MAP ESTIMATION OF THE INPUT OF AN OVERSAMPLED FILTER BANK FROM NOISY SUBBANDS BY BELIEF PROPAGATION

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ABSTRACT

Oversampled filter banks perform a subband decomposition with redundancy representation. This redundancy has been shown to be useful to combat channel impairments, when the subbands are transmitted over a wireless channel, as well as quantization noise. This paper describes an implementation of the maximum *a posteriori* and the minimum mean-square error (MMSE) estimators of the input signal from the noisy quantized subbands obtained at the output of some transmission channel. The relations between the input samples and the noisy subband samples are described using a factor graph. Belief propagation is then applied to get the posterior marginals of the input samples. The experimental results show that when the channel is clear, a linear MMSE estimate performs quite well but the proposed approaches perform significantly better than a reconstruction using the linear MMSE estimator when the channel is noisy: a gain in terms of channel SNR of more than 2 dB is observed.

1. INTRODUCTION

Recently a growing interest has been dedicated to communication systems performing jointly source and channel coding [1]. Such schemes cope better with unknown and changing channel characteristics than the classical tandem schemes. In this context, multirate systems and more particularly Oversampled Filter Banks (OFB) [2,3] are attractive solutions since they provide an overcomplete representation of the input signal by introducing some structured redundancy among the output subbands. OFB may then be seen as error-correcting codes in the real field as evidenced in [4–6]. OFB may correct transmission errors left by channel decoders and mitigate a part of the quantization noise [7]. Specific decoding techniques have been developed for OFB. Hypotheses testing and maximum likelihood estimation are considered in [4]. Kalman filtering is considered in [6]. A consistent reconstruction technique accounting for the bounded nature of the quantization noise is introduced in [8].

This work considers the maximum *a posteriori* (MAP) and MMSE estimation of the input of an OFB, when its output subbands are quantized and transmitted over a noisy channel. The computation of the exact MAP estimator is intractable in general, even for moderate-size input signals. When the OFB consists of finite impulse response filters, a factor graph may describe the relations between the input samples and the noisy subband samples. *Belief propagation* (BP) may then be used to compute the posterior probability distribution (PPD) of each entry of the input vector knowing the noisy subbands. This approach is inspired from [9, 10] where the problem of estimating some input vector $\mathbf{x} \in \mathbb{R}^n$ from noisy observations $\mathbf{y} \in \mathbb{R}^m$ of linear measurements $\mathbf{z} = \Phi\mathbf{x}$ of $\mathbf{x}$ has been addressed with BP. This problem is known as a *linear mixing estimation* problem. Via BP, the linear relations between the variables are exploited to update their PPD. This is done by passing *messages* on the variable states along a graph [11–13]. This message passing algorithm (MPA) operating in real field is similar to MPA for LDPC codes which work in finite fields [14]. The exact implementation of BP for dense mixing matrices is computationally very complex as it involves high-dimensional integrations for the PPD calculation. Implementations of BP based on Gaussian approximations have proven to be efficient and accurate as for example the Generalized Approximate Message Passing (GAMP) algorithm [10].

When the length of the impulse response of the filters involved in the OFB is not too large, the Φ matrix associated to the OFB may be quite sparse. Approximate implementation of the BP algorithm using discretized probability density functions becomes then tractable and has been considered here.

The rest of the paper is organized as follows. The considered communication scheme is presented in Section 2. The link between the input estimation of OFBs from noisy subbands and the linear mixing estimation problems is detailed in Section 3. The MAP and MMSE estimations using BP are then described in Section 4. Finally, experimental results are presented in Section 5 before drawing some conclusions in Section 6.

2. TRANSMISSION SCHEME

The communication scheme considered here is depicted in Figure 1. The random input vector $\mathbf{x} \in \mathbb{R}^n$ has i.i.d. components with prior

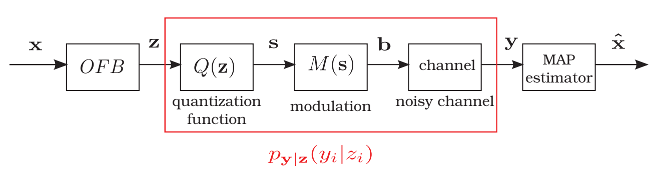

Fig. 1. Transmission scheme based on an OFB

probability density function (pdf) $p_X(x_j), j \in \{0, \dots, n-1\}$. This vector passes first through an OFB introducing a redundancy $\rho = m/n$. The resulting vector $\mathbf{z} \in \mathbb{R}^m$ is then quantized to get a vector of quantization indexes $\mathbf{s}$. The quantization function is denoted by $Q(\mathbf{z})$ and the modulation function by $M(\mathbf{s})$. The modulated sequence corresponding to $\mathbf{s}$ and denoted by $\mathbf{b}$ is transmitted over a memoryless channel. Finally the observation $\mathbf{y}$ of real (or complex) values is obtained at the output of this channel.

