# A SUBSPACE-BASED METHOD FOR BLIND CFO ESTIMATION IN OFDM SYSTEMS

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*Abstract*—Recently, a blind subspace channel estimation algorithm in which using only a few received OFDM blocks is proposed. Based on this earlier work, we introduce a novel subspace-based blind carrier frequency offset (CFO) estimation method for OFDM systems. The CFO is obtained by solving the nullspace of the proposed rank-reduced matrix and the CFO estimate is given in closed form. Moreover, we do not make the assumption that the modulation symbols are white or constant-modulus. Simulation results show that the proposed method performs better than the existing methods.

*Index Terms*—Carrier frequency offset (CFO), orthogonal frequency division multiplexing (OFDM), blind method

## I. INTRODUCTION

Orthogonal Frequency division multiplexing (OFDM) modulation has been widely employed in modern transmission systems due to its ease of channel equalization, high spectrum efficiency, and flexible data rate. It is well known that OFDM systems are very sensitive to the carrier frequency offset (CFO). CFO will destroy the subcarrier orthogonality. Many CFO estimation algorithms have been proposed in the past. These methods can be generally divided into the data-aided and the blind schemes depending on whether or not a training sequence is used.

Blind CFO estimation has received a great attention in the last decade ([1]-[4], [6]) because of its bandwidth efficiency. Some of the methods are based on the constant-modulus (CM) constellations or on the knowledge of the second-order statistics of the transmitted symbols, *e.g.* the least-square based (LS-based) method in [2], the subspace method in [3], and the smoothing power spectrum (SPS) approach in [6]. Their performances degrade when a non-CM signal constellation is used. On the other hand, constellation-independent approaches were found in [1] and [4]. The redundant information within the cyclic prefix (CP) is exploited in [1]. But for channels with long impulse responses, their CFO estimate will be inaccurate. In [4], a CFO estimate was derived by minimizing the off-diagonal terms of the covariance matrix of the received signal. However, a very large amount of symbols is needed for the accurate sampled covariance matrix.

The system parameter called repetition index (denoted $Q$ throughout the following) was introduced in [5] for blind channel identification. By keeping the CP symbols, the authors proposed a structure such that the information within each received block can be utilized $Q$ times. As a result, the performance can be greatly improved when the number of received blocks is limited. Inspired by [5], a novel subspace-based blind CFO estimation algorithm is proposed in this paper. We first build a new matrix structure using $Q$, whose rank is invariant with CFO. Then we show that the rank of the proposed structure can be reduced by $Q-1$ after a simple matrix addition. By solving the nullspace of this matrix addition, the CFO estimate can be obtained. We do not assume that the modulation symbols are

constant-modulus or white. As shown in the simulation, our method performs better than the existing works.

The rest of this paper is organized as follows. Section II reviews the system model. The proposed matrix structure and the CFO estimation algorithm are derived in Section III. Simulation results and comparisons are shown in Section IV. Concluding remarks are given in Section V.

*Notations:* The scalar $j = \sqrt{-1}$ and $(\cdot)^*$ denotes complex conjugation. Column vectors (matrices) are indicated by lowercase (uppercase) boldfaced letters. $\mathbf{A}^T$, $\mathbf{A}^\dagger$, and $\mathbf{A}^\sharp$ denote the transpose, conjugate-transpose, and pseudoinverse of $\mathbf{A}$, respectively. The notation $\mathbf{W}_M$ denotes the $M \times M$ normalized DFT matrix whose $(k,l)$th element is $(1/\sqrt{M})e^{-j2\pi(k-1)(l-1)/M}$. We also adopt some notations from [5]. For a vector $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}^T$, we use $\mathcal{T}_n(\mathbf{a})$ to denote the $(m+n-1) \times n$ full-banded Toeplitz matrix

$$\mathcal{T}_n(\mathbf{a}) = \begin{bmatrix} a_1 & & \mathbf{0} \\ \vdots & \ddots & \\ a_m & & a_1 \\ & \ddots & \vdots \\ \mathbf{0} & & a_m \end{bmatrix}. \quad (1)$$

The sub-vector $[\mathbf{a}]_l^k$ of $\mathbf{a}$ is defined by $[\mathbf{a}]_l^k = \begin{bmatrix} a_k & a_{k+1} & \cdots & a_l \end{bmatrix}^T$ for $k \leq l$. Due to the special property of cyclic prefixes, this definition is extended to arbitrary pair of integers $k$ and $l$ satisfying $k \leq l$ by defining $a_i$ as $a_{(i-1 \bmod m)+1}$ for $i > m$ or $i < 1$. For example, if $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$, then $[\mathbf{a}]_4^{-1} = \begin{bmatrix} a_2 & a_3 & a_1 & a_2 & a_3 & a_1 \end{bmatrix}^T$.

