REACHING CONSENSUS ON A BINARY STATE BY EXCHANGING BINARY ACTIONS

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ABSTRACT

In this paper, we study the problem of distributed hypothesis testing in cooperative networks of agents. All agents are trying to reach consensus on the state of nature by their private signals and the binary actions of their neighbors. This is a challenging problem because the exchanged information of the agents is highly compressed. We propose a gossip-type method where every agent's decision converges in probability to the optimal decision held by a fictitious fusion center. We prove the asymptotical property of the proposed method and provide simulation results that demonstrate the communication cost and convergence time of the method.

Index Terms— Binary consensus, gossip algorithm, distributed detection, multiagent system.

1. INTRODUCTION

In this paper we study a distributed algorithm for binary consensus for connected networks of Bayes agents. It is assumed that the information exchanged among the agents is highly compressed. Initially, the agents get independently their private observations from the environment and act based on the signals and their priors. Then in every time slot, a pair of adjacent agents is selected to iteratively update their beliefs and exchange their actions until they reach consensus. It is shown that a system of a connected network of multiple agents will reach a decision consensus that is the same as the optimal decision rule of a fictitious fusion center obtained by the Bayes' rule.

This problem has been of wide interest in various fields including distributed detection, control theory, artificial intelligence and biology. In [1], e.g., the authors regard the consensus to be the majority's choice and solve the problem with noisy links. In [2], the authors study a similar problem but the solution is not based on the majority's choice. In [3], an algorithm for belief consensus based on average consensus is proposed, where it is assumed that the information exchanged among agents is not compressed.

In this paper, we study the case when agents receive private signals about the binary state of nature and repeatedly exchange actions with their neighbors as in a gossip algorithm. We aim at developing a method that will allow the agents to reach a consensus that is the same as the decision of a fictitious fusion center. The main contribution of the paper is in the proposed method and in proving that the method leads to optimal consensus in probability.

The paper is organized as follows. In the next section we state the problem. In Section 3, we provide a brief review of the gossip algorithm. In the following section we present the proposed method. The proof of convergence is outlined in Section 5, and simulation results are shown in Section 6. In Section 7, we provide some concluding remarks.

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2. PROBLEM STATEMENT

We consider a hypothesis testing problem in a network of Bayesian agents A_n, $n \in N_A = \{1, 2, ..., N\}$. The connections among agents are described by an undirected graph $G = (N_A, E)$, where N_A is the set of vertices (agents), and E is a set of edges, where each edge indicates a communication link between two agents. We assume that the topology of the network is time invariant and that the communication between any two communicating agents is perfect.

Each agent receives its independent private signal y_n that may be generated according to either $\mathcal{H}_1$ or $\mathcal{H}_0$. All the private signals received by the agents are generated by the same hypothesis. The agents are allowed to repeatedly make decisions on the state of nature and modify their private beliefs. At time slot $t \in \mathbb{N}$, for any $n \in N_A$, the A_n's decision $\alpha_n(t)$ is given by,

$$\alpha_n(t) = \begin{cases} 1, & if\ \pi_n(t) \geq 0.5 \\ 0, & if\ \pi_n(t) < 0.5 \end{cases}, \tag{1}$$

where $\pi_n(t)$ is the A_n's belief in $\mathcal{H}_1$ at t. When $t = 1$, we define $\pi_n(1) = P(\mathcal{H}_1|y_n)$. If a fictitious fusion center knows $\pi_n(1)$, $\forall n$, it can be shown that its optimal decision is given by

$$\pi_o = \frac{1}{1 + \left(\prod_{n=1}^{N} \frac{\pi_n(1)}{1-\pi_n(1)}\right)^{-1}}.$$

