HEURISTIC RATIONAL MODELS IN SOCIAL NETWORKS

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ABSTRACT

A network of social agents wants to minimize a global cost given by a sum of local terms involving convex nonlinear functions of self and neighboring variables. Agents update their variables at random times according to a random heuristic rule that is on average optimal with respect to the local cost given values of neighboring agents. When all agents apply heuristic rational optimization, convergence result shows that global cost visits a neighborhood of optimal cost infinitely often with probability 1. An exponential probability bound on the worst deviation from optimality between visits to near optimal operating points is also presented. Models of opinion propagation and voting are cast in the language of heuristic rational optimization. Numerical results are presented for the opinion propagation model on both geometric and small-world network structures.

***Index Terms*— Distributed network optimization, social networks.**

1. INTRODUCTION

Social network models entail a set of agents with different individual interests and limited information gathering capabilities. A central question in social networks concerns the translation of local actions into social welfare of the entire network. Existing work frames this question as a distributed network optimization problem where global objective –welfare of the society– is determined by the agents through iterative application of local optimization rules that update local variables relying on information received from neighboring agents. However, models of social network behavior as distributed network optimization often rely on unrealistic assumptions on local optimization and update rules, connectivity structure, and information passing. While existing results, either in the social systems [1, 2] or network optimization [3, 4] literature, question some of these assumptions, it is always assumed that agents take actions that are optimal with respect to the observed variables. This is not very realistic in social networks as optimal behavior requires improbable foresight.

We propose and study more realistic models whereby agents execute actions that are optimal in an average sense only. We name these rules and the agents that use them as heuristic rational, since we think of them as the application of a heuristic rule that is intent on being optimal even though it may not be so. We show that models commonly used to study propagation of opinions in social networks [5–7] can be cast in the language of heuristic rational optimization. We then move on to study the behavior of networks composed of heuristic rational agents and show that: (i) The global network behavior visits a neighborhood of optimality infinitely often. (ii) The probability of straying away from this neighborhood by more than a given amount is exponentially bounded. These results can be interpreted as an explanation for the emergence [cf. (i)] and sustenance [cf. (ii)] of global network behavior that is close to optimal despite imperfect decision making of individual agents in social systems.

The paper begins by describing the induction of global behavior through the minimization of a cost given by a sum of local terms involving nonlinear functions of self and neighboring variables. At random times, agents observe current values of their neighbors' variables and apply a heuristic rule with the intent of minimizing the global cost with respect to the selection of their local variables. These heuristic rules need not be optimal but we assume that they are so in expectation (Section 2). We proceed to describe how opinion propagation [5] and voter [6,7] models in social networks can be interpreted as a heuristic rational version of local averaging models [3] (Section 2.1). Because of the randomness associated with heuristic rational rules we do not expect convergence to optimal global behavior. Consequently, we characterize the difference in the yield of optimal variables and the values achieved by heuristic rational rules by showing that a neighborhood of optimality is visited infinitely often with probability 1 (Theorem 1, Section 3). We further show that between visits to optimality the probability of the gap in the yield of agents' variables exceeding a given value is bounded exponentially (Theorem 2, Section 4). We close the paper with numerical results for opinion propagation model in social networks (Section 5).

2. LOCAL HEURISTIC RATIONAL OPTIMIZATION

Consider a network of N agents represented by the symmetric graph $G = (V, E)$ where vertices $i \in V$ denote agents and edges $(i, j) \in E$ connections between them. Agent i can only interact with neighboring nodes $n(i) = \{j : (j, i) \in E\}$ that form an edge with her. We denote as $N_i := \#(n(i))$ the cardinality of the number of neighbors. Each of the agents $i \in V$ is associated with corresponding variable $x_i \in \mathbb{R}^n$ and a convex function $f_{0i}(x_i)$. Each of the edges $(i, j) \in E$ is affiliated with a convex function $f_{ij}(x_i, x_j)$ that depends on the agent variables at the vertices of the given edge. To maintain symmetry we require that functions $f_{ij}(x_i, x_j)$ and $f_{ji}(x_j, x_i)$ be equal,

$$f_{ij}(x_i, x_j) = f_{ji}(x_j, x_i), \quad \text{for all } i, j \in n(i). \tag{1}$$

