# AVERAGING STABILITY ANALYSIS OF A NEW ATTITUDE ESTIMATION ALGORITHM

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**ABSTRACT**

The most common approach to attitude estimation involves direct estimation of the rotation matrix or its associated quaternion. This results in a difficult constrained estimation problem. Here instead we develop a new online algorithm that directly estimates the angular velocity which is unconstrained. The rotation matrix or quaternion is then easily obtained from the kinematics. We sketch an averaging analysis of stability of the new algorithm.

***Index Terms***— attitude estimation, rotation matrix, SO(3), averaging

## 1. INTRODUCTION

Attitude estimation is concerned with determining the orientation and angular velocity of a moving rigid body from body centered measurements as well as known reference measurements such as star-sight measurements.

The attitude or pose of a moving rigid body is the orientation of each axis of a body fixed frame fixed with respect to the axes of a temporally fixed reference frame. The attitude can be represented by a rotation matrix which is an orthogonal $3 \times 3$ matrix with unit determinant. The set of rotation matrices has group structure and is denoted as the special orthogonal group, SO(3). Associated with the rotation matrix is a derived quantity, the angular velocity. Typical attitude measurements are available in body fixed coordinates of constant quantities in inertial coordinates. Also rate gyro measurements are often available. All these measurements are noisy and the rate gyro measurements are biassed [1].

The usual approach to attitude estimation involves direct estimation of the rotation matrix or its quaternion equivalent [1],[2],[3]. This is not straightforward since the algorithms must ensure the rotation matrix or quaternion lie in the appropriate spaces i.e. S0(3) and a hypersphere respectively.

In this paper we take a different approach. We construct an adaptive i.e. online algorithm that directly estimates the angular velocity. The angular velocity is unconstrained and that makes the estimation easier. The rotation matrix or quaternion is then easily constructed from the kinematics.

The remainder of the paper is organised as follows. In section 2 we set up the problem and construct the adaptive algorithm. In section 3 we provide an introduction to averaging stability analysis. In section 4 we sketch an averaging analysis of stabilty of the new algorithm. Conclusions are offered in section 5.

**Notation and Acronyms**.
$\theta = \| \omega \|, \theta_o = \| \omega_o \|, \theta_* = \| \omega_* \|$; $\bar{\omega} = \frac{\omega}{\|\omega\|}$; $S(\omega)$ or $S_\omega$ denotes a skew symmetric matrix;wp1 = with probability one; superscript H denotes complex conjugate transpose; superscript $v^*$ denotes complex conjugate; subscript $v_*$ denotes a fixed or $'$frozen$'$ value; we may then have $v_*^*$.

## 2. ADAPTIVE ALGORITHM CONSTRUCTION

We assume two sets of noisy measurements

$$y(t) = \mu(t) + n_y(t), z(t) = m(t) + n_z(t)$$

in body fixed coordinates of known constant quantities in inertial coordinates so that $\mu_e = R(t)\mu(t) \Rightarrow \mu(t) = R^T(t)\mu_e$ where $R(t)$ is the rotation matrix and $\mu_e$ is a fixed known reference; similarly $m(t) = R^T(t)m_e$. Also $n_y(t), n_z(t)$ are independent measurement noises. It is typical to also assume noisy rate gyro measurements $y_\omega(t) = \omega(t) + b(t) + n_\omega(t)$ where $b(t)$ is the bias or drift. However this generates additional difficulties and will be considered elsewhere. The problem then is to estimate the rotation matrix from the noisy measurement signals $y(t), z(t)$.

Since $R^T(t)R(t) = I_3$ and $detR(t) = 1$, the rotation matrix obeys the kinematics

$$\frac{dR(t)}{dt} = RS(\omega) \tag{2.1}$$

where $S(\omega)$ is a skew symmetric matrix

$$S(\omega) = S_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

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and $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity vector.