The fictitious fusion center makes decision $\alpha_o = 1$ if $\pi_o \geq 0.5$, and $\alpha_o = 0$ otherwise. Thus, if every agent can obtain the average value of the log-belief ratio (LBR), $\bar{l} = \frac{1}{N}\Sigma_{n=1}^{N} \log(\frac{\pi_n(1)}{1-\pi_n(1)})$, the agents will reach consensus in decision, which is the same as that of the fictitious fusion center. This average consensus problem can be solved by an average consensus [4] or a gossip algorithm [5]. However, if the communication between the agents is constrained so that it is quantized, neither a quantized average consensus algorithm [6] nor a quantized gossip algorithm [7] can guarantee the consensus in decision. In the following section, we propose an algorithm which has the following property:

$$\lim_{t\to\infty} P(\alpha_n(t) = \alpha_o) = 1, \forall\, n \in N_A. \tag{2}$$

3. A BRIEF REVIEW OF THE GOSSIP ALGORITHM

The gossip algorithm is an asynchronous solution to an averaging problem. Each node (in an N-agent network) has initially a scalar measurement $x_i(1)$, and the goal is to have every agent compute the average $\bar{x} = (1/n)\Sigma_{n=1}^{N} x_i(1)$, which is the average consensus.

As introduced in [5], this algorithm is carried out by linear iterations. In every iteration, an agent is randomly chosen with probability $\frac{1}{n}$ (e.g., A_i). Then A_i randomly selects an agent from its neighbors, say A_j, with probability P_{ij}, and they exchange their current estimates of $\bar{x}$. Upon the exchange, the estimates are updated by

$$x_i(t+1) = x_j(t+1) = \frac{x_i(t) + x_j(t)}{2}.$$

It can be shown that with this algorithm, for any agent $A_n \in N_A$, the convergence is guaranteed [5], i.e., $\lim_{t\to\infty} x_n(t) = \bar{x}$.

The algorithm guarantees convergence, but is based on the assumption that real numbers are exchanged among agents. If this assumption is relaxed, one can apply a quantized gossip algorithm. For the one proposed in [7], both the communication messages and the estimates of the agents are quantized. In [8], a stochastic gossip algorithm was analyzed whose update equations are given by

$$\begin{aligned} x_i(t+1) &= x_i(t) + \epsilon \times (Q[x_j(t)] - Q[x_i(t)]) \\ x_j(t+1) &= x_j(t) + \epsilon \times (Q[x_i(t)] - Q[x_j(t)]), \end{aligned}$$

where $Q(\cdot)$ is a uniform quantizer rounding each number to its nearest integer and $\epsilon \leq \frac{1}{2}$. The main result of the analysis is that for arbitrary initial value $\mathbf{x}(0) \in \mathbb{R}^N$, there exists a natural number t_0 such that:

$$\lim_{t\to\infty} |x_n(t) - \bar{x}| < 1, \forall t > t_0, \forall n \in N_A. \quad (3)$$

The rate of convergence of the algorithm can be found in [9].

4. THE PROPOSED METHOD

We propose a gossip style algorithm where the agents exchange quantized messages and yet, the probability that their decisions reach a consensus representing the optimal decision converges to one. With the same settings as for the gossip algorithm, one pair of agents is selected per time slot, and let they be A_i and A_j. Let also $U_n^{(k)}$ and $L_n^{(k)}$ respectively be the upper and lower bounds of the LBR $l_n(t)$ of A_n ($n \in \{i,j\}$), and $\gamma_n^{(k)}$ be the threshold for decision making, all defined at time slot t and iteration k and estimated by the communicating neighbor. That is, A_i estimates $U_j^{(k)}$, $L_j^{(k)}$ and $\gamma_j^{(k)}$ and vice versa. The agents A_i and A_j then implement the following algorithm (referred to as Local Consensus Algorithm (LCA)):

Step 1 (Initialization): $\forall n \in \{i,j\}, \gamma_n^{(1)} = 0, U_n^{(0)} = +\infty, L_n^{(0)} = -\infty$.