Variables x_i are also constrained to the convex set $\mathcal{X}_i$ in that allowable values satisfy $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^n$. Define the vectors $x := \{x_i\}_{i \in V}$ grouping all network variables and $x_{n(i)} := \{x_j\}_{j \in n(i)}$ containing the variables of all neighbors of i. Further introduce the set $\mathcal{X} := \prod_{i \in V} \mathcal{X}_i$ to represent the Cartesian product of sets $\mathcal{X}_i$.

The function

$$f_i(x_i, x_{n(i)}) := f_{0i}(x_i) + \sum_{j \in n(i)} f_{ij}(x_i, x_j) \tag{2}$$

represents a cost that agent i would like to make as small as possible by proper selection of its variable $x_i \in \mathcal{X}_i$. Since this cost depends on neighboring variables $x_{n(i)}$, it follows that x_i and x_j for $j \in n(i)$ have to be jointly chosen. But these neighboring variables are jointly chosen with their respective neighbors, which depend on the values of their corresponding neighbors, and so on. It follows that as long as the network is fully connected, cost minimization requires simultaneous selection of all variables x_i. This is not a plausible model of network behavior.

Alternatively, suppose that at random time $t \in \mathbb{R}^+$, agent i observes the values of neighboring variables $x_{n(i)}(t)$. Given the interest in minimizing the local cost $f_i(x_i, x_{n(i)})$ in (2), a *rational* action for this agent is to update her variable by selecting the value that minimizes $f_i(x_i, x_{n(i)})$ given the observed values of neighboring variables,

$$\tilde{x}_i(t) = \underset{x_i \in \mathcal{X}_i}{\operatorname{argmin}}\, f_i\big(x_i, x_{n(i)}(t)\big). \tag{3}$$

Since the update in (3) is based on information that can be locally acquired and is unilaterally executed by i it constitutes a possible model for network optimization, which has indeed been used to model, e.g., the propagation of opinions in a social network; see [5] and Section 2.1. However, it is not always accurate to assume that agents apply optimal policies perfectly. In, e.g., social systems, agents apply heuristic rules in

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their decision making which are prone to randomness and suboptimality. To model this type of network we introduce the concept of heuristic rational actions as random actions that are optimal on average as we formally define next.

Definition 1 *Consider network agent i associated with variable x_i and denote as $x_{n(i)}(t)$ the values of neighboring variables at time t. We say that a probabilistic rule $x_i(t) \in \mathcal{X}_i$ is heuristic rational if and only if its expectation is a rational action as defined in* (3)*,*

$$\mathbb{E}\left[x_i(t) \,\middle|\, x_{n(i)}(t)\right] = \tilde{x}_i(t) = \operatorname*{argmin}_{x_i \in \mathcal{X}_i} f_i\big(x_i, x_{n(i)}(t)\big). \tag{4}$$

This paper considers network optimization models that consist of a random activation rule that determines when agents modify their variables and a heuristic rational rule that determines how the active agent updates her local values. Activations are indexed by the non-negative integer variable $k \in \mathbb{N}$ with $k = 0$ denoting the initial state. Variable $k \neq 0$ denotes the kth activation that occurs at time t_k. Each agent has a positive probability of becoming active at any given time interval. Hence, kth activation almost surely involves a unique agent $i = i_k$ modifying her local variable $x_i = x_{i_k}$. When an activation occurs, variables $x_i(t_k)$ stay unchanged for all agents $i \neq i_k$ and are updated to $x_{i_k}(t_k)$ for terminal i_k. Update rules are restricted to depend only on neighboring variables $x_{n(i_k)}(t_k)$ and are assumed heuristic rational in the sense of Definition 1.