## II. SYSTEM MODEL

In an OFDM system, the vector $\mathbf{s}_M(n) = \begin{bmatrix} s_1(n) & s_2(n) & \cdots & s_M(n) \end{bmatrix}^T$ is modulated by an $M \times M$ IDFT matrix $\mathbf{W}_M^\dagger$, producing $\mathbf{x}_M(n) = \begin{bmatrix} x_1(n) & x_2(n) & \cdots & x_M(n) \end{bmatrix}^T$. A cyclic prefix of length $L$ is added before the transmission. Let the transmitted signal be the $(M+L) \times 1$ vector:

$$\mathbf{x}(n) = \begin{bmatrix} \mathbf{x}_{\mathrm{cp}}(n) \\ \mathbf{x}_M(n) \end{bmatrix} = [\mathbf{x}_M(n)]_M^{-L+1}. \quad (2)$$

We assume that the channel $\mathbf{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_L \end{bmatrix}^T$. The received vector is $\mathbf{y}(n) = \begin{bmatrix} \mathbf{y}_{\mathrm{cp}}^T(n) & \mathbf{y}_M^T(n) \end{bmatrix}^T$, where the $M \times 1$ vector $\mathbf{y}_M(n)$ is

$$\mathbf{y}_M(n) = \mathbf{H}_{\mathrm{cir}}\mathbf{x}_M(n) + \mathbf{q}_M(n), \quad (3)$$

and $\mathbf{H}_{\mathrm{cir}}$ is the $M \times M$ circulant matrix whose first column is $\begin{bmatrix} h_0 & h_1 & \cdots & h_L & 0 & \cdots & 0 \end{bmatrix}^T$ and $\mathbf{q}_M(n)$ is the white noise component. The $L \times 1$ vector $\mathbf{y}_{\mathrm{cp}}(n)$ contains inter-block interference (IBI) and can be written as

$$\mathbf{y}_{\mathrm{cp}}(n) = \mathbf{H}_l\mathbf{x}_{\mathrm{cp}}(n) + \mathbf{H}_u\mathbf{x}_{\mathrm{cp}}(n-1) + \mathbf{q}_{\mathrm{cp}}(n), \quad (4)$$

where $\mathbf{H}_l$ is an $L \times L$ lower-triangular Toeplitz matrix whose first column is $\begin{bmatrix} h_0 & \cdots & h_{L-1} \end{bmatrix}^T$, $\mathbf{H}_u$ is an $L \times L$ upper-triangular Toeplitz matrix whose first row is $\begin{bmatrix} h_L & \cdots & h_1 \end{bmatrix}$, and $\mathbf{q}_{cp}(n)$ is another white noise component. Now suppose that the OFDM system suffers from CFO. Let $\Delta f$ be the unknown CFO in Hz and $T_s$ be the sampling period. Then the received vector becomes

$$\mathbf{z}(n) = \begin{bmatrix} \mathbf{z}_{cp}(n) \\ \mathbf{z}_M(n) \end{bmatrix} = c_\theta(n) \begin{bmatrix} e^{-j2\pi L\theta/M}\mathbf{E}_L(\theta)\mathbf{y}_{cp}(n) \\ \mathbf{E}_M(\theta)\mathbf{y}_M(n) \end{bmatrix}, \quad (5)$$

where $\theta = \Delta f M T_s$ is the normalized CFO parameter, $c_\theta(n) = e^{j2\pi(n(M+L)+L)\theta/M}$ represents the phase rotate accumulated from previous blocks, and $\mathbf{E}_k(\theta)$ is the $k \times k$ diagonal matrix:

$$\mathbf{E}_k(\theta) = \text{diag}(1, e^{j2\pi\theta/M}, \ldots, e^{j2\pi(k-1)\theta/M}).$$

## III. PROPOSED CFO ESTIMATION ALGORITHM

Below we will first propose a matrix structure using the CFO-free signal $\mathbf{y}(n)$. Then its relation to the CFO-corrupted signal $\mathbf{z}(n)$ is examined and a subspace-based blind CFO estimation algorithm is derived.

