\mathbb{R}$ and for every $x \in L^2(\mathbb{R})$, its *Fourier transform* is defined by

$$\widehat{x}(\omega) = (\mathcal{F}\,x)(\omega) = \textstyle\int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\ , \quad \omega \in \mathbb{R}\ .$$

We write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ for the unit circle in the complex plane, and $L^2(\mathbb{T}, \mathbb{C}^N)$ is the Hilbert space of Lebesgue integrable functions on $\mathbb{T}$ with values in $\mathbb{C}^N$ equipped with the scaler product

$$\langle \mathbf{x}, \mathbf{y}\rangle_{L^2(\mathbb{T},\mathbb{C}^N)} = \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} \langle \mathbf{x}(\mathrm{e}^{\mathrm{i}\theta}), \mathbf{y}(\mathrm{e}^{\mathrm{i}\theta})\rangle_{\mathbb{C}^N}\,\mathrm{d}\theta\ .$$

Similarly, $\ell^2(\mathbb{C}^N)$ stands for the Hilbert space of square summable sequences in $\mathbb{C}^N$. For $N = 1$, we simply write $L^2(\mathbb{T})$ and ℓ^2. Every $\mathbf{x} \in L^2(\mathbb{T}, \mathbb{C}^N)$ can be written as a Fourier series as

$$\mathbf{x}(\mathrm{e}^{\mathrm{i}\theta}) = \textstyle\sum_{k\in\mathbb{Z}} \mathbf{x}_k\,\mathrm{e}^{\mathrm{i}k\theta} \ \text{ with } \ \mathbf{x}_k = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{x}(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\theta$$

and with the sequence $\{\mathbf{x}_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}^N)$ of *Fourier coefficients*. It is well known, that the above equations constitute a Hilbert space isomorphism between $L^2(\mathbb{T}, \mathbb{C}^N)$ and $\ell^2(\mathbb{C}^N)$.

The *(left) shift operator* $\mathrm{S} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined as

$$\mathrm{S} : \textstyle\sum_{k\in\mathbb{Z}} c_k\,\mathrm{e}^{\mathrm{i}k\theta} \mapsto \sum_{k\in\mathbb{Z}} c_{k+1}\,\mathrm{e}^{\mathrm{i}k\theta} \qquad (1)$$

or equivalently by $(\mathrm{S}f)(\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}\theta}$. For any positive integer R the *decimation operator* $\mathrm{D}_R : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined as

$$\mathrm{D}_R \;:\; \textstyle\sum_{k\in\mathbb{Z}} c_k\,\mathrm{e}^{\mathrm{i}k\theta} \mapsto \sum_{k\in\mathbb{Z}} c_{Rk}\,\mathrm{e}^{\mathrm{i}k\theta}$$

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or equivalently as $(\mathrm{D}_R f)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{R}\sum_{k=0}^{R-1} f\left(\mathrm{e}^{\mathrm{i}(\theta+k2\pi)/R}\right)$. Whenever one of these operators is applied to a vector or matrix valued function, it is understood that the operator is applied to each vector or matrix entry.

Finally, we remark that we will frequently need the notion of *stationary sequences* and the concept of *frames* and *Riesz bases* in Hilbert spaces. Because of the limited space, however, we only refer to corresponding literature on these topics, and especially to [7] where these concepts are explained in the spirit of U-invariant sampling as far it will be needed here.

Signal Space Let $\mathcal{H}$ be an arbitrary Hilbert space. We consider signals in atomic subspaces $\mathcal{A}$ of $\mathcal{H}$ [6]. In our case, these subspaces are characterized by a unitary operator U on $\mathcal{H}$, by a positive integer R, and by a set $\phi = \{\phi^{(l)}\}_{l=1}^{L}$ of functions in $\mathcal{H}$. We set

$$\phi_k^{(l)} := \mathrm{U}^{Rk}\phi^{(l)} , \qquad l = 1,2,\ldots,L\,;\ k \in \mathbb{Z}\,, \tag{2}$$

and define our signal space $\mathcal{A}$ as the closed linear span of $\{\phi_k^{(l)}\}$

$$\mathcal{A} = \mathcal{A}_{\mathrm{U}}(\phi, R) = \overline{\operatorname{span}}\{\phi_k^{(l)}\ :\ l = 1,\ldots,L;\ k \in \mathbb{Z}\}\,. \tag{3}$$