Summing up the local costs $f_i(x_i, x_{n(i)})$ in (2) yields the global cost

$$f(x) := \sum_{i \in V} f_i(x_i, x_{n(i)}) = \sum_{i \in V} f_{0i}(x_i) + \sum_{i \in V, j \in n(i)} f_{ij}(x_i, x_j), \tag{5}$$

that measures the optimality of configuration $x := \{x_i\}_{i \in V}$ from a global perspective – as opposed to $f_i(x_i, x_{n(i)})$ that measures the optimality of configuration x_i from a local perspective. In particular, there exist globally optimal configurations x^* that achieve the minimum possible cost $p^* = f(x^*)$ given by

$$p^* := \min_{x \in \mathcal{X}} f(x) = \min_{x \in \mathcal{X}} \sum_{i \in V} f_i(x_i, x_{n(i)}). \tag{6}$$

The goal of this paper is to compare the sequence of iterates $x(t_k)$ generated by recursive application of heuristic rational rules with the optimal configuration x^*. More to the point, we define the stochastic process $\{F_k\}_{k \in \mathbb{N}}$ of optimality gaps with elements

$$F_k := f(x(t_k)) - p^*. \tag{7}$$

Our results will establish that the optimality gap F_k achieves a small value with probability 1 infinitely often (Theorem 1, Section 3). We will also establish that the largest value achieved in each of these excursions follows an exponential probability bound (Theorem 2, Section 4). Before proceeding with the analysis, we discuss examples of network optimization with heuristic rational agents in social networks.

2.1. Opinion Propagation

The propagation of opinions in a social network can be cast in the language of heuristic rational optimization. In this context we interpret $x_i \in [-1, 1]$ as the opinion of a social agent. Consider a social network where a subset $\mathcal{S}$ of agents are stubborn and have fixed extreme opinions $x_i = \{-1, 1\}$ for all $i \in \mathcal{S}$ while other agents are compliant i.e. value agreement with friends with whom they are directly connected [5]. We model the desire for agreement through the penalty function $f_{ij}(x_i, x_j) = (1/4)(x_i - x_j)^2$. The resulting cost for disagreement for agent i is $f_i(x_i, x_{n(i)}) = (1/4)\sum_{j \in n(i)}(x_i - x_j)^2$ as follows from (2) in which function $f_{0i}(x_i) = 0$. Through minimization of this quadratic cost we have that the rational action, as defined by (3), for agent i at time t is

$$\tilde{x}_i(t) = \frac{1}{N_i} \sum_{j \in n(i)} x_j(t). \tag{8}$$

This action amounts to taking a local average of opinions in the network [3]. A heuristic rational rule $x_i(t)$ randomizes $\tilde{x}_i(t)$ to account for the fact that the average in (8) is not computed exactly but rather guessed. The presumption in Definition 1 is that these guesses are correct on average in that $\mathbb{E}[x_i(t)] = \tilde{x}_i(t)$.

A more interesting example of heuristic rationality stems from the observation that agents are not likely to consider opinions of all of their neighbors at each decision but rather rely on interactions with random subsets of friends. Accounting for the fact that interactions occur between a member of the network and subsets of her friends is the intent of voter models [6, 7]. The model of opinion propagation in this case replaces the average in (8) by the average of a random sample of friends

$$x_i(t) = \frac{1}{\#(\tilde{n}_i(t))} \sum_{j \in \tilde{n}_i(t)} x_j(t), \tag{9}$$

where $\tilde{n}_i(t) \subseteq n_i$ denotes the random interaction group at time t. If all subsets of friends are equally likely to be chosen it follows that actions $\tilde{x}_i(t)$ in (8) and actions $x_i(t)$ in (9) are such that $\mathbb{E}[x_i(t)] = \tilde{x}_i(t)$. Thus, we can think of voter models [cf. (9), [6, 7]] as heuristic rational rules for the local averaging model [cf. (8), [3]].

