MODIFIED ZIV-ZAKAI BOUND FOR TIME-OF-ARRIVAL ESTIMATION OF GNSS SIGNAL-IN-SPACE

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ABSTRACT

Signal Time-Of-Arrival (TOA) estimation accuracy is fundamental to the functioning of Global Navigation Satellite Systems (GNSSs). This work investigates a variant of the Ziv-Zakai bound (ZZB) named *modified* ZZB (MZZB) as a theoretical performance limit in TOA estimation, to overcome the heavy computational effort caused by the presence of nuisance parameters (carrier amplitude/phase, channel coefficients). (M)ZZB is adopted to analyze the theoretical performance of signal delay estimators in the different phases of acquisition and tracking, and numerical results are shown for the main GNSS standard signal formats: BPSK and (filtered) Binary Offset Carriers (BOC) modulations in Additive White Gaussian Noise (AWGN) channel.

Index Terms— Modified Ziv-Zakai bound, Time-of-Arrival (TOA) estimation, acquisition, tracking, Binary Offset Carrier (BOC).

1. INTRODUCTION

"One-way signal Time-Of-Arrival (TOA)" estimation represents the basis of all current Global Navigation Satellite Systems (GNSSs). The accuracy of user position is directly related to the (pseudo-)ranges estimation performed by the receiver via TOA estimation. Commonly the well known Cramér-Rao bound (CRB) is adopted as mean square error (MSE) theoretical benchmark for unbiased estimators, for its ease of calculation. Unfortunately, it requires sufficiently smooth signal waveform and possibly a differentiable parameter probability density function (pdf). In some cases of practical interest, both these conditions are not satisfied, especially as far as the standard GNSS Signal-In-Space are concerned. GPS, GLONASS, Galileo, and other GNSSs adopt Binary Phase Shift Keying (BPSK) and Binary Offset Carriers (BOC) modulations [1] with (theoretically) rectangular pulses, so that the CRB is not applicable if they are not filtered. Other bounds can be found in literature, which prove to be tighter than the CRB, but cannot in general be easily cast into a simple closed form expression. One of these is the Ziv-Zakai bound (ZZB) [2], [3] that stems out of detection theory and also considers possible parameter *a priori* information. The ZZB shows no constraints of parameter pdf, signal shape or SNR value resulting a very interesting MSE benchmark for any signal format. Unfortunately, as well as other bounds, computing the ZZB in the presence of *nuisance parameters* is very hard. In this contribution a *modified* version of the bound is adopted, i.e. the *modified* ZZB (MZZB) [4],[5], [6], whose computation in the presence of nuisance parameters is much simpler. Besides, whenever the size of the nuisance vector gets large the gap between the two versions shows to be negligible [4], [5]. We use the MZZB here to evaluate the performance of TOA estimation during signal acquisition and tracking for standard GNSS SIS (BPSK, BOC). In particular, assuming the proper *a priori* information, we can evaluate the minimum C/N_0 threshold that is needed to acquire or track the signal delay with an MSE lower than a fixed value.

2. MODIFIED ZZB FOR TOA ESTIMATION

The problem considered here is TOA estimation for positioning systems for a generic signal in Additive White Gaussian Noise (AWGN). The model of the received signal is $r(t) = s(t-\tau,\gamma)+w(t)$, where $s(t,\gamma)$ is the transmitted signal, τ is the signal delay with a uniform distribution in [0,T_x] (different distributions could be considered as well), γ is an array of "stray" (nuisance) parameters, and $w(t)$ is a white Gaussian process with Power Spectral Density (PSD) equal to $N_0/2$. In [2],[3],[4] and [5] the ZZB and its modified version are defined step by step for this scenario. The final expression of the *modified* ZZB runs as follows:

$$MZZB(\tau) \triangleq \frac{1}{T_x}\int_0^{T_x} \Delta \int_0^{T_x-\Delta} Q\left(\sqrt{\frac{\mathbb{E}_\gamma\{d^2(\Delta,h|\gamma)\}}{2N_0}}\right) dh d\Delta \quad (1)$$