Every signal $x \in \mathcal{A}_{\mathrm{U}}(\phi, R)$ has thus the form

$$x = \sum_{k\in\mathbb{Z}}\sum_{l=1}^{L} x_k^{(l)}\,\mathrm{U}^{Rk}\phi^{(l)} = \sum_{k\in\mathbb{Z}}\sum_{l=1}^{L} x_k^{(l)}\,\phi_k^{(l)} = \sum_{k\in\mathbb{Z}} \mathbf{x}_k^{\mathrm{T}}\,\boldsymbol{\phi}_k \tag{4}$$

where $\boldsymbol{x} = \{\mathbf{x}_k := [x_k^{(1)},\ldots,x_k^{(L)}]^{\mathrm{T}}\}_{k\in\mathbb{Z}}$ is a sequence in $\mathbb{C}^L$ containing the coefficients of the signal x. The sequence $\boldsymbol{\phi} = \{\boldsymbol{\phi}_k = [\phi_k^{(1)},\phi_k^{(2)},\cdots,\phi_k^{(L)}]^{\mathrm{T}}\}_{k\in\mathbb{Z}}$ is said to be the *generator sequence* of $\mathcal{A}_{\mathrm{U}}(\phi, R)$. Clearly, $\boldsymbol{\phi}$ is an L-dimensional stationary sequence in $\mathcal{H}$, and it is always assumed that $\boldsymbol{\phi}$ is a Riesz basis for its closed linear span $\mathcal{A}_{\mathrm{U}}(\phi, R)$ [7, 8]. This assumption implies that $\boldsymbol{x} \in \ell^2(\mathbb{C}^L)$ and that every signal $x \in \mathcal{A}_{\mathrm{U}}(\phi, R)$ is uniquely defined by its coefficients $\boldsymbol{x} \in \ell^2(\mathbb{C}^L)$. Therefore, it will later be sufficient to determine the coefficients $\boldsymbol{x} = \{\mathbf{x}_k\}_{k\in\mathbb{Z}}$ to reconstruct $x \in \mathcal{A}_{\mathrm{U}}(\phi, R)$.

Sampling Space In modern sampling theory, the sampling of a signal $x \in \mathcal{H}$ is often described by an evaluation of inner products $c_k = \langle x, s_k\rangle$ with a set of sampling functions $\{s_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ [3], and where $\{c_k\}_{k\in\mathbb{Z}}$ are said to be the *(generalized) samples* of x. Here we consider U-invariant sampling schemes [7] in which the sampling functions have the following particular form: $s_k^{(m)} = \mathrm{V}^k s^{(m)}$, where $s^{(m)} \in \mathcal{H}$, with $m = 1,\ldots,M$, is a set of M different vectors in $\mathcal{H}$, and where V is a unitary operator on $\mathcal{H}$.

In particular, we assume that our sampling scheme is matched to the signal space $\mathcal{A}_{\mathrm{U}}(\phi, R)$ in such a way that both, the signal space and the sampling functions are generated by powers of the same unitary operator U. Specifically, we assume that $\mathrm{V} = \mathrm{U}^Q$ where $Q \in \mathbb{N}$ is a certain positive integer, and U is the same unitary operator used for the definition of the signal space (2) and (3). Thus, our sampling functions have the form

$$s_k^{(m)} := \mathrm{U}^{Qk}s^{(m)} , \qquad m = 1,\ldots,M\,;\ k \in \mathbb{Z}\,, \tag{5}$$

and the subspace

$$\mathcal{S} = \mathcal{S}_{\mathrm{U}}(s, Q) = \overline{\operatorname{span}}\{s_k^{(m)}\ :\ m = 1,\ldots,M;\ k \in \mathbb{Z}\} \tag{6}$$

of $\mathcal{H}$ is called the *sampling space* associated with our sampling scheme. We collect the sampling functions is an M-dimensional stationary sequence $\boldsymbol{s} = \{\boldsymbol{s}_k = [s_k^{(1)}, s_k^{(2)},\cdots,s_k^{(M)}]^{\mathrm{T}}\}_{k\in\mathbb{Z}}$.