Step 2 (comparison): In iteration k, the agents act, and their action, denoted by $\delta_n^{(k)}$, is based on their individual LBRs and the thresholds, i.e., $\forall n \in \{i,j\}$,

$$\delta_n^{(k)} = \begin{cases} 1, & \text{if } l_n(t) \geq \gamma_n^{(k)} \\ 0, & \text{otherwise} \end{cases}.$$

Subsequently, each agent transmits its action $\delta_n^{(k)}$ to the communicating agent and updates the threshold by $\gamma_n^{(k+1)} = f(L_n^{(k)}, U_n^{(k)})$. Here we assume that $f(\cdot)$ is identical and known to every agent.

Step 3 (Interval update): The agents update their estimates of L_n and U_n of the communicating agents, that is $\forall n \in \{i,j\}$,

$$[L_n^{(k+1)}, U_n^{(k+1)}] = \begin{cases} [\gamma_n^{(k)}, U_n^{(k)}], & \text{if } \delta_n^{(k)} = 1 \\ [L_n^{(k)}, \gamma_n^{(k)}], & \text{if } \delta_n^{(k)} = 0 \end{cases}. \quad (4)$$

Step 4 (update): If $\delta_i^{(k)} = \delta_j^{(k)}$, stop the communication and set $l_n(t+1) = l_n(t) - \gamma_n^{(k)}$. By (1), $\alpha_n(t+1) = \delta_n^{(k)}$, $n \in \{i,j\}$. Otherwise, set the iteration number to $k+1$, and go back to Step 2.

Remark: The key feature of the algorithm is in letting the agents' belief ratios converge to a fixed region rather than a scalar. It is very important that the threshold update function f in Step 3 is identical and known to every agent.

5. ANALYSIS

The main result on the action gossip algorithm is the theorem presented below. Before we state it, we define a set of initial values of l_i for which the algorithm may fail to converge. We will show that the choice of decision threshold γ must meet some requirements. However, given such choice, it turns out that during the execution of the proposed method, if $l_i(t) + l_j(t) = 0$, the local algorithm will not stop. Let $\mathbf{l}(1)$ be the vector of the initial LBRs of the N agents, and $F \subseteq \mathbb{R}^N$ be the set of values of $\mathbf{l}(1)$ defined by $F = \{\mathbf{l}(1) | \exists i,j \in N_A, M \in \mathbb{N}, \text{ s.t. } l_i(1) + l_j(1) + \sum_{m=1}^{M} \gamma^{(k_m)} = 0\}$, where k_m can be any positive natural number, and $\gamma^{(k_m)}$ is any possible threshold after k_m iterations. Then if any $\mathbf{l}(1) \in F$, there may be no convergence to the optimal consensus.

Although there is an infinite number of failing values, since F is a countable subset of $\mathbb{R}^N$, the probability $P(\mathbf{l}(1) \in F) = 0$ if $\mathbf{l}(1)$ is an N-dimensional continuous random vector. In practice, due to the constraint in storage and computation, the initial value of $\mathbf{l}(1)$ is an element of a countable set. Therefore, we must set a limit for the number of action exchanges between two selected agents. With this modification, the agents will converge with high probability.

Theorem 1 *For any given initial vector $\mathbf{l}(1) \in \mathbb{R}^{N_A} \backslash F$, if the LBR $\mathbf{l}(t)$ is updated by using the LCA, then the optimal consensus can be reached in probability, i.e., $\lim_{t\to\infty} P(\alpha(t) = \alpha_o \cdot \mathbf{1}) = 1$, as long as the function $f(\cdot,\cdot)$ that updates the threshold $\gamma_n^{(k)}$ satisfies the following two properties:*

$(P1)$ $\gamma_i^{(k)} = -\gamma_j^{(k)}$, for any $k = 1, 2, \cdots$

$(P2)$ If $\delta_i^{(k)} \neq \delta_j^{(k)}$ for any $k \in \{1, 2, ..., m\}$, then

$$\lim_{m\to\infty} U_n^{(m)} - L_n^{(m)} = 0, \quad \forall n \in \{i,j\}.$$

Property $(P1)$ guarantees the average value of $\mathbf{l}(t)$ be a constant at any time, and $(P2)$ is the condition for convergence of the LCA.