Note that in view of (2.1) we have, $\frac{d\mu}{dt} = \dot{R}^T \mu_e = S_\omega^T R^T \mu_e = -S_\omega \mu$ so that we obtain the kinematics for $\mu$,

$$\dot{\mu} + S(\omega)\mu = 0 \tag{2.2}$$

And similarly for $m(t)$. It is instructive to analyse the stability of this equation for fixed $\omega$. The characteristic equation is $|pI + S_\omega| = 0 = p(p^2 + \theta^2)$ where $\theta = \| \omega \|$. This has roots $p = 0, \pm j\theta$ and of course corresponds to sinusoidal motion. It is in any case clear that the motion does not explode since

$$\frac{d\mu^T \mu}{dt} = \dot{\mu}^T \mu + \mu^T \dot{\mu} = -\mu^T (S_\omega^T + S_\omega)\mu = 0$$

Further so long as $\mu(0) = \mu_e$ obeys $\mu_e^T \mu_e = 1$ we have $\mu^T \mu = 1$ for all $t$; and similarly for $m(t)$.

Rather than estimate $R(t)$ or its quaternion equivalent directly we estimate $\omega(t)$ and then generate $R(t)$ from the kinematics (2.1).

Before developing an algorithm we recall some well known facts. Since $\mu(t)$ has only two free components while $\omega(t)$ has three we cannot estimate $\omega(t)$ from one measurement signal such as $y(t)$ alone. It turns out that one must use one measurement signal say $y_t$ to estimate the orientation $\bar{\omega}(t) = \frac{\omega(t)}{\theta(t)}$ and the other measurement $z(t)$ to estimate the magnitude $\theta(t) = \| \omega(t) \|$.

### 2.1. Getting $\bar{\omega}(t)$

Our approach to estimating $\bar{\omega}(t)$ is a modification of standard instantaneous steepest descent [4]. We introduce the least squares criterion $J(\bar{\omega}) = \frac{1}{2} \| y - \mu \|^2 = \frac{1}{2} y^T y - \mu^T y + \frac{1}{2} \mu^T mu = \frac{1}{2} y^T y - \mu^T y + \frac{1}{2}$ and then the adpative steepest descent algorithm is

$$\begin{aligned} \frac{d\bar{\omega}}{dt} &= -\gamma \frac{dJ(\bar{\omega})}{d\bar{\omega}} \\ &= \gamma K^T y \end{aligned} \tag{2.3}$$

where we have introduced the so-called sensitivity matrix $K = \frac{d\mu}{d\bar{\omega}^T}$. The evolution equation for this can be found by differentiating through the $\mu$-kinematics to find

$$\dot{K} + \theta S(\bar{\omega})K = \theta S(\mu)$$

where we have used the fact that $S(\omega)\mu = -S(\mu)\omega$. Unfortunately, in view of our stability analysis of (2.2) it follows that this equation must explode.

To overcome this we modify the equation by introducing some damping as follows

$$\dot{K} + \zeta\theta K + \theta S(\bar{\omega})K = \theta S(\mu) \tag{2.4}$$

where $\zeta$ is a damping ratio. Checking the stability for fixed $\bar{\omega}$ we find

$$|pI + \zeta\theta I + \theta S(\bar{\omega})| = 0 = (p + \zeta\theta)[(p + \zeta\theta)^2 + \theta^2]$$

which has roots $-\zeta\theta, -\zeta\theta \pm j\theta$ which are stable. When $\bar{\omega}$ is time varying, then as long as it varies sufficiently slowly ([5],[6]) the equation will still be stable.

The adaptive algorithm for $\bar{\omega}(t)$ now consists of the triplet, (2.2),(2.3),(2.4).

To avoid confusion between the estimated and true quantities we denote the true angular velocity as $\omega_o = \omega_o(t)$ and the true noise free measurement as $\mu_o = \mu_o(t)$ which has kinematics $\dot{\mu}_o + S(\omega_o)\mu_o = 0$. The measurements are then $y(t) = \mu_o(t) + n_y(t)$ and $z(t) = m_o(t) + n_z(t)$. We similarly have $\theta_o = \theta_o(t), \bar{\omega}_o = \bar{\omega}_o(t)$.