where $\mathbb{E}_\gamma\{\cdot\}$ indicates statistical expectation over all possible values of γ, T_x is the maximum uncertainty on the delay and d is the euclidean distance between the two (equiprobable) delayed signals $s(t-h|\gamma)$ and $s(t-h-\Delta|\gamma)$, conditioned to the particular γ. When the estimation time T_0 is large, the squared distance between the signal replicas turns out to be

$$d^2(\Delta, h|\gamma) = 2E_{T_0}(1-\rho_{T_0}(\Delta|\gamma)) \quad (2)$$

where the (conditional) signal correlation function defined as $\rho_{T_0}(\Delta|\gamma) = \Re\left\{\int_0^{T_0} s(t|\gamma)\, s^*(t-\Delta|\gamma)dt\right\}/E_{T_0}$ is normalized to the signal energy $E_{T_0} = P_x T_0$ with P_x the signal power (which is usually also called C).

3. APPLICATION OF MZZB TO STANDARD SIGNAL-IN-SPACE

Let us consider now the performance of TOA estimation for BPSK and BOC signals in an Additive White Gaussian Noise (AWGN) channel. The $BOC(f_s, f_c)$ modulations [1] consist of superposing a square wave subcarrier of frequency $f_s = mf_g$ to the spreading code of a standard Spread-Spectrum BPSK (SS-BPSK) of rate $f_c = nf_g = 1/T_c$, where m and n are two integers and $f_g = 1.023MHz$. The chip time T_c is sliced in Ω half-cycle times $\frac{T_s}{2}$ of the square wave and $\Omega = 2\frac{f_s}{f_c} = 2\frac{m}{n}$ is the modulation order. The $BOC(f_s, f_c)$ signal can be written as

$$s_{BOC}(t) = s_{BPSK}(t)\, sign\left[\sin(2\pi f_s t)\right] \quad (3)$$

with $s_{BPSK}(t) = \sqrt{2P_x}\sum_k a_k rect\left(\frac{t-kT_c-T_c/2}{T_c}\right)$, where P_x is the signal power, a_k are the independent and identically distributed (i.i.d.) $\{\pm 1\}$ chips.

The superposition (product) with the square wave leads to splitting and shifting the baseband SS-BPSK spectrum. This allows for better performance in terms of tracking accuracy than the original BPSK owing to Gabor bandwidth enhancement, but at the cost of a worsening of the correlation function which contains multiple peaks that lead to potential acquisition and tracking ambiguities. One of the scopes of this work is to emphasize the capability of the MZZB bound to take into account these ambiguities.

The distance needed in (2) can be easily computed starting from (3), evaluating the conditional squared distance (2), averaging on nuisance parameters $\gamma = \bar{a}$ that in our case are the i.i.d. code chips. The final expression of the bound in (1) is reported for clarity in the (4), where $\rho(\Delta)$ is the theoretical BOC correlation function, $E_c = C \cdot T_c$ is the signal energy per chip, L is the number of observed chips and C is the power of the received signal.

Figure 1 shows the normalized autocorrelation functions of the theoretical BOC signals and of BPSK, with different chip rates, so that the 99% power bandwidth $B_{99\%}$ is the same for all signals. The BOC autocorrelation function runs out in a single chip time, with a number of side lobes, $\Omega - 1$ for each side, that have non negligible relative peaks compared to the main lobe at $\tau = 0$. If we assume a delay uncertainty equal or greater than a chip time, the estimation will be certainly impaired by the ambiguities caused by these side lobes.

In a more realistic scenario, *filtered* signals have to be considered. In this case the correlation becomes $\rho^F(\Delta) = \rho(\Delta) \otimes h(\Delta) \otimes h(-\Delta)$ where $h(t)$ is the impulse response of the filter, that we assume low-pass with a $-3dB$ bandwidth BW here chosen equal to the $B_{99\%}$ of the signals. Maintaining the previous hypotheses on binary random chips, the filtered correlation function is the only difference in the MZZB resulting expression to be substituted in (4).