For the sake of understanding, we provide the following example of an update rule for $\gamma_n^{(k)}, \forall n \in \{i,j\}$:

Example 1 *If $U_n^{(k)} = +\infty$ or $L_n^{(k)} = -\infty$, $\gamma_n^{(k+1)} = \gamma_n^{(k)} + (\delta_n^{(k)} - 0.5)\Delta$; otherwise, $\gamma_n^{(k+1)} = \dfrac{L_n^{(k)} + U_n^{(k)}}{2}$, where Δ can be any positive real number.*

Before proving the Theorem, we first prove the following two lemmas. They clarify some properties of the proposed method.

Lemma 1 *In the execution of LCA, the mean value of two LBRs does not change and is given by $\dfrac{l_i(t) + l_j(t)}{2} = \dfrac{l_i(t+1) + l_j(t+1)}{2}$. Besides, given any $l_i(t) \neq -l_j(t)$, if the update of γ meets the two properties from Theorem 1, then there exists a finite time $\tilde{k}$ such that $\alpha_n(t+1) = \delta_i^{(\tilde{k})} = \delta_j^{(\tilde{k})} = \delta_o$, $\forall n \in \{i,j\}$, where*

$$\delta_o = \begin{cases} 1, & \text{if } \frac{l_i(t)+l_j(t)}{2} > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Proof: First, note that $(P1)$ requires $\gamma_i^{(k)} = -\gamma_j^{(k)}$, then Step 4 in LCA shows $\frac{l_i(t)+l_j(t)}{2} = \frac{l_i(t+1)+l_j(t+1)}{2}$. As a result, once the local consensus is reached, the decision $\alpha_n(t+1)$ equals the local optimal decision consensus.

When $l_i(1)l_j(1) > 0$, the agents have already reached consensus before any information exchange, and then the lemma is true. If we assume that no iterartion $\tilde{k}$ exists when agents reach consensus, then the proposed method will keep running. By (P2) in Theorem 1, $U_n^{(k)}$ and $L_n^{(k)}$ will asymptotically reach each other. Noting that according to (4), both $l_i(t)$ and $l_j(t)$ must be in the estimated interval, it can be inferred that $\lim_{k\to\infty} L_n^{(k)} = \lim_{k\to\infty} U_n^{(k)} = l_n(t)$. However, since $l_i(t) \neq -l_j(t)$, the estimated interval will be in contradiction to the symmetry requirement of $(P1)$. Therefore, there exists such $\tilde{k}$. □

Lemma 2 *Assuming $\mathbf{l}(1) \in \mathbb{R}^N \backslash F$, let the state of the multi-agent system be $\alpha(t) = [\alpha_1(t), \alpha_2(t), ..., \alpha_N(t)]^\top$, and $\mathbf{1} = [1, 1, ..., 1]^\top \in \mathbb{R}^N$. The proposed method has the following properties:*

$(G1)$ *For any given initial state $\alpha(1)$ of the multi-agent system, at any time during the execution of the proposed method, the state $\alpha(t)$ lies in a finite set Φ.*

$(G2)$ *For any state $\alpha(t) = \alpha \neq \alpha_o \cdot \mathbf{1}$, there exists a finite time t_α such that $P(\alpha(t+t_\alpha) = \alpha_o \cdot \mathbf{1} | \alpha(t) = \alpha) > 0$.*

$(G3)$ *The state $\alpha_o \cdot \mathbf{1}$ is an absorbing state, namely, $P(\alpha(t') \neq \alpha_o \cdot \mathbf{1} | \alpha(t) = \alpha_o \cdot \mathbf{1}) = 0$, for any $t' > t$.*

