ROBUST MAXIMIN MIMO PRECODING FOR ARBITRARY CONVEX UNCERTAINTY SETS

Jiaheng Wang[1,2] *Mats Bengtsson*[2] *Björn Ottersten*[2,3] *Daniel P. Palomar*[4]

[1]National Mobile Communications Research Lab, Southeast University
[2]Signal Processing Lab, ACCESS Linnaeus Center, KTH Royal Institute of Technology
[3]Interdisciplinary Centre for Security, Reliability and Trust (SnT), University of Luxembourg
[4]ECE Department, Hong Kong University of Science and Technology

ABSTRACT

We consider a worst-case robust precoding design for multi-input multi-output (MIMO) communication systems with imperfect channel state information at the transmitter (CSIT). Instead of a particular choice, we consider a general imperfect CSIT model that only assumes the channel errors to be within a convex set, which includes most common imperfect CSIT models as special cases. The robust precoding design is formulated as a maximin problem, aiming at maximizing the worst-case received signal-to-noise ratio or minimizing the worst-case error probability. It is shown that the robust precoder can be easily obtained by solving a convex problem. We further provide an equivalent but more practical form of the convex problem that can be efficiently handled with common optimization methods and software packages.

Index Terms— Convex uncertainty sets, imperfect CSIT, maximin, MIMO, worst-case robustness.

1. INTRODUCTION

It is well known that the performance of multi-input multi-output (MIMO) communication systems depends, to a substantial extent, on the quality of the channel state information (CSI). The full benefit of a MIMO channel is achieved by exploiting CSI at the transmitter (CSIT) and adopting proper precoding techniques. With perfect CSIT, the optimal MIMO precoding has been well studied under various criteria [1]. In practice, however, CSIT is seldom perfect due to many practical issues, such as inaccurate channel estimation, quantization of CSI, erroneous or outdated feedback, and time delays or frequency offsets between the reciprocal channels. Therefore, the imperfection of CSIT has to be considered in MIMO precoding designs so that the system, on one hand, can fully utilize CSIT, and on the other hand, is robust to the imperfection of CSIT.

Following a common deterministic imperfect CSI model [2–9], we assume that the actual channel lies in the neighborhood, often called the uncertainty set or region, of a nominal channel known by the transmitter. The size of this set represents the amount of uncertainty on the channel, i.e., the bigger the set is the more uncertainty there is. This model is suitable to characterize instantaneous CSI with errors. In this case, a precoding design is said to be robust if it can achieve the best performance in the worst channel within the uncertainty set, which is referred as worst-case robustness. Such robust precoding designs can be obtained by optimizing the worst-case performance [2–9], leading to a maximin or minimax problem.

The philosophy of worst-case robustness has been widely used in MIMO precoding designs. Specifically, the worst-case robust minimum mean square error (MSE) precoder was studied in [2] and later generalized by [9] to include transmit power constraints. In [3] and [4], the authors tried to maximize the worst-case received signal-to-noise ratio (SNR) but only focused on a simplified power allocation problem by imposing some transmit directions. Interestingly, it was recently found in [7] and [8] that the transmit directions imposed in [3] and [4] are optimal in some situations, which leads to fully analytical robust precoders as well as some interesting insights. The worst-case robust precoders for MIMO multiaccess and broadcasting channels were studied in [5, 6].

In this paper, we consider a robust MIMO precoding design to maximize the worst-case received SNR or to minimize the worst-case pairwise error probability (PEP) if a space-time block code (STBC) [10] is used. The robust precoding design is formulated as a maximin problem. In contrast with the existing works, e.g., [3, 4, 7, 8], that depended on some particular uncertainty set (e.g., defined by a matrix norm), we consider a general convex uncertainty set, which covers almost all common uncertainty sets as special cases, thus pro-

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viding a general framework.

We show that the robust MIMO precoder for a general convex uncertainty set is given by the optimal Lagrange multiplier of a simple convex problem, which, at least in theory, can be efficiently solved. Considering practical issues, we further reformulate this convex problem into an equivalent but more tractable form that is solvable by most common numerical methods as well as software packages. A dual perspective link between the reformulated problem and the original convex problem is then provided.

