ASYMPTOTIC ANALYSIS OF A PARTIAL FEEDBACK OFDMA SYSTEM EMPLOYING SPATIAL, SPECTRAL, AND MULTIUSER DIVERSITY

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ABSTRACT

Spatial and multiuser diversity are two types of diversity techniques for delivering reliable high-date-rate services. Spectral diversity comes from opportunistic scheduling in the frequency domain enabled by the OFDMA technique, and is influenced by partial feedback design. By employing the best-M partial feedback strategy, we provide a unified view of spatial, spectral, and multiuser diversity through asymptotic (in users) analysis. We examine the tail behavior of the distribution of the received channel quality information (CQI) at the scheduler to prove the type of convergence as well as to derive the asymptotic approximations for the average spectral efficiency under partial feedback. We investigate the application of our analysis to different spatial diversity schemes. Our derived results can be used to quickly determine the minimum required partial feedback in a general multiuser MIMO-OFDMA system.

***Index Terms*—** Partial feedback, multiuser MIMO, OFDMA, spatial diversity, asymptotic analysis

1. INTRODUCTION

Current wireless systems leverage various types of diversity to achieve reliable and high throughput services. The nature of the various forms of diversity can be broadly classified into the following two types. The first type, such as spatial diversity [1], normally utilizes physical layer techniques to provide reliable transmission by mitigating the detrimental effect of channel fading. On the contrary, the other type such as multiuser diversity, exploits the beneficial effect of channel variation to render opportunistic scheduling gain in the media access control (MAC) layer. With the emerging OFDMA-based wireless systems, scheduling decision is performed across users as well as resource blocks. This induces another cross-layer diversity in the frequency domain: the spectral diversity. Spectral diversity is closely related to partial feedback design wherein only the channel quality information (CQI) of certain favorable resource block is conveyed back to the scheduler in order to save close-loop feedback resource [2]. In this paper, we rely on one promising partial feedback strategy which is currently considered in practical systems such as LTE. It is called the best-M partial feedback strategy, where users order and convey the M best CQI among the total resource blocks to the scheduler.

The analysis of the general best-M partial feedback strategy is challenging and analytical expressions in the literature are confined to the special full feedback and best-1 feedback case. Only recently closed form results have been proposed by Hur and Rao [3] for the general best-M partial feedback. However, the results are based on exact analysis which turn out to be cumbersome and do not easily lend themselves to providing insight for further analysis. To enhance tractability, asymptotic analysis based on extreme value theory [4] can be utilized to obtain interpretable results. Zhou and Dai [5] analyze the interaction between spatial and multiuser diversity by examining the scheduling gain. The approach proposed by Song and Li [6] studies the asymptotic throughput in the single carrier system. To the best of our knowledge, almost all the previous treatment of asymptotic analysis are based on the assumption of full feedback. In [7], we derived asymptotic expressions and proposed a general theorem to incorporate the general best-M partial feedback design. However, a SISO framework was assumed in [7] and the general analysis with MIMO systems is still lacking. In this paper, our goal is to incorporate the available spatial dimension to provide a unified asymptotic view of spatial, spectral, and multiuser diversity. This unified analysis enables us to quickly determine the minimum required partial feedback from the asymptotic approximations.

2. SYSTEM MODEL

We consider a downlink multiuser MIMO OFDMA system with one base station equipped with N_t transmit antennas and K users each equipped with N_r receive antennas. We assume that the system consists of R resource blocks with one resource block as the basic feedback and scheduling unit. We consider an asymmetric scenario where different users can have diverse large scale channel gains and denote G_k as the large scale channel gain between the transmitter and user k. The large scale channel gain may consist of path loss, antenna gain, and shadowing which is assumed known in advance by the transmitter through infrequent feedback or location awareness [8]. We denote the frequency domain channel transfer function between transmit antenna j and receive antenna i of user k at resource block r as $H_{k,r,i,j}$. $H_{k,r,i,j}$ is modeled as complex Gaussian distributed random variable with zero mean and unit variance. We also assume that $H_{k,r,i,j}$ is independent across users, resource blocks, and antennas. The received signal of user k at resource block r is given by