Fig.2 depicts the theoretical performance for these signals in terms of RMSE. The uncertainty on the delay for the MZZB computation is fixed to one chip time ($T_x = T_c$), different for each signal, so the integration in (4) on Δ consider the contributions of all of the correlation side lobes.

For (very) low C/N_0 (SNR), the RMSE tends to $T_c/\sqrt{12}$, i.e. the standard deviation of a uniform random variable τ in $[0, T_c]$. In this region, the optimum estimator actually uses the *a priori* information on τ, estimating the variable with its mean value, and neglecting received noise-corrupted data. For a larger C/N_0, the MZZB curves decrease proportionally to $(C/N_0)^{-1}$, perfectly matching with the MCRB [7]. The boundary of the two regions is a threshold, and the higher the BOC modulation order, the higher the number of ambiguities in the correlation function and the higher the C/N_0 threshold to attain the "high-SNR" zone. For low and medium C/N_0 values, as expected, the mismatch between the MZZB and the MCRB curves is due to the absence of a priori information for the latter.

4. A BOUND FOR SIGNAL ACQUISITION AND TRACKING PERFORMANCE

One advantage of using the (M)ZZB is that by properly selecting the a-priori parameter uncertainty, which depends on the particular stage and scheme of estimation, we can model the two different stages of initial acquisition (large uncertainty) and steady-state tracking (smaller uncertainty).

Let's start from TOA acquisition. Assuming no information on the delay, we can consider it as a random variable uniformly distributed on a chip code period ($T_x = NT_c$). Once the uncertainty is fixed, the MZZB can be computed for the acquisition performance and the curve of RMSE can be plotted wrt the C/N_0 ratio, to find the best operating range in which the optimum estimator can achieve a pre-set accuracy during this phase. We assume also that signal is acquired when the estimation error ε falls within a pre-set range *r*, $|\varepsilon| \leq [r/2]$, event which is associated to a so called *probability of detection*: $P_d = \Pr\{|\varepsilon| \leq [r/2]\}$. Assuming that the error ε is a Gaussian random variable[1] $N(0, \sigma_0)$, then $P_d = 1 - 2Q\left(\frac{[r/2]}{\sigma_0}\right)$. Inverting this relation, a maximum standard deviation threshold for the error and a minimum $C/N_0^{Acq}(P_d, r)$ threshold can be found from the RMSE curve of the MZZB.

[1]This is true for instance for Maximum-Likelihood estimator on a large estimation window

$$MZZB(\tau)=\frac{1}{T_x}\int_0^{T_x}\Delta\,(T_x-\Delta)Q\left(\sqrt{\frac{LE_c}{N_0}\,(1-\rho\,(\Delta))}\right)d\Delta=\frac{1}{T_x}\int_0^{T_x}\Delta\,(T_x-\Delta)Q\left(\sqrt{\frac{CT_0}{N_0}\,(1-\rho\,(\Delta))}\right)d\Delta \tag{4}$$

Of course, the MZZB only depends on the starting uncertainty interval and not on the acquisition scheme adopted. Usually, during signal acquisition the search of the coarse delay is done on a limited number of "cells" with a duration δT. The total uncertainty interval is partitioned, and the higher the number of cells, the more accurate the estimation. On the other hand, the higher the number of cells, the longer the acquisition time, which impacts the time to first fix. Once the signal is acquired, the (residual) error will be $|\varepsilon| \leq [\delta T/2]$ $(r=\delta T)$. Usually, δT is a fraction of the *pull-in-range* (*PIR*) of the estimator used for the tracking of the signal, to ensure the tracking is initiated with a sufficiently small error. After acquisition is (successfully) accomplished, we have to update the uncertainty for the residual TOA to just the width of a time cell, and a new MZZB, that applies during tracking, has to be computed.

For this analysis we assume a conventional Early-Late estimator, whose pull-in-range, here defined as its linear S-curve non ambiguous region, is approximated by the early-late spacing d, which in turn is usually chosen equal to an half of the signal autocorrelation main lobe width (named *ACW*). To sum up, once we have the ACW, we can evaluate the relevant acquisition performance through the MZZB (with full uncertainty) and the needed residual error range given by $\delta T=PIR=d=\frac{ACW}{2}$. The RMSE curve of the bound will depend on the particular shaping of the autocorrelation function of the signal, and so the threshold $C/N_0^{Acq}(P_d,PIR)$ will depend on its characteristics.