### A. Proposed structure

We define a $2(M+L) \times 1$ composite block vector

$$\bar{\mathbf{y}}(n) = \begin{bmatrix} \mathbf{y}(n-1) \\ \mathbf{y}(n) \end{bmatrix} \quad (6)$$

and the corresponding $Q$-repeated block matrix

$$\mathbf{Y}_Q(n) = \begin{bmatrix} \mathbf{0}_{(2M+L+Q-1)\times L} & \mathbf{I}_{2M+L+Q-1} \end{bmatrix} \mathcal{T}_Q(\bar{\mathbf{y}}(n)), \quad (7)$$

where the notation $\mathcal{T}_Q(\cdot)$ is defined in (1). From above, $\mathbf{Y}_Q(n)$ is defined as the last $2M+L+Q-1$ rows of $\mathcal{T}_Q(\bar{\mathbf{y}}(n))$. Below we will study the rank of $\mathbf{Y}_Q(n)$, which is useful for CFO estimation in Sec. 3.2.

It is shown in the Appendix that the modified channel equation can be written as

$$\mathbf{Y}_Q(n) = \mathbf{A}_Q \mathbf{X}_Q(n), \quad (8)$$

where $\mathbf{A}_Q$ is a $(2M+L+Q-1)\times(2M+3(Q-1))$ matrix in (31) and $\mathbf{X}_Q(n)$ is a $(2M+3(Q-1))\times Q$ composite signal matrix in (31). Define the matrix

$$\mathbf{Y}_{Q,\text{low}}(n) = \begin{bmatrix} \mathbf{0}_{(2M+L)\times Q} \\ \hline \begin{matrix} y_1(n) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ y_{Q-1}(n) & \cdots & y_1(n) & 0 \end{matrix} \end{bmatrix}, \quad (9)$$

where $y_i(n)$ is the $i$th entry of $\mathbf{y}_M(n)$. Adding $\mathbf{Y}_{Q,\text{low}}(n)$ to $\mathbf{Y}_Q(n)$, we get

$$\tilde{\mathbf{Y}}_Q(n) = \mathbf{Y}_Q(n) + \mathbf{Y}_{Q,\text{low}}(n) = \mathbf{B}_Q \tilde{\mathbf{X}}_Q(n), \quad (10)$$

where $\mathbf{B}_Q$ is a $(2M+L+Q-1)\times(2M+2(Q-1))$ matrix in (33) and $\tilde{\mathbf{X}}_Q(n)$ is the matrix consisting of the first $2M+2(Q-1)$ rows of $\mathbf{X}_Q(n)$. After collecting $J$ received blocks, we have[1]

$$\tilde{\mathbf{Y}}_Q^{(J)} = \mathbf{Y}_Q^{(J)} + \mathbf{Y}_{Q,\text{low}}^{(J)}, \quad (11)$$

where

$$\mathbf{Y}_Q^{(J)} = \begin{bmatrix} \mathbf{Y}_Q(1) & \mathbf{Y}_Q(2) & \cdots & \mathbf{Y}_Q(J-1) \end{bmatrix}, \quad (12)$$

$$\tilde{\mathbf{Y}}_Q^{(J)} = \begin{bmatrix} \tilde{\mathbf{Y}}_Q(1) & \tilde{\mathbf{Y}}_Q(2) & \cdots & \tilde{\mathbf{Y}}_Q(J-1) \end{bmatrix}, \quad (13)$$

$$\mathbf{Y}_{Q,\text{low}}^{(J)} = \begin{bmatrix} \mathbf{Y}_{Q,\text{low}}(1) & \mathbf{Y}_{Q,\text{low}}(2) & \cdots & \mathbf{Y}_{Q,\text{low}}(J-1) \end{bmatrix}. \quad (14)$$

[1]Due to (6), we can form only $J-1$ many $\mathbf{Y}_Q(n)$ for $J$ received blocks.

Using (8) and (10), it can be proved that the ranks of $\mathbf{Y}_Q^{(J)}$ and $\tilde{\mathbf{Y}}_Q^{(J)}$ satisfy the following theorem:

*Theorem 1:* Assume that (i) $J$ is large enough such that $\begin{bmatrix} \mathbf{X}_Q(1) & \cdots & \mathbf{X}_Q(J-1) \end{bmatrix}$ has full row rank $2M+3(Q-1)$, (ii) $h_0 \neq 0$ and $H(e^{j2\pi l/M}) \neq 0$ for $0 \leq l \leq M-1$, and (iii) $Q \leq L/2+1$. Then $\text{rank}(\mathbf{Y}_Q^{(J)}) - \text{rank}(\tilde{\mathbf{Y}}_Q^{(J)}) = Q-1$.

The proof is omitted due to the space limitation. Theorem 1 shows that by adding $\mathbf{Y}_{Q,\text{low}}^{(J)}$ to $\mathbf{Y}_Q^{(J)}$, we can reduce the rank of $\mathbf{Y}_Q^{(J)}$ by $Q-1$. Below we will show how to exploit this rank-reduction criterion for CFO estimation.