Degree of Freedom The signal space $\mathcal{A}$ and the sampling space $\mathcal{S}$ have a similar structure. The main differences between both spaces are the dimensions L and M and the integers R and Q. These parameters characterize the *degree of freedom* in both spaces.

Consider the case where $\mathrm{U} = \mathrm{T}_1 : f(t) \mapsto f(t-1)$ is the translation operator and assume that the variable t stands for "time". Then the sampling scheme, defined by (5), takes M samples every Q time steps, i.e. the average sampling rate is $\sigma = M/Q$, and we will say that the corresponding sampling space $\mathcal{S}$ has a degree of freedom of $\sigma_{\mathcal{S}} = M/Q$. Similarly, every signal $x \in \mathcal{A}_{\mathrm{U}}(\{\phi^{(l)}\}_{l=1}^{L}, R)$ generates L symbols every R time steps (cf. (4)). This corresponds to a *rate of innovation* [9] of $\sigma = L/R$, and we will say that the signal space $\mathcal{A}$ has a degree of freedom of $\sigma_{\mathcal{A}} = L/R$. These intuitive notations are carried over to spaces generated by arbitrary unitary operators U:

Definition: *An atomic space $\mathcal{A}$ of the form* (3) *is said to have a* degree of freedom *of $\sigma_{\mathcal{A}} = M/Q$. If $\mathcal{A}$ is a sampling space then $\sigma_{\mathcal{A}}$ is also be called the* sampling rate*, and if $\mathcal{A}$ is a signal space then $\sigma_{\mathcal{A}}$ is also be called the* rate of innovation*.*

3. RECOVERY CONDITIONS

Fix a Hilbert space $\mathcal{H}$, a signal space $\mathcal{A} \subset \mathcal{H}$ of the form (3) with L generators $\{\phi^{(l)}\}_{l=1}^{L}$, and a sampling space $\mathcal{S} \subset \mathcal{H}$ of the form (6) with M generators $\{s^{(m)}\}_{m=1}^{M}$. We seek a necessary and sufficient condition on the generators $\{\phi^{(l)}\}_{l=1}^{L}$ and $\{s^{(m)}\}_{m=1}^{M}$ such that every $x \in \mathcal{A}$ can perfectly be reconstructed from its generalized samples $c_k^{(m)} = \langle x, s_k^{(m)}\rangle$, $m = 1,\ldots,M$; $k \in \mathbb{Z}$.

First we notice that $\boldsymbol{\phi} = \{\boldsymbol{\phi}_k = [\phi_k^{(1)},\phi_k^{(2)},\cdots,\phi_k^{(L)}]^{\mathrm{T}}\}_{k\in\mathbb{Z}}$ and $\boldsymbol{s} = \{\boldsymbol{s}_k = [s_k^{(1)}, s_k^{(2)},\cdots,s_k^{(M)}]^{\mathrm{T}}\}_{k\in\mathbb{Z}}$ are stationary sequences in $\mathcal{H}$ (cf., e.g., [7, 8]). However, for $R \neq Q$ both sequences are not stationary correlated, since for fixed l and m, the correlation function of $\{\phi_k^{(l)}\}_{k\in\mathbb{Z}}$ and $\{s_k^{(m)}\}_{k\in\mathbb{Z}}$, which is given by

$$\begin{aligned}\langle\phi_k^{(l)}, s_n^{(m)}\rangle &= \langle \mathrm{U}^{Rk}\phi^{(l)}, \mathrm{U}^{Qn}s^{(m)}\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(Rk-Qn)\theta}\,\Phi_{\phi,s}^{(l,m)}(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta\,,\end{aligned} \tag{7}$$