3. NEAR OPTIMALITY

The sequence of iterates $x(t_k)$ generated by recursive application of heuristic rational rules is akin to a stochastic version of block coordinate descent on the function $f(x)$. In coordinate descent algorithms minimization is attempted by alternation between descents on different subsets of variables chosen according to a given rule. In the case of heuristic rational optimization we can identify agents' variables as coordinate blocks and random activation as the selection rule. The structure of the local cost $f_i(x_i, x_{n(i)})$ in (2) allows for the distributed implementation of block coordinate descent. Given this correspondence, we present results that show convergence to a neighborhood of the optimal configuration x^*, [cf. (6)] in some sense if the following assumptions on the cost function $f(x)$ and the random activation rule are satisfied.

(A1) Strong convexity. The global cost $f(x)$ is strongly convex in that there exists a constant $m > 0$ such that for any pair of points $x \in \mathcal{X}$ and $y \in \mathcal{X}$ it holds

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2}\|y - x\|^2. \tag{10}$$

(A2) Lipschitz gradients. Gradients of the global cost $f(x)$ are Lipschitz in that there exists a constant $M > 0$ such that for any pair of points $x \in \mathcal{X}$ and $y \in \mathcal{X}$ it holds

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{M}{2}\|y - x\|^2. \tag{11}$$

(A3) Random activation. At any given time t, all agents are equally likely to become active.

(A4) Bounded variance The mean square error of the heuristic rational action $x_{i_k}(t_k)$ with respect to the corresponding rational action $\tilde{x}_{i_k}(t_k)$ is bounded [cf. (4)].

$$\mathbb{E}\left[\|x_{i_k}(t_k) - \tilde{x}_{i_k}(t_k)\|^2\right] \leq \sigma^2. \tag{12}$$

Assumptions (A1) and (A2) are typical in convergence analysis of descent algorithms. They are satisfied by the examples discussed in Sections 2.1. Assumption (A3) states that activations occur at random times and that all agents are equally likely to become active in any given time interval. This assumption is also common; see, e.g., [6]. Among other possibilities it can be satisfied if all agents have an activation clock based on independent exponential waiting times with equal means. This is more a matter of simplifying discussion than a fundamental requirement. It can be substituted by laxer conditions as we discuss in Remark 1. Assumption (A4) bounds the average irrationality of each agent by bounding the deviation from the rational decision (3). We emphasize that this bound holds on a mean square sense. It is possible to have isolated actions that are arbitrarily bad. Our results are parametric on the irrationality bound σ^2. As time increases the optimality gap F_k of the global network behavior approaches a neighborhood of zero whose size is determined by the irrationality bound σ^2.

The following result characterizes the convergence behavior of sequence of heuristic rational updates with respect to optimality – see [8] for proof.

Theorem 1 *Consider the heuristic rational sequence of iterates $x(t_k)$ such that at time t_k agent i_k updates her local variables $x_i(t_k)$ according to a heuristic rational update [cf. Definition 1] with corresponding optimality gaps F_k [cf. (7)]. Define the best optimality gap by time t_k as $F_k^{\text{best}} := \min_{l\in[0,k]} F_l$. If assumptions (A1)-(A4) hold, it follows that*

$$\lim_{k\to\infty} F_k^{\text{best}} \leq \frac{M\sigma^2}{2\beta}, \quad \textit{a.s.} \tag{13}$$

i.e. the optimality gap becomes smaller than $M\sigma^2/2\beta$ at least once for almost all realizations.

According to Theorem 1 it holds that for almost all realizations the optimality gap F_k approaches or becomes smaller than $\sigma^2 M/2\beta$ at least once as k grows. Theorem 1 also implies that this happens infinitely often. Indeed, if at given time k_0 we have $F_{k_0} > \sigma^2 M/2\beta$ a simple time shift in Theorem 1 permits concluding that there exists a future time k at which $F_k \leq \sigma^2 M/2\beta$.