2. PROBLEM STATEMENT

Consider a narrowband point-to-point MIMO communication system equipped with N transmit and M receive antennas. Mathematically, the baseband, symbol-sampled system can be represented by a linear model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^N$ and $\mathbf{y} \in \mathbb{C}^M$ are the transmitted and received signals, respectively, $\mathbf{H} \in \mathbb{C}^{M\times N}$ is the channel matrix, and $\mathbf{n} \in \mathbb{C}^M$ is a circularly symmetric complex Gaussian noise vector with zero mean and covariance matrix $\sigma_n^2\mathbf{I}$, i.e., $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I})$. The transmit strategy or precoding is determined by the transmit covariance matrix $\mathbf{Q} = E\{\mathbf{x}\mathbf{x}^H\}$. Indeed, via decomposing $\mathbf{Q} = \mathbf{F}\mathbf{F}^H$, the transmitted symbol vector $\mathbf{s}$, with $E\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$, can be linearly precoded by $\mathbf{F}$, resulting in $\mathbf{x} = \mathbf{F}\mathbf{s}$. In practice, the transmitter should satisfy the power constraint $\mathbf{Q} \in \mathcal{Q}$ where

$$\mathcal{Q} \triangleq \{\mathbf{Q} : \mathbf{Q} \succeq \mathbf{0},\ \mathrm{Tr}(\mathbf{Q}) \leq P\} \tag{2}$$

and P is the budget on the total transmit power.

Due to many practical issues, CSIT is seldom perfectly known, which thus calls for robust precoding that can utilize CSIT and meanwhile combat against its imperfection. As a common imperfect CSI model [2–9], people often assume that the actual channel can be expressed as $\mathbf{H} = \hat{\mathbf{H}} - \mathbf{\Delta}$, where $\hat{\mathbf{H}}$ is a nominal channel (e.g., an estimate or feedback of $\mathbf{H}$) known by the transmitter, and $\mathbf{\Delta}$ is the error between $\hat{\mathbf{H}}$ and $\mathbf{H}$ and belongs to an uncertainty set $\mathcal{E}$, i.e., $\mathbf{\Delta} \in \mathcal{E}$. Then, according to the philosophy of worst-case robustness, a precoder is said to be robust if it can achieve the best performance for the worst channel error in $\mathcal{E}$ [2–9].

In this paper we assume perfect CSI at the receiver (CSIR) and adopt the following performance measure:

$$\Psi(\mathbf{Q}, \mathbf{H}) \triangleq \mathrm{Tr}(\mathbf{H}\mathbf{Q}\mathbf{H}^H). \tag{3}$$

It has been verified in [7] that maximizing $\Psi(\mathbf{Q}, \mathbf{H})$ corresponds to:1) maximizing the received SNR; 2) minimizing the PEP of an STBC; 3) maximizing a low-SNR approximation of the mutual information; 4) minimizing a low-SNR approximation of the MSE. Therefore, by definition, the worst-case robust MIMO precoder is given by the solution to the following maximin problem:

$$\max_{\mathbf{Q}\in\mathcal{Q}} \min_{\mathbf{\Delta}\in\mathcal{E}}\ \mathrm{Tr}\left((\hat{\mathbf{H}} - \mathbf{\Delta})\mathbf{Q}(\hat{\mathbf{H}} - \mathbf{\Delta})^H\right) \triangleq \Psi(\mathbf{Q}, \mathbf{\Delta}). \tag{4}$$

In the literature, there are many particular choices of the uncertainty set $\mathcal{E}$. For example, the most frequently used uncertainty set is defined by some matrix norm as

$$\mathcal{E}_n \triangleq \{\mathbf{\Delta} : \|\mathbf{\Delta}\| \leq \varepsilon\} \tag{5}$$

where ε is the error radius, and the matrix norm $\|\cdot\|$ could be the (weighted) Frobenius norm [2, 4–7, 9] or the (weighted) spectral norm [8]. As another example, if $\hat{\mathbf{H}}$ results from uniformly quantizing the elements of $\mathbf{H}$ with a stepsize ρ, the uncertainty set can be defined as [3]

$$\mathcal{E}_q \triangleq \left\{\mathbf{\Delta} : |\mathrm{Re}\{[\mathbf{\Delta}]_{ij}\}| \leq \frac{\rho}{2},\ |\mathrm{Im}\{[\mathbf{\Delta}]_{ij}\}| \leq \frac{\rho}{2},\ \forall i, j\right\}. \tag{6}$$

Instead of a particular choice, in this paper we consider a general convex uncertainty set that covers all common uncertainty models, e.g., [2–9], as special cases. To be more exact, we only assume that $\mathcal{E}$ is a nonempty compact convex set. If the maximin robust design problem (4) with a general convex uncertainty set can be solved, so can the special cases. We show that this goal can be efficiently achieved by solving just a simple convex optimization problem.