is not a function of the difference $k-n$ alone, but depends on k and n. Therein $\Phi_{\phi,s}^{(l,m)}$ stands for the cross spectral density of the stationary correlated sequences $\{\mathrm{U}^k\phi^{(l)}\}_{k\in\mathbb{Z}}$ and $\{\mathrm{U}^k s^{(m)}\}_{k\in\mathbb{Z}}$, and we write $\boldsymbol{\Phi}_{\phi,s}$ for the $M \times L$ matrix whose entry in the mth row and lth column is

$$[\boldsymbol{\Phi}_{\phi,s}(\mathrm{e}^{\mathrm{i}\theta})]_{m,l} = \Phi_{\phi,s}^{(l,m)}(\mathrm{e}^{\mathrm{i}\theta})\,. \tag{8}$$

The following theorem provides a necessary and sufficient condition such that every signal $x \in \mathcal{A}$ can be reconstructed from its generalized samples $c_k^{(m)} = \langle x, s_k^{(m)}\rangle$.

Theorem 1 (Conditions for Reconstruction): *Let $\mathcal{A}$ and $\mathcal{S}$ be a signal and a sampling space of the form* (3) *and* (6)*, respectively, and assume that the L-dimensional generator sequence $\boldsymbol{\phi}$ forms a Riesz basis for $\mathcal{A}$. Let $\boldsymbol{\Phi}_{\phi,s}$ be the $M \times L$ matrix as defined in* (8)*. Set*

$$\boldsymbol{\Psi}_{r,q}(\mathrm{e}^{\mathrm{i}\theta}) := (\mathrm{D}_{RQ}\mathrm{S}^{rQ-qR}\boldsymbol{\Phi}_{\phi,s})(\mathrm{e}^{\mathrm{i}\theta})\,,$$

and define the $RM \times QL$ matrix $\boldsymbol{\Psi}(\mathrm{e}^{\mathrm{i}\theta})$ as a block matrix by

$$\boldsymbol{\Psi}(\mathrm{e}^{\mathrm{i}\theta}) = \begin{pmatrix} \boldsymbol{\Psi}_{0,0}(\mathrm{e}^{\mathrm{i}\theta}) & \ldots & \boldsymbol{\Psi}_{0,Q-1}(\mathrm{e}^{\mathrm{i}\theta}) \\ \vdots & & \vdots \\ \boldsymbol{\Psi}_{R-1,0}(\mathrm{e}^{\mathrm{i}\theta}) & \ldots & \boldsymbol{\Psi}_{R-1,Q-1}(\mathrm{e}^{\mathrm{i}\theta}) \end{pmatrix}. \tag{9}$$

Then every $x \in \mathcal{A}$ can be recovered from its generalized samples $c_k^{(m)} = \langle x, s_k^{(m)} \rangle_{\mathcal{H}}$ by means of a bounded linear operator if and only if there exist constants $0 < A \leq B < \infty$ such that

$$A \leq \mathbf{\Psi}^*(\zeta)\mathbf{\Psi}(\zeta) \leq B \quad \textit{for almost all } \zeta \in \mathbb{T} \,. \tag{10}$$

Sketch of proof: Let $\mathbf{c}_k = [c_k^{(1)}, \ldots, c_k^{(M)}]^{\mathrm{T}}$ be the M-dimensional vector containing all M signal samples taken at (time) instant k, and let $\boldsymbol{c} = \{\mathbf{c}_k\}_{k\in\mathbb{Z}}$ be the M-dimensional sequence containing all signal samples. As before $\boldsymbol{x} = \{\mathbf{x}_k\}_{k\in\mathbb{Z}}$ is the L-dimensional sequence containing the coefficients of the signal $x \in \mathcal{A}$. Out of $\boldsymbol{c}$ and $\boldsymbol{x}$, we define the subsequences

$$\boldsymbol{c}^{(r)} := \{\mathbf{c}_k^{(r)} := \mathbf{c}_{kR+r}\}_{k\in\mathbb{Z}} \,, \quad r = 0, 1, \ldots, R-1 \tag{11}$$