Proof: Since the number of agents N is finite, the state is given by $\alpha(t) = [\alpha_1(t), \alpha_2(t), ..., \alpha_N(t)]^\top$, and during the execution of the proposed method, the state $\alpha(t)$ is in a finite set Φ of size 2^N. Hence, the property $(G1)$ is satisfied. Once the state $\alpha(t)$ becomes $\alpha_o \cdot \mathbf{1}$, every agent makes identical decision as α_o, and regardless which edge e_{ij} is selected, A_i and A_j will not change their decisions. The reason for the latter is that the local consensus between them has already been achieved, showing that $(G3)$ is satisfied.

The claim that the proposed method satisfies $(G2)$ can be proved by mathematical induction.

Basis: we first show that the statement holds for $N = 2$. When $N = 2$, only A_1 and A_2 can be selected at time slot 1. By Lemma 1, their actions will converge to α_o at time 2, which shows that the probability for converging in finite time is positive.

Inductive step: We show that if $(G2)$ is met when $N = m$, then it also holds when $N = m + 1$.

Since every edge has a positive probability to be selected in every time slot, if we can prove that for any state α, there exists an edge sequence such that after these selections the global consensus is reached, we prove (G2).

Without loss of generality, we assume that $\alpha_o = 1$. Since the graph G with $N = m + 1$ nodes is connected, there exists a connected subgraph $G_w = G \backslash A_w$ with m nodes, where A_w is the only left node. For any $\alpha(t) = \alpha$, by the assumption in the inductive step, the probability that the m nodes reach consensus in finite time is positive. Then there exists a sequence of edges such that consensus among these m agents is reached after time t_α. There are three possible outcomes.

Case 1: $\alpha_w(t) = 1$, and the optimal consensus of the remaining agents is also 1. In such cases the consensus is obtained after time t_α, showing that $(G2)$ is satisfied.

Case 2: $\alpha_w(t) = 1$, while the optimal consensus of the remaining agents is 0. Let A_v be a neighbor node of A_w, with e_{vw} being the edge in between. Consider the choice of edges $\{e_{vw},\ e_{(r_1,s_1):(r_{t_1},s_{t_1})},\ e_{vw},\ e_{(r_2,s_2):(r_{t_2},s_{t_2})}, e_{vw}...\}$ in the proposed algorithm, where $e_{(r_1,s_1):(r_{t_1},s_{t_1})}$ represents a sequence of edges, where none of the edge vertices is equal to v and where the sequence produces a consensus in G_w. The consensus among G_w is assured by the assumption that (G2) holds when $N = m$. We will show that by this selection, the system converges to $\alpha_o \cdot \mathbf{1}$.

Let the time slots when edge e_{vw} is selected be $\{t_1, t_2...\}$. If we assume the optimal consensus $\alpha_o \cdot \mathbf{1}$ cannot be reached in finite time, the decision of A_v must be zero at these time slots for the summation of all LBRs is positive. We let the evolution of threshold for agent $A_n \in G$ at the kth iteration be described by a function of its action sequence, $\gamma_n^{(k)}(\delta_n^{(1)}, \delta_n^{(2)}, ..., \delta_n^{(k-1)})$, $\forall n \in N_A$. By LCA, if $l_w(t_i) > |\gamma_v^{(2)}(0)|$, it holds that $l_w(t_i) - l_w(t_{i+1}) > \max\{|\gamma_v^{(2)}(0)|, |l_v(t_i)|\}$. Then we have $l_w(t_i) - l_w(t_{i+1}) > |\gamma_v^{(2)}(0)|$ if $l_w(t_i) > |\gamma_v^{(2)}(0)|$. This implies that there exists a finite number i_1 such that $l_w(t_{i_1}) < |\gamma_v^{(2)}(0)|$. Then at time t_{i_1+1}, the action sequence of A_v will be $\delta_v^{(1)} = 0$, $\delta_v^{(2)} = 1$. By the same reason, there exists a finite number i_2, such that $l_w(t_{i_2}) < |\gamma_v^{(3)}(0, 1)|$. It follows that there always exists a time i_m, such that

$$l_w(t_{i_m}) < |\gamma_v^{(m+1)}(0, 1_{1:m-1})|, \quad \forall m \in 2, 3, \cdots, \tag{5}$$

where $1_{1:m-1}$ represents a consecutive sequence of $m - 1$ ones.