### 2.2. Getting $\theta(t)$

To estimate $\theta(t)$ it seems necessary to isolate its periodic aspect; we proceed heuristically. Holding $\bar{\omega}, \theta$ fixed differentiate the first order dynamics $\dot{m} + \theta S(\bar{\omega})m = 0$ to obtain

$$\begin{aligned} 0 &= \ddot{m} + \theta S(\bar{\omega})\dot{m} \\ \Rightarrow 0 &= \ddot{m} + \theta S(\bar{\omega})(-\theta S(\bar{\omega}))m \\ \Rightarrow 0 &= \ddot{m} + \theta^2 (I - \bar{\omega}\bar{\omega}^T)m \end{aligned}$$

via the well known property $S^2(\bar{\omega}) = \bar{\omega}\bar{\omega}^T - I$. Now it turns out that the compact way to proceed is to make use of the eigenvectors $v$ of $S(\bar{\omega})$. Again holding $v$ fixed (since $\bar{\omega}$ was fixed) and multiplying through by $v^T$ yields, since $v^T \bar{\omega} = 0$

$$v^T \ddot{m} + \theta^2 v^T m = 0$$

so that $v^T m(t)$ is a combination of sines and cosines. Indeed from Rodriguez formula for $\bar{\omega}, v$ fixed,

$$\begin{aligned} v^T m(t) &= v^T m_e cos(\theta t) - v^T S(\bar{\omega}) m_e sin(\theta t) \\ &= v^T m_e cos(\theta t) + j v^T m_e sin(\theta t) \\ &= v^T m_e e^{j\theta t} \end{aligned}$$

With these heuristics in mind we propose to estimate $\theta$ as follows. We draw on a body of work in the adaptive literature concerned with estimation of frequency. In particular we modify the algorithm developed by [7] to the current setting.

There are three steps. Firstly given $\bar{\omega}(t)$ from (2.3) we compute $v(t) = v(\bar{\omega}(t))$.Secondly we generate an auxiliary signal from the following stable filtering operation,

$$\ddot{x} + 2\zeta\theta\dot{x} + \theta^2 x = 2\zeta\theta^2 Re(v^T z) \tag{2.5}$$

Then the adaptive algorithm is

$$\dot{\theta} = -\gamma 2\zeta\theta x[\theta Re(v^T z) - \dot{x}] \tag{2.6}$$

So the complete algorithm consists of (2.3),(2.6) together with the auxiliary equations (2.2),(2.4),(2.5) and the equation $v(t) = v(\bar{\omega}(t))$. We now turn to the stability analysis.

## 3. AVERAGING ANALYSIS OF STABILITY: PRELIMINARIES

The algorithm is highly nonlinear and a direct stability analysis would be a difficult task. Instead we carry out an averaging analysis of the algorithm stability [4],[8],[9].

Avaraging analysis is an advanced form of perturbation analysis applying to systems of differential or difference equations with small parameters (in this case $\gamma$). The idea is to replace the primary system of interest with a simpler system whose stability analysis is more manageable. One then relates the behaviour of the two systems together by means of a Hovering theorem [4]. Averaging has a long history in applied mathematics particularly nonlinear mechanics; see historical notes in [8],[4].

[4] delivers results only in discrete time. However the techniques developed there extend easily to continuous time as illustrated in [10],[11]. We use that approach here.

Since averaging analysis does not seem to have been applied before in attitude estimation we provide a brief heuristic derivation of the averaging equations before proceeding to a more formal analysis.

The adaptive algorithm (2.3),(2.6) together with the auxiliary equations (2.5)(2.2),(2.4), form a mixed (aka two) time-scale system. This is because we assume the gain $\gamma$ is small so that $\omega$ changes slowly (the slow time-scale) whereas $K, \mu$ change rapidly (the fast time-scale ).

To keep the heuristic discussion brief we consider only the algorithm for $\bar{\omega}(t)$ and assume $\theta$ is constant. But we treat the full algorithm in the next section. To begin the heuristic discussion then, integrate (2.3) from an arbitary time $t$ over an interval $T$ to get,

$$\bar{\omega}(t+T) - \bar{\omega}(t) = \gamma \int_t^{t+T} K^T(u) y(u) du$$

If $\bar{\omega}(t)$ changes slowly (since $\gamma$ is small) and $T$ is not too large then $\mu(t)$ will not differ too much from the value it would have if we integrated (2.3) from time $t$ with $\bar{\omega}$ fixed at the value at the start of the interval, namely $\bar{\omega}(t)$. We denote this value as $\mu(\bar{\omega}(t), u)$ and it obeys

$$\frac{d\mu}{du} + \theta S(\bar{\omega}(t))\mu(\bar{\omega}(t), u) = 0, t \leq u \leq t+T$$