$$\boldsymbol{x}^{(q)} := \{\mathbf{x}_k^{(q)} := \mathbf{x}_{kQ+q}\}_{k\in\mathbb{Z}} \,, \quad q = 0, 1, \ldots, Q-1 \tag{12}$$

with Fourier transforms $\boldsymbol{C}^{(r)}(\mathrm{e}^{\mathrm{i}\theta})$ and $\boldsymbol{X}^{(q)}(\mathrm{e}^{\mathrm{i}\theta})$, which we stack in vectors of length RM and QL as

$$\widetilde{\boldsymbol{C}} = [\boldsymbol{C}^{(0)}, \ldots, \boldsymbol{C}^{(R-1)}]^{\mathrm{T}} \;\text{ and }\; \widetilde{\boldsymbol{X}} = [\boldsymbol{X}^{(0)}, \ldots, \boldsymbol{X}^{(Q-1)}]^{\mathrm{T}},$$

respectively. A straight forward calculation, using (7), shows that the samples and the coefficients are related by

$$\widetilde{\boldsymbol{C}}(\mathrm{e}^{\mathrm{i}\theta}) = \mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})\, \widetilde{\boldsymbol{X}}(\mathrm{e}^{\mathrm{i}\theta}) \tag{13}$$

where $\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})$ is the $RM \times QL$ matrix (9). Now the statement of the theorem follows by standard arguments. ∎

The fact that every $x \in \mathcal{A}$ can be reconstructed from its generalized samples by means of a bounded operator is equivalent to the statement that the sampling functions $\boldsymbol{s} = \{s_k^{(m)}\}_{k\in\mathbb{Z}}^{m=1,\ldots,M}$ form a pseudoframe [10] for the subspace $\mathcal{A}$. Moreover, it is possible to give conditions under which $\boldsymbol{s}$ is even a pseudo-Riesz basis for $\mathcal{A}$.

Corollary 2: *Let $\mathcal{A}$, of form* (3)*, be an atomic subspace of $\mathcal{H}$ and assume that the L-dimensional generator sequence $\boldsymbol{\phi}$ forms a Riesz basis for $\mathcal{A}$. Let $\boldsymbol{s} = \{s_k^{(m)} = \mathrm{U}^{Qk} s^{(m)}\}_{k\in\mathbb{Z}}^{m=1,\ldots,M}$ be an M-dimensional stationary sequence in $\mathcal{H}$ and denote by $\mathbf{\Psi}$ the matrix defined in* (9)*. Then*
1) $\boldsymbol{s}$ is a pseudoframe for $\mathcal{A}$ if and only if there exist positive constants $A_{\mathbf{\Psi}} \leq B_{\mathbf{\Psi}}$ such that

$$A_{\mathbf{\Psi}} \leq \mathbf{\Psi}^*(\zeta)\mathbf{\Psi}(\zeta) \leq B_{\mathbf{\Psi}} \quad \textit{for almost all } \zeta \in \mathbb{T} \,.$$

2) $\boldsymbol{s}$ is a pseudo-Riesz basis for $\mathcal{A}$ if and only if in addition to 1) also

$$A_{\mathbf{\Psi}} \leq \mathbf{\Psi}(\zeta)\mathbf{\Psi}^*(\zeta) \leq B_{\mathbf{\Psi}} \quad \textit{for almost all } \zeta \in \mathbb{T} \,.$$

The existents of a lower bound in (10) implies that the $RM \times QL$ matrix $\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})$ has rank QL for almost all $\theta \in [-\pi, \pi)$, i.e. that $RM \geq QL$. As a consequence we have:

Corollary 3 (Necessary Sampling Rate): *Let $\mathcal{A}$ and $\mathcal{S}$ be a signal and a sampling space, respectively, as in Theorem 1. In order that every $x \in \mathcal{A}$ can be recovered from its generalized samples $c_k^{(m)} = \langle x, s_k^{(m)} \rangle_{\mathcal{H}}$, it is necessary that the sampling rate of $\mathcal{S}$ is larger or equal than the rate of innovation of $\mathcal{A}$, i.e.*

$$\sigma_{\mathcal{S}} = \tfrac{M}{Q} \geq \tfrac{L}{R} = \sigma_{\mathcal{A}} \,.$$