### B. CFO estimator

When there is CFO, the received vector is $\mathbf{z}(n)$, not $\mathbf{y}(n)$. Similar to (6), let us form the composite block $\bar{\mathbf{z}}(n) = \begin{bmatrix} \mathbf{z}^T(n-1) & \mathbf{z}^T(n) \end{bmatrix}^T$. Then it is easy to verify that for any nonnegative integer $Q$, we can write

$$\mathcal{T}_Q(\bar{\mathbf{z}}(n)) = c_\theta(n)e^{-j2\pi L\theta/M}\mathbf{E}_{2(M+L)+Q-1}(\theta)\mathcal{T}_Q(\bar{\mathbf{y}}(n))\mathbf{E}_Q(-\theta) \quad (15)$$

Define the following $(2M+L+Q-1)\times(J-1)Q$ Toeplitz-cascading matrix:

$$\mathbf{Z}_Q^{(J)} = \begin{bmatrix} \mathbf{Z}_Q(1) & \mathbf{Z}_Q(2) & \cdots & \mathbf{Z}_Q(J-1) \end{bmatrix}, \quad (16)$$

where $\mathbf{Z}_Q(n)$ is formulated as the structure in (6)-(7) using the CFO-corrupted vector $\bar{\mathbf{z}}(n)$. By combining (7) and (15), and using the fact that $c_\theta(n+1) = e^{j2\pi(M+L)\theta/M}c_\theta(n)$, we can rewrite (16) as

$$\mathbf{Z}_Q^{(J)} = c_\theta(n)\mathbf{E}_{2M+L+Q-1}(\theta)\mathbf{Y}_Q^{(J)}\left(\mathbf{E}_{J-1}((M+L)\theta) \otimes \mathbf{E}_Q(-\theta)\right), \quad (17)$$

where $\otimes$ denotes the Kronecker product and $\mathbf{Y}_Q^{(J)}$ is the CFO-free matrix in (12). Let us define another $(2M+L+Q-1)\times(J-1)Q$ matrix by replacing $\mathbf{Y}_Q^{(J)}$ in (17) with $\tilde{\mathbf{Y}}_Q^{(J)}$ defined in (13):

$$\tilde{\mathbf{Z}}_Q^{(J)} \triangleq c_\theta(n)\mathbf{E}_{2M+L+Q-1}(\theta)\tilde{\mathbf{Y}}_Q^{(J)}\left(\mathbf{E}_{J-1}((M+L)\theta) \otimes \mathbf{E}_Q(-\theta)\right). \quad (18)$$

From the formulation in (17) and (18), because both diagonal CFO matrices $\mathbf{E}_{2M+L+Q-1}(\theta)$ and $\left(\mathbf{E}_{J-1}((M+L)\theta) \otimes \mathbf{E}_Q(-\theta)\right)$ are invertible, we have $\text{rank}(\mathbf{Z}_Q^{(J)}) = \text{rank}(\mathbf{Y}_Q^{(J)})$ and $\text{rank}(\tilde{\mathbf{Z}}_Q^{(J)}) = \text{rank}(\tilde{\mathbf{Y}}_Q^{(J)})$, *i.e.* the rank of $\mathbf{Y}_Q^{(J)}$ and $\tilde{\mathbf{Y}}_Q^{(J)}$ are invariant with CFO. From Theorem 1, we obtain

$$\text{rank}(\mathbf{Z}_Q^{(J)}) - \text{rank}(\tilde{\mathbf{Z}}_Q^{(J)}) = Q-1. \quad (19)$$

Moreover, $\tilde{\mathbf{Z}}_Q^{(J)}$ can be expressed in terms of $\mathbf{Z}_Q^{(J)}$. By substituting (11) into (18) and using (5) and (17), it can be verified that

$$\tilde{\mathbf{Z}}_Q^{(J)} = \mathbf{Z}_Q^{(J)} + e^{j2\pi\theta}\mathbf{Z}_{Q,\text{low}}^{(J)}, \quad (20)$$

where $\mathbf{Z}_{Q,\text{low}}^{(J)}$ has the same structure as (14) with elements replaced by the CFO-corrupted symbols. From (19) and (20), we observe that the rank of $\mathbf{Z}_Q^{(J)}$ will decrease by $Q-1$ if we add $e^{j2\pi\theta}\mathbf{Z}_{Q,\text{low}}^{(J)}$ to $\mathbf{Z}_Q^{(J)}$. This rank-reduction property will be exploited for finding the CFO $\theta$ below.