This called the <u>frozen state</u> equation. Similarly we introduce a frozen state equation generating $K(\bar{\omega}(t), u)$. We thus get

$$\bar{\omega}(t+T) - \bar{\omega}(t) \approx \gamma \int_t^{t+T} K^T(\bar{\omega}(t), u) y(u) du$$

Next we assume that for <u>fixed</u> $\bar{\omega}$ the following limit exists uniformly [1] in $t$,

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} K^T(\bar{\omega}, u)\mu(\bar{\omega}, u) du \Rightarrow < K^T, \mu > (\bar{\omega})$$

[1] The alert reader may be worried about the uniformity in a stochastic context. It cannot hold but can be gotten around without affecting the result

And similarly we assume existence of $< K^T, y > (\bar{\omega})$. Then if $T$ is not too large we have

$$\bar{\omega}(t+T) - \bar{\omega}(t) \approx \gamma T < K^T, y > (\bar{\omega}(t))$$

and dividing by $T$ and letting $T \to 0$ leads to the averaged system

$$\dot{\bar{\omega}}_{av} = \gamma < K^T, y > (\bar{\omega}_{av}(t)) \tag{3.1}$$

This is a simpler system than the primary system (2.3) since it is an autonomous system. Finally we need to link the primary and averaged systems by a Hovering theorem [4] e.g.

$$sup_{0\leq t \leq \frac{T}{\gamma}} |\bar{\omega}(t) - \bar{\omega}_{av}(t)| \leq C_T(\gamma) \;\; wp1 \tag{3.2}$$

This result is valid for fixed $\gamma$ but it has the crucial property that $C_T(\gamma) \to 0$ wp1 as $\gamma \to 0$. This Hovering theorem is a so-called finite time averaging result. It can be extended to an infinite interval result [8],[4]. For lack of space we treat only finite time averaging.

Actually there are two stages to the finite time averaging. Firstly we do a stability analysis assuming the true angular velocity is fixed. Then we treat the case of time-varying angular velocity i.e. tracking. For lack of space we deal only with the case of constant true angular velocity. The time-varying case will be developed elsewhere but follows the lines of [4] e.g. as in [10].

## 4. AVERAGING ANALYSIS

Unfortunately lack of space prevents inclusion of all the details which will be given elsewhere. Here we sketch the results.

### 4.1. Averaging $\bar{\omega}(t)$

It can be shown that the averaged system for $\bar{\omega}(t)$ is,

$$\dot{\bar{\omega}}_{av} = \gamma B(\bar{\omega}_{av})(\bar{\omega}_{av} - \bar{\omega}_o) \tag{4.1}$$

where

$$B(\bar{\omega}_*) = R_*(\zeta S(\bar{\omega}_*) - (I - \bar{\omega}_*\bar{\omega}_*^T)), R_* = \frac{\bar{\omega}_o^T \mu_e \bar{\omega}_*^T \mu_e}{(1+\zeta^2)\theta_*}$$

We now carry out a linearized stability analysis of this system. The linearized system is

$$\dot{\bar{\omega}}_{av} = \gamma B(\bar{\omega}_o)(\bar{\omega}_{av} - \bar{\omega}_o)$$

$\gamma B(\bar{\omega}_o)$ has eigenvalues $p = 0$ and $p = -\gamma R_o \pm j\gamma\zeta R_o$ where $R_o = \frac{(\bar{\omega}_o^T \mu_e)^2}{(1+\zeta^2)\theta_o} > 0$. Thus we have two stable roots and one marginally stable.