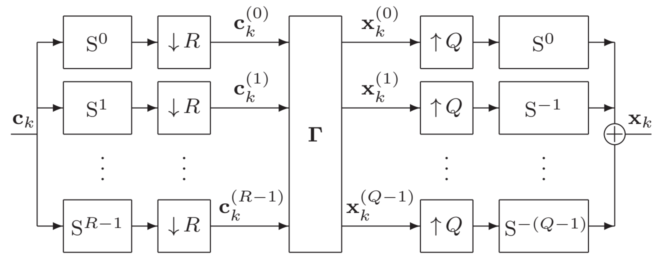

Fig. 1. Reconstruction as a multirate filter bank implementation.

4. RECOVERY FILTER

The proof of Theorem 1 already showed a method how the signal $x \in \mathcal{A}$ can be reconstructed from its samples. It follows from (13): Under condition (10) of Theorem 1 a left inverse of the matrix $\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})$ exists for almost all $\theta \in [-\pi, \pi)$. Consequently, the Fourier series $\boldsymbol{X}^{(q)}$ of the coefficient sequences can be determined from the Fourier series of the generalized samples $\boldsymbol{C}^{(r)}$ and so one can recover the coefficients $\{x_k^{(l)}\}$ of the signal from its samples $\{c_k^{(m)}\}$. For a more formal statement, we define the RM- and QL-dimensional sequences

$$\begin{aligned} \widetilde{\mathbf{c}}_k &= [\mathbf{c}_k^{(0)}, \mathbf{c}_k^{(1)}, \ldots, \mathbf{c}_k^{(R-1)}]^{\mathrm{T}} \qquad \text{and} \\ \widetilde{\mathbf{x}}_k &= [\mathbf{x}_k^{(0)}, \mathbf{x}_k^{(1)}, \ldots, \mathbf{x}_k^{(Q-1)}]^{\mathrm{T}} \,, \quad k \in \mathbb{Z} \,, \end{aligned}$$

respectively, with $\{\mathbf{c}_k^{(r)}\}_{k\in\mathbb{Z}}$ and $\{\mathbf{x}_k^{(q)}\}_{k\in\mathbb{Z}}$ as defined in (11) and (12), respectively.

Corollary 4 (Reconstruction Filter): *Let $x \in \mathcal{A}_{\mathrm{U}}(\phi, R)$ be of the form* (4) *and let $\{c_k^{(m)} = \langle x, s_k^{(m)} \rangle\}_{k\in\mathbb{Z}}^{m=1,\ldots,M}$ be its generalized samples. Assume that condition* (10) *of Theorem 1 is satisfied. Then a linear filter which recovers the coefficient sequence $\{\widetilde{\mathbf{x}}_k\}_{k\in\mathbb{Z}}$ from the sequence $\{\widetilde{\mathbf{c}}_k\}_{k\in\mathbb{Z}}$ of generalized samples is given by*

$$\widetilde{\mathbf{x}}_n = \textstyle\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \, \widetilde{\mathbf{c}}_{n-k} \tag{14}$$

where $\{\mathbf{\Gamma}_k\}_{k\in\mathbb{Z}}$ is a sequence of $QL \times RM$ matrices with transfer function

$$\mathbf{\Gamma}(\mathrm{e}^{\mathrm{i}\theta}) = \textstyle\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \, \mathrm{e}^{\mathrm{i}k\theta} = [\mathbf{\Psi}^*(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})]^{-1}\mathbf{\Psi}^*(\mathrm{e}^{\mathrm{i}\theta}) \,.$$

Remark: Note that $\mathbf{\Gamma}$ only needs to be a left inverse of $\mathbf{\Psi}$. The above corollary takes the so-called Moore-Penrose inverse for this purpose. However, other choices are possible.