3. OPTIMAL ROBUST PRECODER

In this section, we provide the optimal solution to the maximin problem (4). Note that, a similar general convex uncertainty set was also considered in [3], but the authors only focused on a power allocation problem, simplified from (4) by imposing possibly suboptimal transmit directions. In contrast, we are interested in finding the globally optimal solution to (4) in an efficient way.

To solve the maximin problem (4), one possible way is to express it as an ordinary maximization problem

$$\underset{\mathbf{Q}\in\mathcal{Q}}{\mathrm{maximize}}\ \Psi^\star(\mathbf{Q}) \triangleq \min_{\mathbf{\Delta}\in\mathcal{E}} \Psi(\mathbf{Q}, \mathbf{\Delta}). \tag{7}$$

Since minimization preserves concavity [11], $\Psi^\star(\mathbf{Q})$ is a concave function and thus (7) is a convex optimization problem. On the other hand, the optimal value function $\Psi^\star(\mathbf{Q})$ is usually nondifferentiable, so the common gradient-based methods, such as the Newton method or gradient-based interior-point methods, are not applicable. In this case, one can exploit subgradient-based methods, e.g., the subgradient projection method [12], which, in addition of the computation of a subgradient, suffers a slow convergence speed (see [13] for more details).

In this paper, we solve the maximin problem (4) in a more elegant way. The following result provides the optimal solution to (4) from the dual perspective of convex optimization.

Proposition 1. *Suppose that $\mathcal{E}$ is a nonempty compact convex set and $\mathcal{Q}$ is defined in (2). Consider the following convex problem:*

$$\begin{array}{ll} \underset{\boldsymbol{\Delta}\in\mathcal{E},t}{\text{minimize}} & Pt \\ \text{subject to} & (\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta}) \preceq t\mathbf{I} \end{array} \tag{8}$$

and let $\mathbf{Z}^\star$ be the optimal Lagrange multiplier associated with the constraint $(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta}) \preceq t\mathbf{I}$. Then, $\mathbf{Z}^\star$ is the optimal solution to the maximin problem (4).

Proof: To show that $\mathbf{Z}^\star$ is a solution to (4), we write the partial Lagrangian of (8) as

$$\begin{aligned} & L(\boldsymbol{\Delta}, t; \mathbf{Z}) \\ = \;& Pt + \mathrm{Tr}\left(\mathbf{Z}((\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta}) - t\mathbf{I})\right) \\ = \;& (P - \mathrm{Tr}(\mathbf{Z}))\,t + \mathrm{Tr}\left(\mathbf{Z}(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta})\right) \end{aligned} \tag{9}$$

with Lagrange multiplier $\mathbf{Z} \succeq \mathbf{0}$. The dual function is given by

$$G(\mathbf{Z}) = \inf_{\boldsymbol{\Delta}\in\mathcal{E},t} L(\boldsymbol{\Delta}, t; \mathbf{Z}) \tag{10}$$

whose domain is $\mathcal{Z} \triangleq \{\mathbf{Z} : \mathbf{Z} \succeq \mathbf{0},\ G(\mathbf{Z}) > -\infty\}$. To guarantee that $G(\mathbf{Z})$ is bounded from below, it follows that $P - \mathrm{Tr}(\mathbf{Z}) = 0$. As a result, we have

$$\mathcal{Z} = \{\mathbf{Z} : \mathbf{Z} \succeq \mathbf{0},\ \mathrm{Tr}(\mathbf{Z}) = P\} \tag{11}$$

$$G(\mathbf{Z}) = \min_{\boldsymbol{\Delta}\in\mathcal{E}} \mathrm{Tr}\left(\mathbf{Z}(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta})\right) \tag{12}$$

so that the dual problem of (8) is

$$\max_{\mathbf{Z}\in\mathcal{Z}} \min_{\boldsymbol{\Delta}\in\mathcal{E}} \ \mathrm{Tr}\left(\mathbf{Z}(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta})\right). \tag{13}$$