For F_k to become small we need to have the current network configuration $x(k)$ close to the optimal configuration x^*. Consequently, Theorem 1 implies that $x(k)$ enters into a neighborhood of the optimal configuration infinitely often. The volume of this neighborhood increases with increasing mean squared error of the heuristic rule σ^2, increasing Lipschitz constant M, or decreasing condition number β. The condition number $\beta := m/(MN)$ is small for functions $f(x)$ having $m \ll M$ corresponding to ill conditioned functions with elongated level sets. Therefore, the dependence on β captures the difficulty of minimizing the cost $f(x)$. The constant M is of little consequence as it plays the role of a normalizing constant. If we multiply the function $f(x)$ with a constant, both, the optimality gaps F_k and the Lipschitz constant M are multiplied by the same constant. The dependence on the mean squared error σ^2 captures the increase in global suboptimality as agents' behaviors become more erratic.

If the optimality gap F_k approaches a small value infinitely often but can stray away from it, the question arises of what the process's behavior is between visits to the optimality neighborhood. We answer this question in the following section after the following remark.

Remark 1 Results in this section follow with slight modifications when the assumption that all agents are equally likely to become active is relaxed to the assumption that all agents have possibly different but strictly positive probabilities of becoming active. This less restrictive assumption still ensures that when the configuration $x(t)$ is not optimal there is always a positive probability of the *rational* rule descending towards the optimum.

4. EXCURSIONS FROM NEAR OPTIMALITY

Although Theorem 1 shows that the network state moves within a close boundary of the optimal configuration almost surely and infinitely often, it does not claim a guarantee on staying close to the optimal value. In fact, it is easy to see that in some particular examples the process F_k is almost sure to move out of the optimality neighborhood $F_k \leq M\sigma^2/2\beta$ and even become arbitrarily bad with small but nonzero probability. This may happen in the unlikely but not impossible situation in which the variations in the heuristic rational rule cancel out the intended drive towards optimality. In this section, we derive an exponential probability bound on these excursions from optimality. The bound shows that while arbitrarily bad excursions may be possible they happen with exponentially small probability.

To formally define excursions away from the optimality neighborhood, suppose that at given iteration k, the optimality gap is $F_k = (1+\rho)M\sigma^2/2\beta$, i.e., larger than the neighborhood border by a factor $\rho > 0$. Further consider a given value $\gamma > F_k$. We define excursion as the trajectory $F_k, F_{k+1}, \ldots, F_{k+L}$ of the optimality gap until the process returns to a value $F_{k+L} < F_k$ smaller than the given gap F_k from which the excursion started. Notice that L is a random stopping time given by $L = \min_{l>0}\left(F_{k+l} < F_k\right)$. In particular, we are interested in the worst value $F_k^\dagger = \max(F_k, F_{k+1}, \ldots, F_{k+L})$ reached during the excursion. In formal terms we define $F_k^\dagger$ as

$$F_k^\dagger := \max_{l\geq 0}\left(F_{k+l}, \text{ for } l \leq \min_{j>0}\left(F_{k+j} < F_k\right)\right). \tag{14}$$

Our goal here is to determine the probability $\mathrm{P}\big(F_k^\dagger \geq \gamma\big)$ that the worst value attained during the excursion exceeds the given γ. To bound the probability $\mathrm{P}\big(F_k^\dagger \geq \gamma\big)$ we need the following additional assumption.

(A5) Bounded Increments. The difference on optimality gaps between successive iterations is almost surely bounded by a finite constant $\kappa > 0$, i.e., for all times k we have that

$$\mathrm{P}\left(|F_{k+1} - F_k| \leq \kappa \,\middle|\, F_k\right) = 1. \tag{15}$$

A particular case in which Assumption (A5) is satisfied is when the functions $f_{ij}(x_i, x_j)$ are bounded for all feasible values $x_i \in \mathcal{X}_i$ and $x_j \in \mathcal{X}_j$. Assumption (A5) can be alternatively satisfied if the differences $\|x_{i_k}(t_k) - \tilde{x}_{i_k}(t_k)\|$ between rational and heuristic rational actions are almost surely bounded. This latter condition is more stringent than the finite variance requirement of Assumption (A4). For the opinion propagation scenario in Section 2.1, the bound in (15) is the maximum number of neighbors, i.e., $\kappa = \max_i(N_i)$. This corresponds to the most connected agent flipping its opinion from -1 to 1.