First note that there exist $Q-1$ linearly independent nonzero vectors $\mathbf{u}_k$ such that

$$\mathbf{u}_k^\dagger\left(\mathbf{Z}_Q^{(J)} + e^{j2\pi\theta}\mathbf{Z}_{Q,\text{low}}^{(J)}\right) = \mathbf{0}, \quad 1 \leq k \leq Q-1. \quad (21)$$

The above equations are identified as the singular pencil. Note that the values of $\mathbf{u}_k$ are irrelevant to the CFO estimation. By denoting $\mathbf{u}_k' = e^{-j2\pi\theta}\mathbf{u}_k$ for $1 \leq k \leq Q-1$, we thus transform (21) into the following canonical form:

$$\left(\mathbf{Z}_{Q,\text{low}}^{(J)\dagger} - \left(-e^{j2\pi\theta}\right)\mathbf{Z}_Q^{(J)\dagger}\right)\mathbf{u}_k' = \mathbf{0}, \quad 1 \leq k \leq Q-1. \quad (22)$$

The problem of finding $e^{j2\pi\theta}$ in the pencil is the generalized eigenvalue problem. Since $\mathbf{Z}_{Q,\text{low}}^{(J)\dagger}$ and $\mathbf{Z}_Q^{(J)\dagger}$ are not square matrices, the algorithms for solving the generalized eigenvalues can not be directly applied. To overcome this issue, we multiply (22) by the pseudoinverse of $\mathbf{Z}_Q^{(J)\dagger}$, resulting

$$\left[\left(\mathbf{Z}_Q^{(J)\dagger}\right)^\sharp \mathbf{Z}_{Q,\text{low}}^{(J)\dagger} - \left(-e^{j2\pi\theta}\right)\mathbf{I}_{2M+Q-1+L}\right]\mathbf{u}_k' \approx \mathbf{0}, \qquad (23)$$

where we use an approximation because $\left(\mathbf{Z}_Q^{(J)\dagger}\right)^\sharp \mathbf{Z}_Q^{(J)\dagger} = \mathbf{I}_{2M+Q-1+L}$ holds only when $Q \geq L/2+1$ (see Theorem 1 and the dimensions of $\mathbf{A}_Q$). When $Q < L/2+1$, $\left(\mathbf{Z}_Q^{(J)\dagger}\right)^\sharp \mathbf{Z}_Q^{(J)\dagger}$ approaches $\mathbf{I}_{2M+Q-1+L}$ in the least square sense. Consequently, solving the CFO is now degenerated to the ordinary eigenvalue problem. Also note from the structures in (9) and (14) that $\mathbf{Z}_{Q,\text{low}}^{(J)\dagger}$ has only $Q-1$ nonzero columns, which means solving $Q-1$ nonzero eigenvalues for each equation in (23) is equivalent to solving the eigenvalues of the submatrix

$$\mathbf{Z}_1 \triangleq \text{the lower right } (Q-1)\times(Q-1) \text{ submatrix of } \left(\mathbf{Z}_Q^{(J)\dagger}\right)^\sharp \mathbf{Z}_{Q,\text{low}}^{(J)\dagger}. \qquad (24)$$

That is, $-e^{j2\pi\theta}$ is the eigenvalue of $\mathbf{Z}_1$ with multiplicity of $Q-1$. Finally, since the sum of eigenvalues is equal to the trace of a matrix, the estimate of the CFO $\theta$ is obtained by

$$\hat{\theta} = \frac{1}{2\pi}\arg\left\{-\left(\text{trace}(\mathbf{Z}_1)\right)\right\}. \qquad (25)$$

The range of CFO estimate is $-0.5 < \hat{\theta} \leq 0.5$.

*C. The limitations on $Q$ and $J$*

Since $\mathbf{Z}_1$ is $(Q-1)\times(Q-1)$, $Q > 1$ is required. Also, from Theorem 1, we conclude that $Q$ should satisfy

$$2 \leq Q \leq \frac{L}{2}+1. \qquad (26)$$

A necessary condition for the $(2M+3(Q-1))\times(J-1)Q$ matrix $\begin{bmatrix}\mathbf{X}_Q'(1) & \cdots & \mathbf{X}_Q'(J-1)\end{bmatrix}$ having full row rank is

$$J \geq \frac{2M-3}{Q}+4. \qquad (27)$$

Combining (26) and (27), the constraint on $J$ can also be expressed as

$$J \geq \frac{4M-6}{L+2}+4. \qquad (28)$$

That is, (28) shows that the smallest $J$ for the proposed algorithm depends on the block size $M$ and the CP length $L$.