Coming now to the tracking stage, the MZZB ingredients are again the delay uncertainty (much narrower now), and the signal autocorrelation function. The curves for tracking are re-computed assuming the residual (acquisition) error as a uniform random variable on the time bin span $T_x=\delta T=\frac{ACW}{2}$. During the tracking phase, the error has to stay inside the estimator PIR, so an operating range can be found choosing the maximum error deviation threshold with the experimental rule $3\sigma_{DLL}\leq\frac{PIR}{2}=\frac{ACW}{4}$, reading from the MZZB curve the minimum C/N_0^{Tr} which ensures the constraint.

4.1. Results

Once the methodology is clear, we analyzed the performance of BPSK and BOC Galileo SIS. The signal parameters that we considered are defined in the Open Service Signal-In-Space Interface Control Document Issue 1 (OS SIS ICD) of February 2010. In particular, we considered the specific ranging code chip rates,the primary code lengths (NT_c) and the receiver bandwidths, computing the correlation ACW from the theoretical signals. For the standard SIS, we reported the maximum standard deviations for the acquisition (for a set of P_d) and tracking, with the respective minimum C/N_0 thresholds. In the analysis we assume the well known equivalence $T_0=1/2B_n$, between the equivalent observation time and DLL noise bandwidth B_n, considering a common value of $B_n=10Hz$ for tracking and a $T_0=0.05$ for acquisition.

Figure 3 shows the curves of RMSE for acquisition and tracking. In addition, Tab. 1 summarizes our results and reports the parameters adopted. Regarding the acquisition performance, the minimum C/N_0^{Acq} threshold increases, as is natural, for increasing P_d, and ranges from 28.3 to 29.7 dBHz. Fine estimation requires a *lower* C/N_0^{Tr} than the previous stage, thanks to its better a priori information, with values range from 16.4 to 23.1 dBHz, i.e. more than 10 dBHz difference compared to the acquisition ones. The σ values reported in the table are the maxima allowable for the minima C/N_0. Obviously better σ values can be achieved with higher C/N_0, following the curves computed in Fig.3.

5. CONCLUSIONS

This work investigated a *modified* version of the Ziv-Zakai bound, the so-called MZZB, that makes it feasible to find the bound in the presence of nuisance parameters, such as the chips of a random ranging code. The MZZB was applied to BPSK and BOC modulations, and allowed to highlight the impact on time estimation of signal autocorrelation side lobes. The related performance loss was shown to be strictly related to the number and the amplitude of the lobes. The MZZB proved also expedient to estimate the minimum C/N_0 thresholds that ensure safe acquisition of the Galileo SIS under a certain probability of detection, and to keep signal tracking.

ACKNOWLEDGEMENT. The content of the present article reflects solely the authors view and by no means represents the official ESA view or the one from ESA Galileo Project Office