$$u_{k,r} = \sqrt{PG_k} H_{k,r} s_{k,r} + v_{k,r}, \tag{1}$$

where P is the transmit power per resource block, $s_{k,r}$ is the transmitted symbol, and $v_{k,r}$ is the additive white noise distributed with $\mathcal{CN}(0, \sigma^2)$. Note that $H_{k,r}$ is the equivalent channel depending on the spatial diversity employed in the system. Therefore, it is a function of $H_{k,r,i,j}$ which is explored in later sections. The signal-to-noise ratio (SNR) can be written as

$$\mathsf{SNR}_{k,r} = \rho_k Z_{k,r}, \tag{2}$$

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where $\rho_k \triangleq \frac{PG_k}{\sigma^2}$ denotes the large scale channel effect of user k, and $Z_{k,r} \triangleq |H_{k,r}|^2$ represents the small scale channel effect of user k at resource block r. Since ρ_k is known in advance by the system, we denote $Z_{k,r}$ as the CQI of user k at resource block r.

3. ASYMPTOTIC ANALYSIS

3.1. Best-M Partial Feedback Strategy

The feedback policy is that users measure CQI for each resource block at their receivers and feed back the CQI values of the best M resource blocks from among the total R values. For each resource block, the scheduling policy chooses the user for transmission with the largest CQI among the users who fed back CQI to the transmitter for that resource block. It can be easily seen that users are equiprobable to be scheduled because the scheduler only takes into account the small scale channel effect. Also, we assume that if no user provides CQI for a certain resource block, then scheduling outage happens and the transmitter does not utilize it for transmission.

We know that $Z_{k,r}$ are independent and identically distributed (i.i.d.) across users and resource blocks. For notational simplicity, we denote it as Z. We denote $Y_{k,r,M}$ as the received CQI at the transmitter for user k at resource block r under the best-M partial feedback policy, and simplify it as Y_M due to the i.i.d. assumption. Due to our scheduling policy, the scheduled user k_r^* at resource block r is selected from the set of users providing CQI for that resource block (we denote the set as $\mathcal{U}_{r,M}$), namely

$$k_r^* = \arg \max_{k \in \mathcal{U}_{r,M}} Y_{k,r,M}, \tag{3}$$

The cumulative distribution function (CDF) of Y_M acts as the building block in deriving the average spectral efficiency of the system. We present the result through the following lemma.

Lemma 1. The CDF of Y_M is given by

$$F_{Y_M}(x) = \sum_{m=0}^{M-1} \xi_1(R, M, m)(F_Z(x))^{R-m}, \tag{4}$$

where $\xi_1(R, M, m) = \sum_{i=m}^{M-1} \frac{M-i}{M} \binom{R}{i} \binom{i}{m} (-1)^{i-m}$.

Proof. See [3]. □

The explicit expression for the average spectral efficiency is presented in [3]. We want to emphasize that the closed form results are cumbersome and do not easily lend themselves to offering insights. Therefore, we resort to asymptotic analysis for further investigation.

3.2. Procedure for Determining the Asymptotic Approximation

In the asymptotic analysis, we first assume symmetric large scale channel effect ρ for notational simplicity. Then we tailor our results to the specific cases with asymmetric large scale channel effects. We aim to find the limiting distribution of the maximum throughput in order to derive the asymptotic expression for the average spectral efficiency. Specifically, we examine the limiting distribution of the throughput W_M,

$$W_M = T(Y_M) = \log_2(1 + \rho Y_M). \tag{5}$$

We provide the following best-M limiting throughput distribution (LTD-M) theorem to the general partial feedback OFDMA system as follows.