Note that the constraint $\mathrm{Tr}(\mathbf{Z}) = P$ in (13) can be relaxed to $\mathrm{Tr}(\mathbf{Z}) \le P$, since the optimal $\mathbf{Z}$ is always achieved with equality. Now, comparing (13) with (4), one can find that they are exactly the same with $\mathbf{Z} = \mathbf{Q}$. Therefore, the optimal Lagrange multiplier $\mathbf{Z}^\star$ is also the optimal solution to (4). ■

Proposition 1 indicates that the robust precoder can by obtained by solving the convex problem (8), which consists of a differentiable objective and a convex feasible set, thus being solvable by the common gradient-based numerical methods. Note that $(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta})$ is a matrix convex function of $\boldsymbol{\Delta}$ in the positive semidefinite space $\mathbb{S}_+^N$ [11], so $(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta}) \preceq t\mathbf{I}$ is a convex constraint and (8) is indeed a convex problem. Moreover, denoting the optimal primal solution to (8) by $\boldsymbol{\Delta}^\star$, the pair $(\mathbf{Z}^\star, \boldsymbol{\Delta}^\star)$ is in fact a saddle point of $\Psi(\mathbf{Q}, \boldsymbol{\Delta})$, for which we refer the interested reader to [13] for more details.

4. PRACTICAL REFORMULATION

So far we have theoretically shown that the solution to the maximin problem (4) can be found by solving (8) instead. However, it should be pointed out that, although (8) is a convex problem, the constraint $(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta}) \preceq t\mathbf{I}$ is given by a matrix convex function. This causes a difficulty in practice, because most optimization methods as well as software packages are not designed to solve a convex problem involving matrix convex functions.

The question now is: Is there a more practical method? In the following, we provide a positive answer to this question by showing that one can solve, instead of (8), an equivalent but more tractable problem.

Proposition 2. *Suppose that $\mathcal{E}$ is a nonempty compact convex set, and consider the following convex problem:*

$$\begin{array}{ll} \underset{\boldsymbol{\Delta}\in\mathcal{E},t}{\text{minimize}} & Pt \\ \text{subject to} & \begin{bmatrix} t\mathbf{I} & (\hat{\mathbf{H}}-\boldsymbol{\Delta})^H \\ \hat{\mathbf{H}}-\boldsymbol{\Delta} & \mathbf{I} \end{bmatrix} \succeq \mathbf{0}. \end{array} \tag{14}$$

Denote its optimal solution by $(\boldsymbol{\Delta}^\star, t^\star)$ and let

$$\mathbf{Y}^\star \triangleq \begin{bmatrix} \mathbf{Y}_{11}^\star & \mathbf{Y}_{12}^\star \\ \mathbf{Y}_{21}^\star & \mathbf{Y}_{22}^\star \end{bmatrix} \in \mathbb{S}_+^{N+M}$$

where $\mathbf{Y}_{11}^\star \in \mathbb{S}_+^N$, $\mathbf{Y}_{22}^\star \in \mathbb{S}_+^M$, and $\mathbf{Y}_{12}^\star = \mathbf{Y}_{21}^{\star H} \in \mathbb{C}^{N\times M}$, be the optimal Lagrange multiplier associated with the constraint

$$\begin{bmatrix} t\mathbf{I} & (\hat{\mathbf{H}}-\boldsymbol{\Delta})^H \\ \hat{\mathbf{H}}-\boldsymbol{\Delta} & \mathbf{I} \end{bmatrix} \succeq \mathbf{0}. \tag{15}$$

Then, $(\boldsymbol{\Delta}^\star, t^\star)$ is also the optimal solution to (8), and $\mathbf{Z}^\star = \mathbf{Y}_{11}^\star$ is the optimal Lagrange multiplier associated with the constraint $(\hat{\mathbf{H}}-\boldsymbol{\Delta})^H(\hat{\mathbf{H}}-\boldsymbol{\Delta}) \preceq t\mathbf{I}$ in (8).