In the particular case of a scalar quantization with the same rate R for each subband sample and a BPSK modulation, each quantized index $s_i, i \in \{0, \ldots, m-1\}$ of $\mathbf{s}$ is represented by a binary sequence $\mathbf{b}_i$ of R elements and the observation $\mathbf{y} \in \mathbb{R}^{m \times R}$ is formed by m vectors $\mathbf{y}_i \in \mathbb{R}^R$ representing the components z_i of $\mathbf{z}$. The considered problems consist in the evaluation of the MAP estimate of $\mathbf{x}$:

$$\widehat{\mathbf{x}}_{\text{MAP}} = \arg \max_{\mathbf{x} \in \mathbb{R}^n} p(\mathbf{x}|\mathbf{y}) \tag{1}$$

and of its MMSE estimate:

$$\widehat{\mathbf{x}}_{\text{MMSE}} = \int_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x} p(\mathbf{x}|\mathbf{y}) d\mathbf{x} \tag{2}$$

The evaluation of $\widehat{\mathbf{x}}_{\text{MAP}}$ and $\widehat{\mathbf{x}}_{\text{MMSE}}$ is intractable in practice when considering high-dimensional input vectors. We show in the next section that these problems may be addressed in the framework of linear mixing problems for which a suboptimal solution can be evaluated using the BP algorithm. More precisely, one is able to estimate marginal PPDs for each entry of $\mathbf{x}$.

3. LINEAR MIXING PROBLEM

3.1. General Scheme

A general linear mixing problem, see, *e.g.*, [9, 10], is presented in Figure 2. The vector $\mathbf{x}$ goes through an $m \times n$ matrix Φ:

$$\mathbf{z} = \Phi \mathbf{x} \tag{3}$$

The output vector $\mathbf{z}$ is then transmitted over a separable measurement channel characterized by its conditional probability $p_{\mathbf{Y}|Z}(\mathbf{y}_i|z_i)$ and delivering the measurements $\mathbf{y}$. Here, the quantization and modulation operations, assumed to be separable are incorporated into the measurement channel, see Figure 2. The difficulty in the estimation

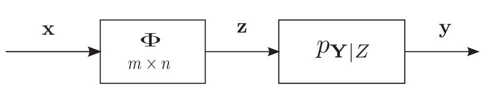

Fig. 2. General linear mixing estimation problem

of $\mathbf{x}$ knowing $\mathbf{y}$ is that Φ mixes $\mathbf{x}$ to get $\mathbf{z}$. Getting the PPD of each element $x_j, j \in \{0, \ldots, n-1\}$ or $z_i, i \in \{0, \ldots, m-1\}$ involves a high-dimensional integral which is difficult to evaluate. Such an estimation problem may be solved using BP, provided that a graph representing the dependencies between the variables is available. BP updates then the PPDs of these variables via a message passing procedure along the edges of this graph [11, 14].

3.2. Linear mixing performed by OFBs

An OFB is a filter bank whose number of output subbands is larger than the downsampling factor. These subbands form then a redundant representation of the input signal. A typical $M-$band OFB with a downsampling factor of $N \leqslant M$ such that $\rho = M/N$, is presented in Figure 3. This OFB is formed by M FIR analysis filters $\{\mathbf{h}_m\}_{m=0}^{M-1}$ with maximal length $N \times (L+1)$. The polyphase representation of this OFB is the matrix:

$$\mathbf{E}(z) = \sum_{\ell=0}^{L} \mathbf{E}_\ell z^{-\ell}$$

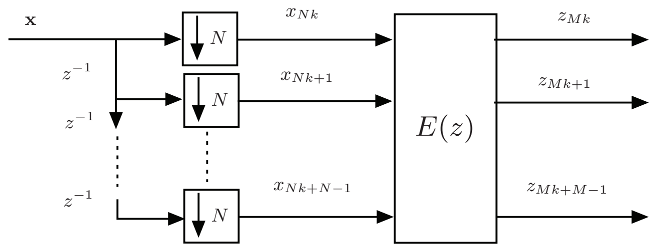

Fig. 3. Oversampled filter bank

where $\mathbf{E}_\ell$, $\ell = 0, \ldots, L$ is a sequence of $M \times N$ matrices which can be constructed from $\{\mathbf{h}_m\}_{m=0}^{M-1}$ [15]. The following polyphase notations are used for the vectors $\mathbf{x}$ and $\mathbf{z}$:

$$\mathbf{x} = \{x_0, \ldots, x_{N-1}, \ldots, x_{Nk}, \ldots, x_{Nk+N-1}, \ldots, x_{n-1}\}$$

$$\mathbf{z} = \{z_0, \ldots, z_{M-1}, \ldots, x_{Mk}, \ldots, x_{Mk+M-1}, \ldots, z_{m-1}\}$$

where k refers to the current instant. At each instant k the input of the OFB is the vector $\mathbf{x}^k = (x_{Nk}, \ldots, x_{Nk+N-1})^T$ and its output is the vector $\mathbf{z}^k = (z_{Mk}, \ldots, z_{Mk+M-1})^T$ obtained as follows:

$$\mathbf{z}^k = \sum_{\ell=0}^{L} \mathbf{E}_\ell \mathbf{x}^{k-\ell} = \mathbf{E}_{L:0} \mathbf{x}^{k-L:k}, \tag{4}$$

where $\mathbf{x}^{k-L:k} = \left(\left(\mathbf{x}^{k-L} \right)^T, \ldots, \left(\mathbf{x}^k \right)^T \right)^T$ contains all the input samples on which the OFB output at time k depends and $\mathbf{E}_{L:0} = (\mathbf{E}_L, \ldots, \mathbf{E}_0)$ is an $M \times (L+1)N$ matrix. One can then write the whole OFB operations as a linear mixing as presented in (3), where

$$\Phi = \begin{bmatrix} \mathbf{E}_L & \cdots & \mathbf{E}_1 & \mathbf{E}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_L & \cdots & \mathbf{E}_1 & \mathbf{E}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{E}_L & \cdots & \mathbf{E}_1 & \mathbf{E}_0 \end{bmatrix}$$

The MAP and MMSE estimation problems formulated in (1) and (2) can then be solved in an approximate way by using the marginal PPDs of $\mathbf{x}$ evaluated with the BP algorithm.

4. MAXIMUM A POSTERIORI ESTIMATION WITH BELIEF PROPAGATION

Belief propagation is an iterative message passing algorithm [14] that associates to a transform matrix Φ a *factor* or *Tanner* graph $\mathcal{G}_\Phi$. An example of such a graph is presented in Figure 4. The graph $\mathcal{G}_\Phi$ is a bipartite graph formed by two kinds of nodes: the variable nodes $j = 0, \ldots, n-1$ corresponding to the input variables x_j and the factor nodes $i = 0, \ldots, m-1$ corresponding to the output measurements $\mathbf{y}_i$ of $\mathbf{z}_i$. An edge between the node j and the node i means that the entry Φ_{ij} is non-zero and thus the variables x_j and $\mathbf{z}_i$ are linearly dependent. The set of variable nodes connected to the factor node i is denoted by $N_{\text{out}}(i)$. Similarly the set of factor nodes connected to the variable node j is denoted by $N_{\text{in}}(j)$. The different nodes talk to each other by sending messages (*beliefs*) on the states of each input variable x_j and the corresponding probabilities.

The steps of the BP algorithm in the real field are inspired by the ones presented by Rangan in [9]. They are summarized as follows:

1. *Initialization*:

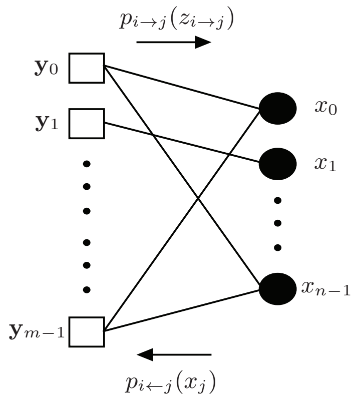

Fig. 4. Factor graph for the linear mixing estimation problem

(a) Set the current iteration $k = 1$.