Theorem 1. (LTD-M Theorem) Assume that under the best-M partial feedback scheme with R resource blocks and K users, the CQI received at the transmitter Y_M is a nonnegative random variable with CDF $F_{Y_M}(x)$ such that $f_{Y_M}(x) = F'_{Y_M}(x) > 0$ and $\omega(F_{Y_M}) \triangleq \sup\{x : F_{Y_M} < 1\} = \infty$. If $F_{Y_M} \in \mathcal{D}(G_3)$, i.e., F_{Y_M} belongs to the domain of attraction of the Gumbel distribution, then the distribution of the throughput $F_{W_M}(r) = F_{Y_M}(T^{-1}(r)) \in \mathcal{D}(G_3)$, i.e., F_{W_M} belongs to the domain of attraction of the Gumbel distribution. Moreover, the normalizing constants for user k are given by

$$\begin{aligned} a_{k:K}(M) &= \log_2(1 + \rho_k F_{Y_M}^{-1}(1 - \tfrac{R}{KM})), \\ b_{k:K}(M) &= \log_2\left(\frac{1+\rho_k F_{Y_M}^{-1}(1-\frac{R}{KMe})}{1+\rho_k F_{Y_M}^{-1}(1-\frac{R}{KM})}\right). \end{aligned} \tag{6}$$

Proof. See [7]. □

The LTD-M theorem enables us to study the distribution of Y_M instead of directly examining F_{W_M}. We can approximate the average spectral efficiency $C(M)$ by the asymptotic expression $\mathcal{C}(M)$:

$$\mathcal{C}(M) = \frac{1}{K}\left(1 - \left(1 - \frac{M}{R}\right)^K\right) \sum_{k=1}^{K} (a_{k:K}(M) + E_0 b_{k:K}(M)), \tag{7}$$

where E_0 is the Euler constant.

We see from (4) that the statistical property of Y_M depends on Z, which relies on different spatial diversity schemes. We know that the asymptotic convergence property involves only the tail property of the distribution. We examine a general form of distribution in the following theorem, which incorporate common fading models and spatial diversity schemes.

Theorem 2. If $f_Z(x)$ is tail equivalence to $x^\alpha e^{-\beta x}$ as $x \to \infty$ with $\beta > 0$ and any α, then $F_{Y_M} \in \mathcal{D}(G_3)$.

Proof. (*Sketch*) In order to prove $F_{Y_M} \in \mathcal{D}(G_3)$, we need to show that $\lim_{x\to\infty} \frac{d}{dx}\left[\frac{1-F_{Y_M}(x)}{f_{Y_M}(x)}\right] = 0$. Carrying out the differentiation, another equivalent condition is $\lim_{x\to\infty} \frac{(F_{Y_M}(x)-1)f'_{Y_M}(x)}{(f_{Y_M}(x))^2} = 1$. We firstly substitute F_{Y_M} from (4) and $f_Z(x) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\beta x}$ to derive f_{Y_M} and f'_{Y_M}, then we can complete the proof by evoking the fact that $\sum_{m=0}^{M-1} \xi_1(R, M, m) = 1$ and applying the L'Hospital's rule. □

Remark: Though we assume Rayleigh fading for channel modeling, Theorem 2 can be developed to address common class of channel models, such as Ricean fading, Nakagami fading, and log-normal fading. Due to space limit, we do not elaborate on this and utilizes Rayleigh fading for further spatial diversity analysis.

The final step to obtain the explicit form of the normalizing constants in Theorem 1 involves in evaluating $F_{Y_M}^{-1}$. Due to the complicated form of F_{Y_M}, the normalizing constants for the general best-M partial feedback can not be expressed in a simple form with elementary functions except for the special full feedback and best-1 feedback cases. Therefore, we provide the following tight functional approximation

$$F_{Y_M}(x) \simeq \tilde{F}_{Y_M}(x) = (F_Z(x))^{\frac{R}{M}}. \tag{8}$$

Up to now, we have developed the sufficient conditions for checking the type of convergence and proposed a tractable functional approximation and the LTD-M theorem to obtain the normalizing constants. These normalizing constants are utilized to calculate the asymptotic approximation to the average spectral efficiency of the system.