## IV. SIMULATION RESULTS AND DISCUSSIONS

Consider an OFDM system with $M = 64$ subcarriers and CP length $L = 16$. The multipath channel is modeled as a 16-order Gaussian random FIR channel whose taps are i.i.d. with variances $Ce^{-k/5}$ for $0 \leq k \leq 16$ and $C = 1/\sum_{k=0}^{16}e^{-k/5}$. The performance of the CFO estimator is evaluated by the mean square error (MSE), which is defined by $\text{MSE}(\theta) = E\{|\hat{\theta}-\theta|^2\}$. All results are averaged through 1000 Monte Carlo trials.

According to (26)-(28), we have $2 \leq Q \leq 9$, and $J \geq 18$. Figure 1 depicts the MSE of CFO estimate vs the SNR for various $J$ with $\theta = 0.3$, for $Q = 2$ (dotted lines) and $Q = 9$ (solid lines). The modulation symbols are QPSK. The settings of $(J,Q) = (20,2)$ and $(50,2)$ do not satisfy (27) and thus the performances are poor for these cases. In order to obtain good performance for $Q = 2$, large $J$ is required. The performances are significantly better when $Q = 9$. We also plot the MSE of CFO estimate vs the actual CFO
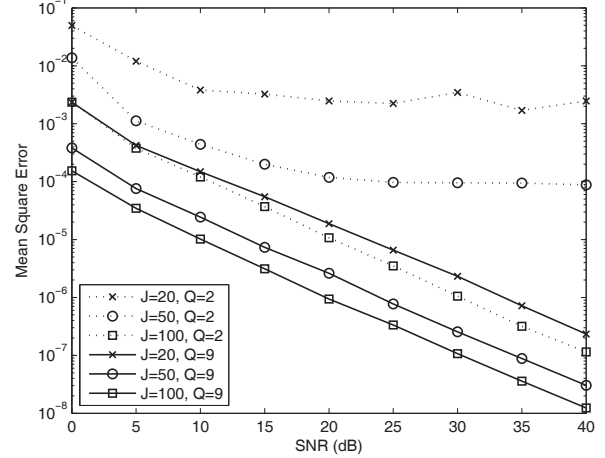


Fig. 1: MSE of CFO estimate vs SNR with $\theta = 0.3$.
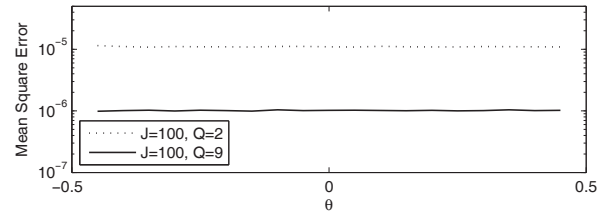


Fig. 2: MSE of CFO estimate vs CFO with SNR = 20dB.

with SNR = 20dB in Figure 2. The proposed method is robust to variety of CFO values.

Figures 3 and 4 compare the proposed method with existing blind CFO estimation methods for QPSK and 16-QAM constellations, respectively. In both figures, we set $\theta = 0.3$ and $J = 100$. In Figure 3, we first see that the 'Diagonality [4]' is inaccurate for $J = 100$ since it relies on the estimate of signal covariance matrix. The 'CP-based [1]' is the best for SNR<10dB, but it soon suffers from IBI and thus degrades at high SNR. The proposed method shows the best performance for SNR$\geq$10dB. Note that although the performance of 'Subspace [3]' is similar to the proposed method at high SNR, it degrades significantly when SNR is low. For 16-QAM, Figure 4 shows that all methods suffer from error floor for SNR>10dB except for the proposed method. The proposed method is significantly better than all other methods. Since we did not post any constraint on the data constellation, our method has the same performance for both QPSK and 16-QAM signals.

## V. CONCLUSIONS

In this paper, we proposed a subspace-based blind CFO estimation algorithm. We formulated a new matrix structure using the repetition index and showed that its rank can be reduced through a simple matrix addition. Based on this, the CFO estimate is obtained by solving the nullspace of the rank-reduced matrix. Simulation results demonstrated that the proposed method performs better than the existing works and it is robust to different signal constellations.