If we assume that the optimal consensus cannot be reached by this edge selection in finite time, the agents in G_w will always reach consensus as 0. From property $(P2)$, we deduce that $\lim_{m\to\infty} |\gamma_v^{(m+1)}(0, 1_{1:m-1})| = 0$. By using (5), we have $\lim_{t\to\infty} l_w(t) = 0$, which is contradicted by the assumption that the average value of $\mathbf{l}(1)$ is positive, i.e., $\alpha_o = 1$. Hence, after finite time t_α, the remaining m agents will reach consensus as 1, which shows that $(G2)$ holds.

Case 3: Now $\alpha_w^{(t)} = 0$, while the optimal consensus of the remaining agents is 1. There exists another subgraph $G_{w'}$ of G with m nodes. Let the only remaining node of this subgraph be $A_{w'}$ with its neighbor $A_{v'}$. Then there exists a sequence of edges that produces consensus on the subgraph $G_{w'}$ which yields either a global consensus or a state, which is the same as in *Case 2*. By the above proof, the system can reach consensus in finite time with positive probability. □

Proof of Theorem 1: By $(G1)$, the state $\alpha(t)$ always lies in a finite set Φ, and $(G2)$ implies that there exists a finite number $T = \max_{\alpha\in\Phi} t_\alpha$ such that $P(\alpha(t+T) = \alpha_o \cdot \mathbf{1} | \alpha(t) = \alpha) > 0$ for any $\alpha \in \Phi$. Let $\varepsilon = \min_{\alpha\in\Phi}\{P(\alpha(t+T) = \alpha_o \cdot \mathbf{1} | \alpha(t) = \alpha)\}$. Then, it follows that $P(\alpha(t+T) \neq \alpha_o \cdot \mathbf{1} | \alpha(t) \neq \alpha_o \cdot \mathbf{1}) \leq 1 - \varepsilon$. By the definition of conditional probability, $P(\alpha(t) \neq \alpha_o \cdot \mathbf{1} | \alpha(1) \neq \alpha_o \cdot \mathbf{1}) \leq (1-\varepsilon)^{\lfloor \frac{t}{T} \rfloor}$, which converges to zero as $t \to \infty$. Note that if $\alpha(1) = \alpha_o \cdot \mathbf{1}$, $(G3)$ implies that $P(\alpha(t) = \alpha_o \cdot \mathbf{1} | \alpha(1) = \alpha_o \cdot \mathbf{1}) = 1$. Therefore, $\lim_{t\to\infty} P(\alpha(t) = \alpha_o \cdot \mathbf{1}) = 1$. □

If $\mathbf{l}(1)$ is noncountable, based on the theorem, the proof of (2) is immediate.

6. SIMULATIONS

In this section, we present some simulations of the proposed algorithm and numerical comparisons between the quantized gossip [7] and the proposed method. In all the experiments, the multi-agent system is modeled as a random geometric graph [10] $G(V, E)$, where the N agents are chosen uniformly and independently in a 1×1 square. Each pair is connected if the Euclidian distance between the nodes is smaller than $r(N)$, where $r(N) = \sqrt{\frac{\log(N)}{N}}$ due to the connectivity requirement.