Now we can determine the minimum required partial feedback. The method is explored in [3] and [7], where the ratio between the average spectral efficiency achieved by minimum required partial feedback M^* and by full feedback exceeds a pre-defined threshold η, namely

$$\text{Find the minimum } M^*, \quad s.t. \ \frac{\mathcal{C}(M^*)}{\mathcal{C}(N)} \geq \eta \tag{9}$$

Due to the cumbersome expressions from the exact analysis, our proposed method is to substitute the asymptotic approximations to determine M^*.

4. DIFFERENT SPATIAL DIVERSITY SCHEMES

Recall that the specific form of Z depends on the spatial diversity scheme in the system. In this section, we consider three spatial diversity schemes: 1) transmit antenna selection/selective combining (TAS/SC) scheme; 2) orthogonal space-time block codes (OSTBC) scheme; 3) transmit antenna selection/maximum ratio combining (TAS/MRC) scheme. Note that all three schemes are adequate to achieve full spatial diversity order over the MIMO channels. Also, we assume that the total transmit power across N_t antennas is the same for the three schemes for fair comparison.

4.1. TAS/SC Scheme

In this scheme, the system selects the transmit receive antenna pair with the strongest small scale channel effect from N_tN_r possible antenna pairs. Thus the equivalent channel and the CQI of user k at block r can be expressed as

$$Z_{k,r} = |H_{k,r}|^2 = \max_{i,j} |H_{k,r,i,j}|^2. \tag{10}$$

We can obtain the CDF of Z from order statistics, $F_Z^{\mathrm{TAS/SC}}(x) = (1-e^{-x})^{N_tN_r}$. Then the PDF of Z can be derived as

$$f_Z^{\mathrm{TAS/SC}}(x) = N_tN_r(1-e^{-x})^{N_tN_r-1}e^{-x}. \tag{11}$$

Therefore, Theorem 2 holds with $\alpha = 0$, and $\beta = 1$. By employing the functional approximation, we can approximate the normalizing constants $a_{k:K}^{\mathrm{TAS/SC}}(M)$, $b_{k:K}^{\mathrm{TAS/SC}}(M)$ as follows

$$\begin{aligned} \tilde{a}_{k:K}^{\mathrm{TAS/SC}}(M) &= \log_2\left(1+\rho_k \ln\left(\frac{1}{1-(1-\frac{R}{KM})^{\frac{M}{N_tN_rR}}}\right)\right), \\ \tilde{b}_{k:K}^{\mathrm{TAS/SC}}(M) &= \log_2\left(\frac{1+\rho_k \ln\left(\frac{1}{1-(1-\frac{R}{KMe})^{\frac{M}{N_tN_rR}}}\right)}{1+\rho_k \ln\left(\frac{1}{1-(1-\frac{R}{KM})^{\frac{M}{N_tN_rR}}}\right)}\right). \end{aligned} \tag{12}$$

We can then substitute the above results into (7) to obtain the asymptotic approximation to the average spectral efficiency of the system.

4.2. OSTBC Scheme

OSTBC scheme leverages coding over transmit antennas and time to realize full spatial diversity order. From [1], the equivalent channel as well as the CQI of user k at block r is the square of the Frobenius norm of the channel matrix normalized by N_t, namely

$$Z_{k,r} = |H_{k,r}|^2 = \frac{1}{N_t}\sum_{i=1}^{N_r}\sum_{j=1}^{N_t} |H_{k,r,i,j}|^2. \tag{13}$$

Thus Z follows the Gamma distribution with $\mathcal{G}(N_tN_r, N_t)$ with the CDF given by the incomplete Gamma function ratio, i.e., $F_Z^{\mathrm{OSTBC}}(x) = \tilde{\Gamma}(N_tN_r, N_tx) = \frac{1}{\Gamma(N_tN_r)}\int_0^{N_tx} t^{N_tN_r-1}e^{-t}dt$. The PDF of Z is derived as

$$f_Z^{\mathrm{OSTBC}}(x) = \frac{N_t^{N_tN_r}}{\Gamma(N_tN_r)} x^{N_tN_r-1}e^{-N_tx}. \tag{14}$$