The reconstruction may be illustrated by means of a multirate filter bank as in Fig. 1. The M-dimensional sequence $\{\mathbf{c}_k\}_{k\in\mathbb{Z}}$ at the input of the filter bank contains the sequences of the M signal samples. The blocks with powers of the operator S, defined in (1), represent delay elements or shift registers, and blocks denoted with $\downarrow R$ stand for a down-sampling by R. These blocks keep only every Rth sample and discard the rest. Due to the previous shifts, different samples are discarded in each branch. The outputs of the down-samplers are the R subsequences $\{\mathbf{c}_k^{(r)}\}_{k\in\mathbb{Z}}$ as defined in (11). The block $\mathbf{\Gamma}$ stands for the linear filter (14). Its outputs are the Q subsequences $\{\mathbf{x}_k^{(q)}\}_{k\in\mathbb{Z}}$, defined in (12). The L-dimensional sequence $\{\mathbf{x}_k\}_{k\in\mathbb{Z}}$, of the desired coefficients, is obtained after an appropriate up-sampling by Q (i.e. adding $Q-1$ zero vectors between consecutive vectors), delaying, and summation of the Q subsequences.

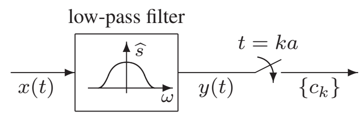

Fig. 2. Low-pass filtering and uniform sampling.

5. PARTICULAR EXAMPLE – SI SPACES

This section illustrates the introduced concepts by means of a particular example. To simplify the exposition, we consider the case where we have only one sampling sequence ($M = 1$) and where the sampling rate $\sigma_{\mathcal{S}} = M/Q$ is normalized to one (i.e. $Q = 1$). Then the matrix $\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})$ of Theorem 1 has size $R \times L$ and its entry on the mth row and lth column is given by

$$\begin{aligned}[\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})]_{m,l} &= (\mathrm{D}_R \mathrm{S}^{m-1} \Phi^{(l)}_{\phi,s})(\mathrm{e}^{\mathrm{i}\theta}) \\ &= \frac{1}{R} \sum_{k=0}^{R-1} \Phi^{(l)}_{\phi,s} \left(\mathrm{e}^{\mathrm{i}(\theta + k2\pi)/R} \right) \mathrm{e}^{-\mathrm{i}(m-1)(\theta+k2\pi)/R} \,.\end{aligned}$$

Therein $\Phi^{(l)}_{\phi,s}$ denotes the cross spectral density of the stationary correlated sequences $\{\mathrm{U}^k \phi^{(l)}\}_{k\in\mathbb{Z}}$ and $\{s_k\}_{k\in\mathbb{Z}}$. In particular, we choose $\mathcal{H} = L^2(\mathbb{R})$ and $\mathrm{U} = \mathrm{T}_a : x(t) \mapsto x(t-a)$ is taken to be the translation operator with shift a. Then the signal space $\mathcal{A}$ reduces to an SI space, and one obtains the classical sampling scheme consisting of a low-pass filter followed by a uniform sampling at rate $1/a$, cf. Fig. 2. It is easily verified [7] that in this case the spectral density $\Phi^{(l)}_{\phi,s}$ is given by

$$\Phi^{(l)}_{\phi,s}(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{a} \sum_{k\in\mathbb{Z}} \widehat{\phi}^{(l)} \left(\tfrac{k2\pi - \theta}{a} \right) \overline{\widehat{s}\left(\tfrac{k2\pi-\theta}{a} \right)}$$

wherein $\widehat{\phi}^{(l)}$ and $\widehat{s}$ are the Fourier transforms of $\phi^{(l)}$ and s, respectively. Therewith, the entries of the matrix $\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})$ become

$$\begin{aligned}[\mathbf{\Psi}(\mathrm{e}^{\mathrm{i}\theta})]_{m,l} &= \frac{\mathrm{e}^{-\mathrm{i}\frac{(m-1)}{R}\theta}}{aR} \sum_{r=0}^{R-1} \mathrm{e}^{-\mathrm{i}2\pi\frac{(m-1)r}{R}} \cdot \\ &\cdot \sum_{k\in\mathbb{Z}} \widehat{\phi}^{(l)} \left(k\tfrac{2\pi}{a} - r\tfrac{2\pi}{aR} - \tfrac{\theta}{aR} \right) \overline{\widehat{s}\left(k\tfrac{2\pi}{a} - r\tfrac{2\pi}{aR} - \tfrac{\theta}{aR} \right)} \,. \quad (15)\end{aligned}$$