We have agents that observe data and they have to choose between the following two hypotheses:

$$\begin{aligned} \mathcal{H}_1 : &\qquad y_i = \theta + w_i \\ \mathcal{H}_0 : &\qquad y_i = w_i, \end{aligned}$$

where θ is known and w_i is a random perturbation modeled as a zero mean Gaussian random variable with known variance σ_w^2, and we assume that $\theta > 0$. For the prior probability of the hypothesis, we let $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$. Without loss of generality, we assume the data are generated from $\mathcal{H}_1$, and we set $\theta = 1$, $\sigma_w = 5$.

In the first experiment, we study the LBR of $N = 15$ agents evolving by quantized gossip and the proposed method. In the quantized gossip algorithm, since it is with high probability that $|l_i(1)| < 0.6$, we use a q bits mid-riser uniform quantizer from -0.6 to 0.6, which is given by:

$$Q(x) = \begin{cases} 0.6, & \text{if} x > 0.6 \\ -0.6, & \text{if} x < -0.6 \\ \frac{1.2}{(2^q-1)} \cdot \left(\lfloor \frac{x}{1.2/(2^q-1)} \rfloor + \frac{1}{2} \right), & \text{otherwise} \end{cases},$$

where $q = 4$ in experiment one.

In the proposed method, we let γ update by the rule introduced in Example 1, where we set $\Delta = 0.2$. In order to make a comparison between the two methods, they are implemented on the same network with identical initial observations and selection of edges in each iteration. The results are shown in Fig. 1, where we see the evolution of the LBRs of five of the 15 agents. It can be seen that the proposed method provides a consensus after $t = 100$, while for the agents using the four bits quantized gossip method, the LBRs keep oscillating.

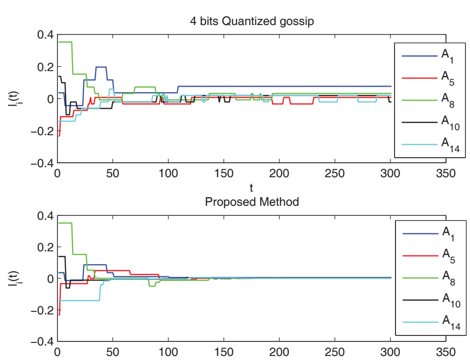

Fig. 1. Evolution of LBR by the proposed and quantized gossip methods.

In the second part, we conducted Monte Carlo simulations, where we ran the above experiment with the same parameters for 1,000 times. In each experiment, given identical initial data, the multi-agent system implemented the proposed method with threshold updated as in Example 1. For comparisons, it also implemented the 1-bit, 3-bit, 5-bit, 8-bit quantized gossip. In Fig. 2, the convergence rate and convergence probability is compared among the methods. It can be seen that the proposed method converges in all trials (as predicted by Theorem 1), whereas none of the quantized gossip algorithms converges in all trials. We also observed that our method converged faster than the 5-bit quantized gossip, and the 1-bit quantized gossip did not reach consensus at all.

If we define the communication cost to be the number of transmissions $\times$ the number of bits per transmission, then the average communication cost for the quantized gossip is q (i.e., 1, 3, 5, or 8) bits in this case. However, the communication cost of our method was merely 1.1468 bits.

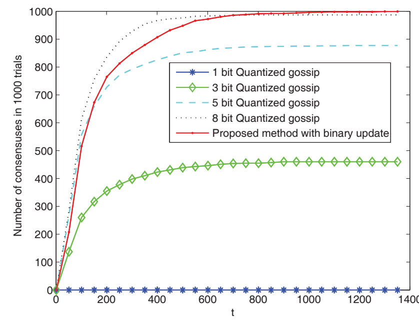

Fig. 2. The convergence rate and consensus probability with different methods.

7. CONCLUSION

In this paper we proposed a method for binary consensus with binary communication. We proved that the consensus is achieved with probability one. We demonstrated the performance of the proposed methods and compared it with the quantized gossip method by simulations. The results show that the proposed method has favorable performance. The method is general and can be utilized in many distributed applications.

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