It is clear that Theorem 2 holds with $\alpha = N_tN_r - 1$, and $\beta = N_t$. By employing the functional approximation, we can approximate the normalizing constants $a_{k:K}^{\mathrm{OSTBC}}(M)$, $b_{k:K}^{\mathrm{OSTBC}}(M)$ as follows

$$\begin{aligned} \tilde{a}_{k:K}^{\mathrm{OSTBC}}(M) &= \log_2\left(1+\rho_k\tilde{\Gamma}^{-1}_{(N_tN_r,N_t)}((1-\tfrac{R}{KM})^{\frac{M}{R}})\right), \\ \tilde{b}_{k:K}^{\mathrm{OSTBC}}(M) &= \log_2\left(\frac{1+\rho_k\tilde{\Gamma}^{-1}_{(N_tN_r,N_t)}((1-\frac{R}{KMe})^{\frac{M}{R}})}{1+\rho_k\tilde{\Gamma}^{-1}_{(N_tN_r,N_t)}((1-\frac{R}{KM})^{\frac{M}{R}})}\right), \end{aligned} \tag{15}$$

where $\tilde{\Gamma}^{-1}_{(\cdot,\cdot)}(\cdot)$ is the inverse incomplete Gamma function. It should briefly be noted that the calculated expression acts as an upper bound because the full code rate for complex OSTBC is only achieved with $N_t = 2$.

4.3. TAS/MRC Scheme

The TAS/MRC scheme realizes antenna selection at the transmitter and MRC at the receiver. The equivalent channel and the CQI of user k at block r can be formulated as

$$Z_{k,r} = |H_{k,r}|^2 = \max_j \sum_{i=1}^{N_r} |H_{k,r,i,j}|^2. \tag{16}$$

We can obtain the CDF of Z as $F_Z^{\mathrm{TAS/MRC}}(x) = (\tilde{\Gamma}(N_r, x))^{N_t}$. The PDF of Z is derived to be

$$f_Z^{\mathrm{TAS/MRC}}(x) = \frac{N_t(\tilde{\Gamma}(N_r,x))^{N_t-1}}{\Gamma(N_r)} x^{N_r-1}e^{-x}. \tag{17}$$

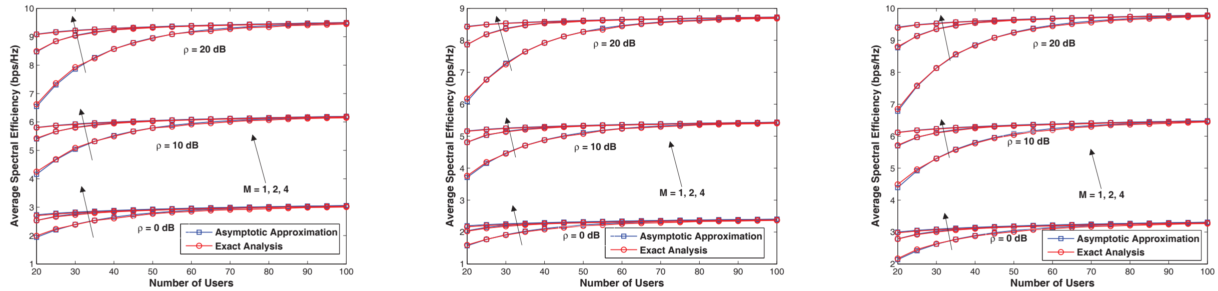
Therefore, Theorem 2 holds with $\alpha = N_r - 1$, and $\beta = 1$. By employing the functional approximation, we can approximate the normalizing constants $a_{k:K}^{\mathrm{TAS/MRC}}(M)$, $b_{k:K}^{\mathrm{TAS/MRC}}(M)$ as follows

$$\begin{aligned} \tilde{a}_{k:K}^{\mathrm{TAS/MRC}}(M) &= \log_2\left(1+\rho_k\tilde{\Gamma}^{-1}_{(N_r,1)}((1-\tfrac{R}{KM})^{\frac{M}{N_tR}})\right), \\ \tilde{b}_{k:K}^{\mathrm{TAS/MRC}}(M) &= \log_2\left(\frac{1+\rho_k\tilde{\Gamma}^{-1}_{(N_r,1)}((1-\frac{R}{KMe})^{\frac{M}{N_tR}})}{1+\rho_k\tilde{\Gamma}^{-1}_{(N_r,1)}((1-\frac{R}{KM})^{\frac{M}{N_tR}})}\right). \end{aligned} \tag{18}$$

We can then substitute the above results into (7) to obtain the asymptotic approximation to the average spectral efficiency of the system.