## APPENDIX

Since the first $L$ rows of $\mathcal{T}_Q(\bar{\mathbf{y}}(n))$ are removed in (7), we can see from (4), (6), and (7) that only the first $Q-1$ rows of $\mathbf{Y}_Q(n)$
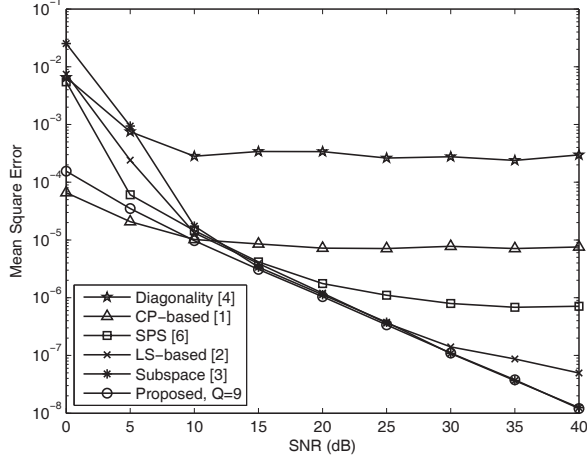
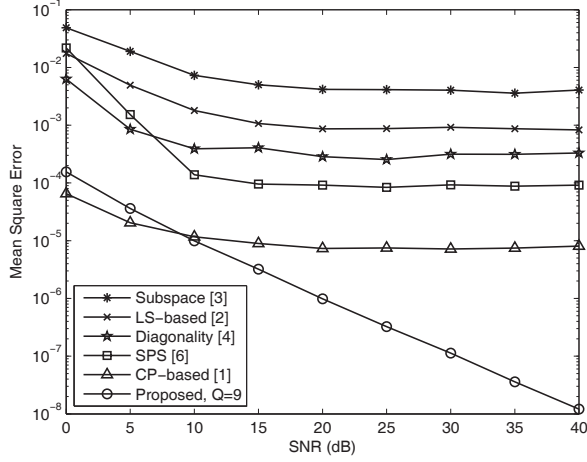Fig. 3: MSE of CFO estimate vs SNR for various methods with $\theta = 0.3$, $J = 100$, and QPSK constellation.



Fig. 4: MSE of CFO estimate vs SNR for various methods with $\theta = 0.3$, $J = 100$, and 16-QAM constellation.

are affected by $\mathbf{x}_{\mathrm{cp}}(n-2)$. By defining the $(Q-1) \times Q$ matrix

$$\mathbf{X}_Q^{\mathrm{lap}}(n-1)$$
$$= \begin{bmatrix} [\mathbf{x}_{\mathrm{cp}}(n-1)]_{Q-1}^1 & [\mathbf{x}_{\mathrm{cp}}(n-2)]_L^L & \cdots & [\mathbf{x}_{\mathrm{cp}}(n-2)]_L^{L-Q+2} \\ & [\mathbf{x}_{\mathrm{cp}}(n-1)]_{Q-2}^1 & & \end{bmatrix},$$

we can decompose $\mathbf{Y}_Q(n)$ as

$$\mathbf{Y}_Q(n) = \underbrace{\begin{bmatrix} \mathcal{H}_{M+L+Q-1} & \\ & \mathbf{H}_{\mathrm{cir2}}^{\mathrm{top}} \\ & & \mathbf{I}_{Q-1} \end{bmatrix}}_{\mathbf{H}_Q} \begin{bmatrix} \mathbf{X}_Q^{\mathrm{lap}}(n-1) \\ \mathbf{X}_Q^{\mathrm{cs}}(n-1) \\ \mathbf{X}_Q^{\mathrm{lap}}(n) \\ \mathbf{X}_Q^{\mathrm{cs2}}(n) \\ \mathbf{Y}_Q^{\mathrm{up}}(n) \end{bmatrix}, \quad (29)$$

where $\mathcal{H}_k$ is defined as a $k \times (L+k)$ upper-triangular Toeplitz matrix whose first row is $\begin{bmatrix} h_L & \cdots & h_0 & \mathbf{0} \end{bmatrix}$ for any $k > 0$, $\mathbf{H}_{\mathrm{cir2}}^{\mathrm{top}}$ denotes the first $M - Q + 1$ rows of the $M \times M$ circulant matrix $\mathbf{H}_{\mathrm{cir2}}$ whose first row is $\begin{bmatrix} h_L & \cdots & h_0 & \mathbf{0} \end{bmatrix}$, $\mathbf{X}_Q^{\mathrm{cs}}(n-1)$ is an $(M + L - Q + 1) \times Q$ matrix whose $i$th column is