Proof: The equivalence between (14) and (8) can be easily proved by using the Schur complement. The difficulty lies in the relation between the optimal Lagrange multipliers of these two problems. This can be achieved by exploring the optimality conditions of (14) and (8). Due to the space limitation, we refer the interested reader to [13] for the detailed proof. ■

Note that the constraint (15) is a linear matrix inequality (LMI), i.e., a very tractable form of convex optimization in practice. Therefore, (14) can be efficiently solved by many software packages, e.g., CVX [14]. Such software packages contain numerical methods, e.g., primal-dual interior-point methods [11], that can provide not only the optimal primal variables but also the optimal dual variables, i.e., Lagrange multipliers. In particular, when the uncertainty set is given by $\mathcal{E}_n$ in (5) or $\mathcal{E}_q$ in (6), (14) is or can be transformed into a semidefinite program (SDP).

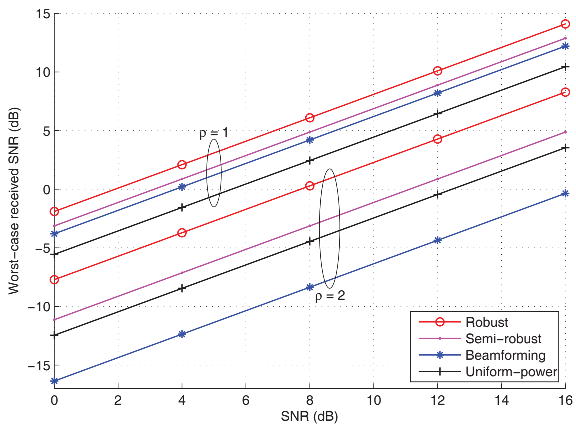

Fig. 1. Worst-case received SNR versus SNR at $\rho = 1$ and 2 for $\mathcal{E} = \mathcal{E}_q$ and $M = N = 4$.

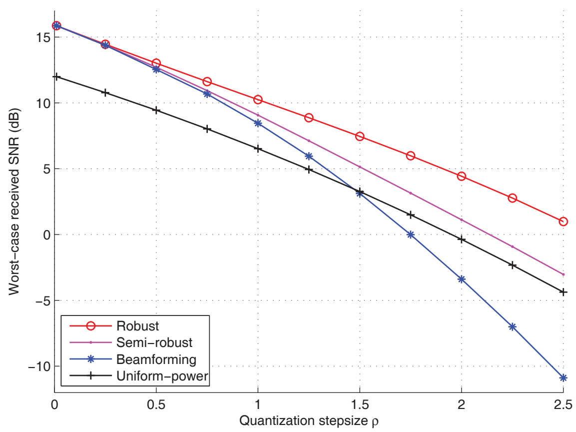

Fig. 2. Worst-case received SNR versus quantization stepsize ρ at SNR = 12dB for $\mathcal{E} = \mathcal{E}_q$ and $M = N = 4$.

5. NUMERICAL RESULTS

To demonstrate the effect of the robust MIMO precoding, we compare different precoding strategies, according to the philosophy of worst-case robustness, through their worst-case performance. Moreover, to take into account different channels, the worst-case performance is averaged over the nominal channel $\hat{\mathbf{H}}$, whose elements are randomly generated according to zero-mean, unit-variance, i.i.d. Gaussian distributions.

Due to the space limitation, we consider only the uncertainty set $\mathcal{E}_q$ in (6) for quantization errors. The robust precoding is compared with the beamforming strategy that transmits only over the maximum eigenmode of $\hat{\mathbf{H}}$, the uniform-power strategy that allocates the transmit power equally over all eigenmodes of $\hat{\mathbf{H}}$, and the semi-robust strategy in [3] that provided a robust power allocation but with fixed (suboptimal) transmit directions. Fig. 1 shows the worst-case received SNRs of the four strategies versus SNR for different quantization stepsizes, and Fig. 2 displays the relation between the worst-case received SNR and the quantization stepsize. It can be clearly seen that the robust strategy always outperforms the non-robust or semi-robust strategies in terms of worst-case performance, and that the gain becomes larger as the uncertainty increases.

6. CONCLUSION

We have considered a robust MIMO precoding design, formulated as a maximin problem, to maximize the worst-case received SNR or minimize the worst-case PEP for an STBC with imperfect CSIT. Instead of a particular choice, we have considered a general convex uncertainty set, which include most commonly used uncertainty models as special cases. The robust MIMO precoder, i.e., the solution to the maximin problem, was found to be the optimal Lagrange multiplier of a simple convex problem. We then reformulated this convex problem into an equivalent form that can be efficiently solved in practice.

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