6. REFERENCES

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Acquisition		Uncertainty NT_c		$P_d = 0.99$		$P_d = 0.995$		$P_d = 0.999$	
Modulation	**Chip Rate**	**Unc** $[T_c]$	**Unc [Km]**	σ **[m]**	C/N_0^{Acq}	σ **[m]**	C/N_0^{Acq}	σ **[m]**	C/N_0^{Acq}
BPSK(5)	5115000	5115	300	11.50	28.30	10.43	28.37	8.98	28.48
BPSK(10)	10230000	10230	300	5.75	28.76	5.22	28.83	4.49	28.94
BOC(1,1)	1023000	4092	1200	19.17	28.59	17,39	28.65	14.96	28.74
BOC(6,1)	1023000	4092	1200	2.50	29.64	2.27	29.70	1.95	29.89
BOC(15,10)	10230000	10230	300	1.15	29.20	1.04	29.25	0.90	29.34
BOC(15,2.5)	2557500	4092	480	1.0	29.52	0.91	29.57	0.781	29.65
BOC(10,5)	5115000	5115	300	1.64	29.13	1.49	29.19	1.28	29.28
Tracking			**AC mainlobe width**		**Uncertainty PIR**		σ		
Modulation	**Chip Rate**	**RX BW [MHz]**	**ACW** $[T_c]$	**ACW [m]**	**Unc** $[T_c]$	**Unc [m]**	σ $[T_c]$	σ **[m]**	C/N_0^{Tr}
BPSK(5)	5115000	40.92	2.0	117.30	1	58.65103	0.167	9.78	16.40
BPSK(10)	10230000	20.46	2.0	58.65	1	29.32551	0.167	4.89	17.27
BOC(1,1)	1023000	24.552	0.667	195.50	0.33	97.75	0.0556	16.29	16.40
BOC(6,1)	1023000	24.552	0.087	25.50	0.0435	12.75	0.00725	2.12	18.89
BOC(15,10)	10230000	51.15	0.4	11.73	0.2	5.87	0.0333	0.978	18.50
BOC(15,2.5)	2557500	24.552	0.087	10.2	0.0435	5.10	0.00725	0.85	23.06
BOC(10,5)	5115000	40.92	0.286	16.78	0.1429	8.379	0.0238	1.396	18.64

Table 1: Acquisition and tracking parameters.

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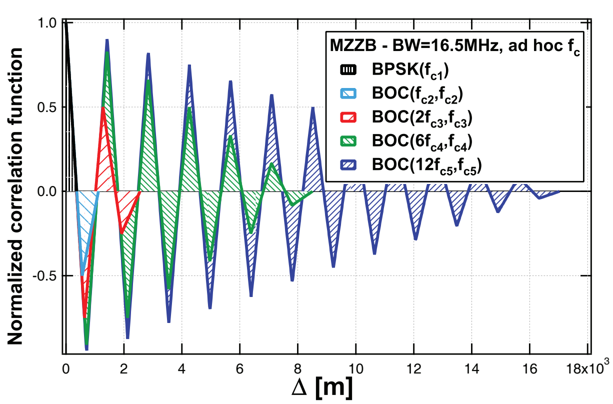

Fig. 1: Theoretical BOC correlation functions.

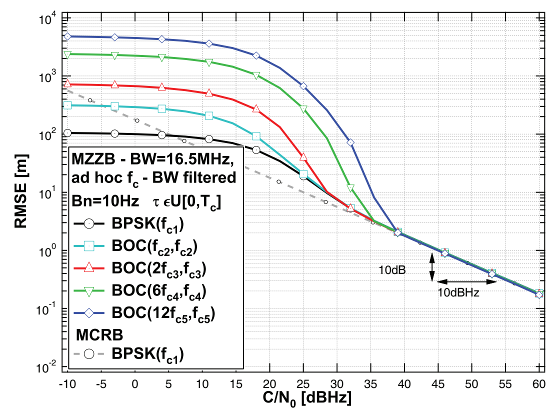

Fig. 2: Multi-peaks effect - filtered BOC modulations.

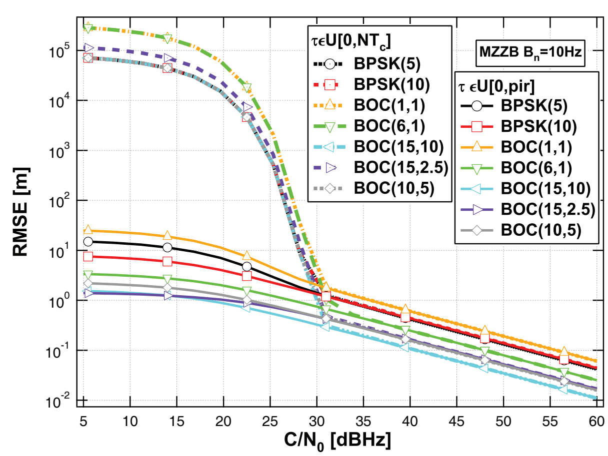

Fig. 3: Theoretical acquisition and tracking performance of Galileo GNSS by means of MZZB.