$[\mathbf{x}(n-1)]_{M+L+i-1}^{Q+i-1}$, and $\mathbf{X}_Q^{\mathrm{cs2}}(n)$ is an $M \times Q$ matrix whose $i$th column is $[\mathbf{x}(n)]_{M+Q-1+i-1}^{Q+i-1}$. Note that the last $Q - 1$ rows of $\mathbf{Y}_Q(n)$, denoted as $\hat{\mathbf{Y}}_Q^{\mathrm{up}}(n)$, is an upper-triangular Toeplitz matrix whose first row is $\begin{bmatrix} 0 & y_M(n) & \cdots & y_{M-Q+2}(n) \end{bmatrix}$. In general, there is no common channel equation for $\mathbf{Y}_Q^{\mathrm{up}}(n)$ if the channel order is greater than 0, *i.e.* $\mathbf{Y}_Q^{\mathrm{up}}(n)$ cannot be decomposed as the channel and the signal matrices with inner dimensions smaller than that of $\mathbf{Y}_Q^{\mathrm{up}}(n)$. Because the columns of $\mathbf{X}_Q^{\mathrm{cs}}(n-1)$ are cyclic-shifts of each other. Using (2), we have

$$\mathbf{X}_Q^{\mathrm{cs}}(n-1) = \bar{\mathbf{W}}_M^\dagger \mathbf{S}_Q^{\mathrm{ps}}(n-1), \quad (30)$$

where $\bar{\mathbf{W}}_Q = \begin{bmatrix} \mathbf{w}_{M-L+Q} & \cdots & \mathbf{w}_M & \mathbf{W}_M \end{bmatrix}$ with $\mathbf{w}_i$ denotes the $i$th column of $\mathbf{W}_M$ and the $(k, l)$th element of the $M \times Q$ matrix $\mathbf{S}_Q^{\mathrm{ps}}(n-1)$ is $e^{-j2\pi(k-1)(l-1)/M} s_k(n-1)$. By substituting (30) into (29), we obtain

$$\mathbf{Y}_Q(n) = \mathbf{H}_Q \underbrace{\begin{bmatrix} \mathbf{I}_{Q-1} & & \\ & \bar{\mathbf{W}}_Q^\dagger & \\ & & \mathbf{I}_{M+2(Q-1)} \end{bmatrix}}_{\mathbf{A}_Q} \underbrace{\begin{bmatrix} \mathbf{X}_Q^{\mathrm{lap}}(n-1) \\ \mathbf{S}_Q^{\mathrm{ps}}(n-1) \\ \mathbf{X}_Q^{\mathrm{lap}}(n) \\ \mathbf{X}_Q^{\mathrm{cs2}}(n) \\ \mathbf{Y}_Q^{\mathrm{up}}(n) \end{bmatrix}}_{\mathbf{X}_Q(n)}.$$
$$(31)$$

Let us denote $\mathbf{H}_{\mathrm{cir2}}^{\mathrm{bot}}$ as the last $Q - 1$ rows of $\mathbf{H}_{\mathrm{cir2}}$, *i.e.* $\mathbf{H}_{\mathrm{cir2}} = \begin{bmatrix} (\mathbf{H}_{\mathrm{cir2}}^{\mathrm{top}})^T & (\mathbf{H}_{\mathrm{cir2}}^{\mathrm{bot}})^T \end{bmatrix}^T$. By comparing the product $\mathbf{H}_{\mathrm{cir2}}^{\mathrm{bot}} \mathbf{X}_Q^{\mathrm{cs2}}(n)$ to $\mathbf{Y}_Q^{\mathrm{up}}(n)$, we find that they are differed by

$$\mathbf{H}_{\mathrm{cir2}}^{\mathrm{bot}} \mathbf{X}_Q^{\mathrm{cs2}}(n) - \mathbf{Y}_Q^{\mathrm{up}}(n) = \begin{bmatrix} y_1(n) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ y_{Q-1}(n) & \cdots & y_1(n) & 0 \end{bmatrix}. \quad (32)$$

By substituting (32) into (31) and using the matrix defined in (9), we define another matrix

$$\tilde{\mathbf{Y}}_Q(n) = \mathbf{Y}_Q(n) + \mathbf{Y}_{Q,\mathrm{low}}(n)$$
$$= \begin{bmatrix} \mathcal{H}_{M+L+Q-1} & \\ & \mathbf{H}_{\mathrm{cir2}} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{I}_{Q-1} & & \\ & \bar{\mathbf{W}}_Q^\dagger & \\ & & \mathbf{I}_{M+Q-1} \end{bmatrix}}_{\mathbf{B}_Q} \tilde{\mathbf{X}}_Q(n),$$
$$(33)$$

where $\tilde{\mathbf{X}}_Q(n)$ is the matrix consisting of the first $2M + 2(Q-1)$ rows of $\mathbf{X}_Q(n)$.

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