If the Fourier transforms of the generators $\widehat{\phi}^{(l)}$ and $\widehat{s}$ are known, one can easily check, using Theorem 1, whether every signal $x \in \mathcal{A}_{\mathrm{T}_a}(\phi, R)$ can be reconstructed from its samples and Corollary 4 gives the transfer function of a reconstruction filter in terms of the matrix (15). Moreover, one may use (15) to design, for a giving sampling function s, appropriate generating functions $\phi^{(l)}$ for the signal space $\mathcal{A}$ such that signal recovery is always possible, or vice versa, for given generators $\phi^{(l)}$ one may design an appropriate sampling filter s. The next lemma, for example, uses (15) to derive necessary conditions on the spectral support of the generators $\phi^{(l)}$ and the filter function s, such that every $x \in \mathcal{A}_{\mathrm{T}_a}(\phi, R)$ can be reconstructed from its generalized samples. Thereby the *spectral support* of a function $s \in L^2(\mathbb{R})$ is the set $\mathbb{S}_s = \{\omega \in \mathbb{R} : |\widehat{s}(\omega)| > 0\}$. If $\phi = \{\phi^{(1)}, \ldots, \phi^{(L)}\}$ is a set of L function in $L^2(\mathbb{R})$, then the spectral support of ϕ is the union of the spectral supports of the individual functions, i.e. $\mathbb{S}_\phi = \bigcup_{l=1}^{L} \mathbb{S}_{\phi^{(l)}}$. Moreover, the Lebesgue measure of $\mathbb{S}$ is denoted by $\lambda(\mathbb{S})$.

Lemma 5: *Let $\mathcal{A}$ be an SI space with L generators and with rate of innovation $\sigma_{\mathcal{A}} = L/R \leq 1$, and let $s \in L^2(\mathbb{R})$ be the impulse response of the filter in Fig. 2. In order that every $x \in \mathcal{A}$ can be recovered from its generalized samples $\{c_k = \langle x, s_k \rangle\}_{k\in\mathbb{Z}}$, where $s_k = \mathrm{T}_a^k s$, it is necessary that*

a) $\lambda(\mathbb{S}_{\phi^{(l)}} \cap \mathbb{S}_s) > 0$ *for every $l = 1, 2, \ldots, L$*
b) $\lambda(\mathbb{S}_\phi \cap \mathbb{S}_s) \geq \frac{2\pi}{a}\, \sigma_{\mathcal{A}}$.

Point $a)$ requires that every generator $\phi^{(l)}$ has at least some common spectral support with the filter s. Otherwise, all information carried by this generator would completely be filtered out. Point $b)$ requires a minimal overlap of the spectral supports of the generators and the sampling filter. This minimal overlap is proportional to the sampling rate $1/a$ and to the rate of innovation $\sigma_{\mathcal{A}}$ of the signal space. Point $b)$ implies in particular a necessary condition on the spectral support of s and ϕ, namely $\lambda(\mathbb{S}_s) \geq \frac{2\pi}{a}\, \sigma_{\mathcal{A}}$ and $\lambda(\mathbb{S}_\phi) \geq \frac{2\pi}{a}\, \sigma_{\mathcal{A}}$.

6. REMARKS AND OUTLOOK

Detailed proofs of our results will be presented elsewhere [11]. Moreover, although Section 5 considered only the popular case where $\mathrm{U} = \mathrm{T}_a$ is the translation operator, there exist others, nontrivial operators U which are of some interest from a practical point of view (e.g. the modulation or dilatation operator) and for which the proposed frameworks gives similar simple results as for the translation operator [7, 11]. An extension of the proposed ideas to signal spaces on which the standard sampling techniques are not applicable [12], may be beneficial as well.

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