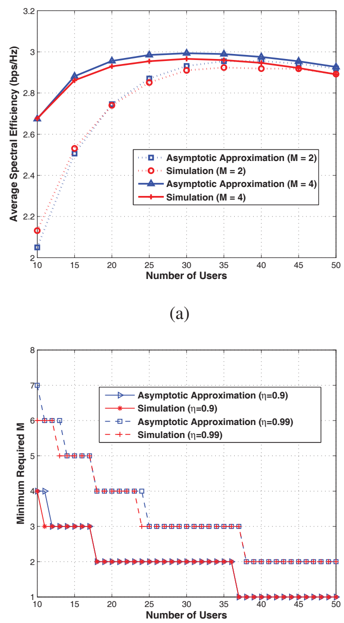
5. NUMERICAL RESULTS

We assume the number of resource blocks $R = 16$, the number of transmit antennas at the base station $N_t = 4$, the number of receive antennas residing in the user side $N_r = 2$. We firstly examine the asymptotic analysis in the symmetric case where users have the same large scale channel effect. In Fig. 1 we compare the average spectral efficiency obtained by the exact analysis and by our proposed asymptotic approximation under different symmetric ρ for the best-M partial feedback strategy. Results are shown for all the aforementioned three spatial diversity schemes. It can be seen that the asymptotic approximation tracks the system performance well even for small number of users. Generally speaking, we can order the performance



(a) (b) (c)

Fig. 1. Comparison of the average spectral efficiency for best-M feedback obtained using the exact analysis and the asymptotic analysis under different ρ for different M with respect to the number of users ($\rho = 0$ dB, 10 dB, 20 dB; $N = 16$; $M = 1, 2, 4$; $N_t = 4, N_r = 2$): (a) TAS/SC scheme; (b) STBC scheme; (c) TAS/MRC scheme.



(a)

(b)

Fig. 2. The TAS/MRC scheme under asymmetric scenario: ($N = 16$; $N_t = 4, N_r = 2$; $\lambda = 0.05$; $\eta = 0.9, 0.99$) (a) The average spectral efficiency with respect to the number of users; (b) Comparison of the minimum required M obtained from simulation and asymptotic analysis under different thresholds.

achieved by the three schemes as: TAS/MRC, TAS/SC, and STBC with TAS/MRC being the best. The converse ranking would be the implementation complexity.

Fig. 2 examines the asymmetric case where users experience different large scale channel effects. We employ the exponential decay model [3]: $\rho_k = \rho e^{-\lambda k}$, such that $\sum_{k=1}^{K} \rho_k = K$. We consider the TAS/MRC scheme with $\lambda = 0.05$. Since our scheduling policy makes decision on the small scale channel effect, it may decrease the average spectral efficiency when the number of users increases. That is due to the exponential decay model which expands the variation of the average spectral efficiency and the concave property of the throughput function. Fig. 2 (a) compares the average spectral efficiency from simulation as well as our proposed asymptotic approximation. It can be seen that there is a good agreement between the two. We also determine the minimum required partial feedback in Fig. 2 (b). We set two thresholds in (9) $\eta = 0.9, 0.99$, and observe that the results obtained using asymptotic analysis tracks the results from simulation very well, especially for lower threshold and large number of users. Therefore, in a practical multiuser MIMO-OFDMA system, our proposed asymptotic analysis can be utilized to quickly determine the minimum required partial feedback given the system operating parameters and the number of associated users.

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