The exponential bound on $\mathrm{P}\big(F_k^\dagger \geq \gamma\big)$ is stated in the following theorem – see [8] for proof.

Theorem 2 *Assume that at time k the value of F_k exceeds the optimality neighborhood of Theorem 1 by a factor $\rho > 0$, i.e., $F_k = (1+\rho)M\sigma^2/2\beta$, and let $F_k^\dagger$ be the worst optimality gap achieved during the subsequent excursion as defined in* (14)*. If assumptions (A1)-(A5) hold, then, for arbitrary given constant γ we have*

$$\mathrm{P}\left(F_k^\dagger \geq \gamma \middle| F_k\right) \leq \mathrm{e}^{-c(\gamma - F_k)}, \tag{16}$$

with $c = 2\rho M\sigma^2/[(\rho M\sigma^2)^2 + \kappa^2]$.

According to Theorem 2 the probability of $F_k^\dagger$ being larger than some arbitrary constant γ decreases exponentially. This is a bound on the worst optimality gap attained during the process starting at a level set $F_k = (1+\rho)M\sigma^2/2\beta$ and ending at or below the starting level set F_k. This result provides a way to characterize process behavior outside the convergence region. The exponential bound given by (16) is dependent on a scaling coefficient c. Scaling coefficient c is inversely proportional with the bound on excursion probability. Accordingly, an increase in any of the constants σ^2, κ, M or ρ decreases scaling coefficient c pushing excursion probability bound (16) up. The effect of increase in mean squared error σ^2 would imply decrease in predictability of individual actions possibly deteriorating optimality gap. Increment bound κ given in (A5) represents the maximum possible change in optimality gap between subsequent steps. If the increment bound κ is larger, the process can jump to a larger optimality gap in one step. In the extreme case that the process can possibly have infinite increments, the excursion bound probability becomes the trivial value of 1. Lipschitz constant M is a property of

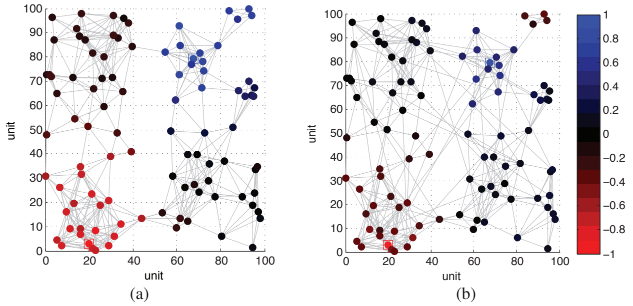

Fig. 1. Examples of two network structures for opinion propagation in social networks. Network (a) is a geometric network. Connections are drawn between agents situated less than 20 units apart. Dotted squares mark two stubborn agents in the set $\mathcal{S} = \{1, 2\}$. Network (b) is a small-world network constructed from network (a) through a cycle of rewiring (cf. Section 5) with probability $p_r = 0.1$. Color encodes opinions at time $t = 100$.

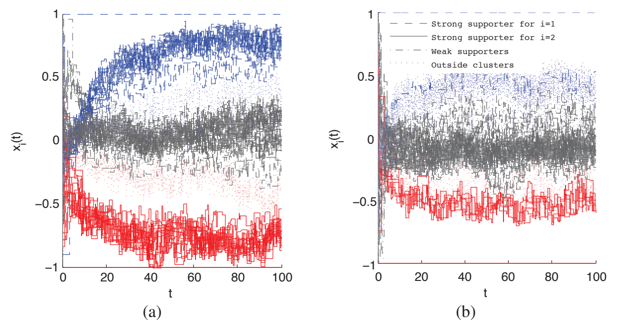

Fig. 2. Agent opinions as a function of time. (a) and (b) refer to opinion evolution for the networks in figs. 1(a)-(b), respectively. Lines represent the path $x_i(t)$ of each agent's opinion up until time $t = 100$.

the objective function and will be large for ill conditioned functions with elongated level sets. Constant ρ indicates how far away $F_k^\dagger$ is at the start of the process from near optimality region. Scaling coefficient decreases linearly as ρ grows but at the same time constant γ and F_k has to grow in parallel canceling the effect of ρ on (16).