Continuing, we then have the solution,

$$\begin{aligned} &\bar{\omega}_{av}(t) - \bar{\omega}_o = e^{\gamma B(\omega_o)t}(w_{av,o} - \bar{\omega}_o) \\ = \;\; & [2Re(v_* v_*^H)e^{-\gamma R_o t} cos(\gamma\zeta R_o t) + \bar{\omega}_o\bar{\omega}_o^T](w_{av,o} - \bar{\omega}_o) \end{aligned}$$

where $w_{av,o} = \bar{\omega}(0)$ is the initial value of $\bar{\omega}_{av}(t)$. Thus $\bar{\omega}_{av}(t)$ converges to $w_{av,e}$ satisfying

$$w_{av,e} = \bar{\omega}_o + \bar{\omega}_o[\bar{\omega}_o^T w_{av,o} - 1] = \bar{\omega}_o \bar{\omega}_o^T w_{av,o}$$

Thus we have local convergence [2] and can determine $\bar{\omega}_o$ from $w_{av,e} / \parallel w_{av,e} \parallel$.

### 4.2. Averaging $\theta(t)$

It can be shown that the the averaged system is

$$\dot{\theta}_{av} = -2\gamma\zeta\theta_{av}^2[2\zeta a_R f^T(\bar{\omega}_{av} - \bar{\omega}_o) + \Gamma(\bar{\omega}_{av})(\theta_{av} - \theta_o)] \tag{4.2}$$

Expressions for $a_R, f, \Gamma(\bar{\omega}_{av})$ are omitted since we don't need them for the ensuing joint analysis.

### 4.3. Joint Averaging

Putting the two averaged systems (4.1),(4.2) together we get,

$$\begin{pmatrix} \dot{\bar{\omega}}_{av} \\ \dot{\theta}_{av} \end{pmatrix} = \gamma \begin{pmatrix} B(\bar{\omega}_{av}) & 0 \\ -4\zeta^2\theta_{av}^2 a_R f^T & -2\zeta\theta_{av}^2\Gamma(\bar{\omega}_{av}) \end{pmatrix} \begin{pmatrix} \bar{\omega}_{av} - \bar{\omega}_o \\ \theta_{av} - \theta_o \end{pmatrix}$$

The linearised averaged system is then

$$\begin{pmatrix} \dot{\bar{\omega}}_{av} \\ \dot{\theta}_{av} \end{pmatrix} = \gamma \begin{pmatrix} B(\bar{\omega}_o) & 0 \\ -4\zeta^2\theta_o^2 a_R f^T & -2\zeta\theta_o^2\Gamma(\omega_o) \end{pmatrix} \begin{pmatrix} \bar{\omega}_{av} - \bar{\omega}_o \\ \theta_{av} - \theta_o \end{pmatrix}$$

where, $\Gamma(\omega_o) = \frac{|v_o^T m_e|^2}{2\theta_o}$. Because of its triangular structure this system has the same eigenvalues as $\gamma B(\omega_o)$ together with an eigenvalue $-2\gamma\zeta\theta_o^2\Gamma(\omega_o) = -\gamma\zeta\theta_o|v_o^T m_e|^2$ which is stable. The eigenvectors of the linearised system matrix are then $\binom{\bar{\omega}_o}{0}, \binom{v_o}{0}, \binom{v_o^*}{0}, \binom{0}{1}$. And we conclude as before that $\bar{\omega}_{av}(t) \to w_{av,e} = \bar{\omega}_o \bar{\omega}_o^T w_{av,o}$. We also conclude $\theta_{av}(t) - \theta_o = e^{-2\zeta\theta_o^2\Gamma(\omega_o)t}(\theta_{av}(0) - \theta_o) \to 0$ as $t \to \infty$. And so we have local convergence.

## 5. SUMMARY

In this paper we have taken a different approach to attitude estimation from body fixed measurements. Rather than estimate the rotation matrix or quaternion as is usally done, we have estimated the unconstrained angular velocity directly. The rotation matrix can then be easily obtained from the kinematics. The construction of the new algorithm required two stages. An estimator of the direction of the angular velocity from one set of measurements and an estimator for the magnitude of the angular velocity which used also a second set of measurements. The direction estimator is a simple instantaneous steepest descent algorithm while the magnitude estimator uses the eigenvector of the associated skew symmetric matrix and is inspired by existing methods of frequency estimation. The resulting algorithm is nonlinear and stability analysis was sketched using finite time averaging analysis. The algorithm was found to be locally stable. Noise aspects, infinite time averaging, tracking behaviour and the use of gyro measurements will be pursued in the future.

[2] In view of the zero eigenvalue we need in fact to carry out a center manifold stability analysis [12] which in view of space limits will be treated elsewhere

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