Next, we give numerical examples for an opinion propagation scenario in which agents are heuristic rational rule decision makers.

5. SIMULATION

We consider the model of opinion propagation with stubborn agents presented in Section 2.1 on two network models. For the first network, connectivity is generated using a geometric model. We drop a group of $N = 100$ agents on a 100unit$\times$100unit two dimensional field. The coordinates r_i of user i are chosen inside this square uniformly at random. The neighborhood set of agent i consists of all agents j positioned within a cut-off distance $d = 20$unit of r_i, i.e., $n(i) = \{j : \|r_i - r_j\| \leq d,\ j \neq i\}$. Fig. 1(a) shows geometric network structure in which lines indicate connections between agents. Second network is a small-world network constructed from the first network in Fig. 1(a) by going through a cycle of edge rewiring. This cycle goes through all nodes in order and each edge that connects the node in consideration to some other node is reconnected to another random node with rewiring probability p_r. The addition of these connections reduces the average path length of the network's graph. A small-world network generated with rewiring probability $p_r = 0.1$ is depicted in Fig. 1(b). Both networks entail two stubborn agents in the set $\mathcal{S} = \{1, 2\}$ marked with dotted squares at locations $r_1 = (67, 79)$ and $r_2 = (20, 3)$. Stubborn agents have set extreme opinions $x_1(t) = 1$ and $x_2(t) = -1$. The remaining agents $i \in V/\mathcal{S}$ are compliant. They start with a random opinion uniformly drawn from $[-1, 1]$. We assume agents become active independent of each other and that times between activations of user i are exponentially distributed with parameter $\mu = 1$. The chosen rate of activation $\mu = 1$ corresponds to an average of 50 activations per agent over the whole time horizon. Opinions are updated using the rational action in (8) superimposed with zero mean noise. The noise is chosen as uniformly distributed in $[-\alpha, \alpha]$ with $\alpha = 0.1$.

The evolution of individual opinions for both geometric and small-world network structures during $t = 100$ time units is presented in figs. 2(a)-(b), respectively. For the geometric network (Fig. 2(a)), the emergence of three opinion clusters is clear after around $t = 20$. Two of these clusters settle on opinions between the intervals $0.5 < x_i(50) < 0.9$ and $-0.9 < x_i(50) < -0.5$ corresponding to strong support for the opinion of agents 1 and 2, respectively. The third cluster settles on opinions between the intervals $-0.25 < x_i(50) < 0.25$ corresponding to weak support for agent 1 or 2. Only 14 agents settle into intermediate opinions not belonging to any of these clusters without themselves clustering around a particular opinion. It is noticeable that agents in the clusters with strong support for either opinion are in close proximity of the corresponding stubborn agent. Strong supporters of agent 1, i.e., those in the cluster $\{i : 0.5 < x_i(50) < 0.9\}$, are located in the upper-right quadrant. Strong supporters of agent 2, i.e, those in the cluster $\{i : -0.9 < x_i(50) < -0.5\}$, are located in the lower-left quadrant. Weak supporters of either agent are located in either upper-left or lower-right quadrant. This outcome is based on how the specific network structure facilitates the propagation of opinions.

For the small-world network (Fig 2(b)), the decrease in average distance between agents decreases the influence of stubborn agents. The total number of strong supporters of either extreme opinion drops from 39 in the geometric network case (cf. Fig. 2(a)) to 8 in the small-world network case (Fig. 2(b)). Further the value of the strongest supporter for opinion 1 drops from 0.89 to 0.52 and similarly the value of the strongest supporter for opinion -1 rises from -0.87 to -0.6.

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