(b) For each variable node j and factor node i forming an edge of $\mathcal{G}_\Phi$ set the messages to the initial distribution of the random variable X_j:

$$p^x_{i\leftarrow j}(k, x_j) = p^x_j(k, x_j) = p_{X_j}(x_j) \tag{5}$$

2. *Linear Mixing*:

(a) Assume that the random variables X_j are independent and that $X_j \sim p^x_{i\leftarrow j}(k, x_j)$

(b) Compute the distributions $p^z_{i\to j}(k, z_{i\to j})$ of the random variables:

$$Z_{i\to j} = \sum_{r\in N_{\text{out}}(i)\setminus j} \Phi_{ir} X_r. \tag{6}$$

3. *Output update*:
For each variable node j and factor node i forming an edge of $\mathcal{G}_\Phi$ compute the likelihood function

$$p^u_{i\to j}(k, u_i) = \int p_{\mathbf{Y}|Z}(\mathbf{y}_i | u_i + z_{i\to j}) \, p^z_{i\to j}(k, z_{i\to j}) dz_{i\to j}. \tag{7}$$

4. *Input update*:

(a) For each variable node j and factor node i forming an edge of $\mathcal{G}_\Phi$ update the message sent by j to i

$$p^x_{i\leftarrow j}(k{+}1, x_j) = \alpha p_{X_j}(x_j) \prod_{\ell\in N_{\text{in}}(j)\setminus i} p^u_{\ell\to j}(k, \Phi_{\ell j} x_j) \tag{8}$$

where α is a normalization constant obtained by imposing that $p^x_{i\leftarrow j}(k+1, x_j)$ sum up to 1.

(b) For each variable node j update the distribution

$$p^x_j(k+1, x_j) = \beta \, p_{X_j}(x_j) \prod_{\ell\in N_{in}(j)} p^u_{\ell\to j}(k, \Phi_{\ell j} x_j) \tag{9}$$

where β is a normalization constant obtained by imposing that $p^x_j(k+1, x_j)$ sum up to 1.

5. *Incrementation*:

(a) $k = k + 1$

(b) Return to Step 2 until a sufficient number of iterations is performed.

The message $p_{i\leftarrow j}(x_j)$ sent by j to i expresses the beliefs of the variable node j about the states in which X_j could be and their corresponding probabilities. The message $p_{i\to j}(z_{i\to j})$ is sent by the factor node i to the variable node j. It allows to compute the likelihood function $p^u_{i\to j}(u_i)$ that evaluates how likely the measurement $\mathbf{y}_i$ is obtained at node i when $X_j = x_j$.

When $\mathcal{G}_\Phi$ does not contain any cycle and after enough iterations, this series of message-passing is likely to converge to a consensus that determines the true marginal $p(x_j|\mathbf{y})$.

In order to estimate the input signal $\mathbf{x}$ of an OFB from its noisy received subbands $\mathbf{y}$, the direct implementation of this BP algorithm to perform the evaluation of the marginal PPD of each component of $\mathbf{x}$ is possible as the correspondant matrix Φ is relatively sparse.

5. EXPERIMENTAL RESULTS

In this section we present the results obtained with the MAP and MMSE estimators using the marginal PPDs of $\mathbf{x}$ provided by the BP algorithm of Section 4.

We have considered an input vector $\mathbf{x} \in \mathbb{R}^8$. The components of $\mathbf{x}$ are i.i.d. zero-mean Gaussian with variance $\sigma_x^2 = 1$. The OFB used is based on the Haar filters with $M = 6$, $N = 4$, and $L = 1$. The corresponding transform matrix Φ is

$$\Phi = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_0 & \mathbf{0}_{6\times 4} \\ \mathbf{0}_{6\times 4} & \mathbf{E}_1 & \mathbf{E}_0 \end{bmatrix}$$

where

$$\mathbf{E}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{E}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vector $\mathbf{z} \in \mathbb{R}^{12}$ obtained at the OFB output is quantized using a scalar quantization function $Q(z)$ with a rate $R = 4$ bits with a quantization step

$$\Delta = 2\sigma_x/(2^R - 1).$$

Quantized samples are then BPSK modulated and transmitted over an AWGN channel with an SNR between 0 dB and 13 dB. For each value of the SNR, the number of noise realizations has been set to 3000.

The MAP and MMSE estimators for $\mathbf{x}$ are compared to a scheme using a linear MMSE (LMMSE) estimator described in Figure 5. A

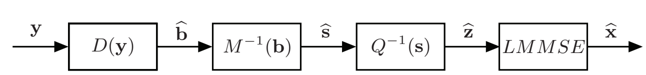

Fig. 5. OFB input estimator based on a LMMSE estimator

classical decoder $D(.)$ takes hard decisions on the received measurements $\mathbf{y}$. After demodulation and inverse quantization, the received vector $\widehat{\mathbf{z}}$ is obtained. The combined effect of the quantization noise and of the errors due to the channel, after inverse quantization, has been shown in [16] to be efficiently represented by a zero-mean Gaussian-Bernoulli-Gaussian noise $\mathbf{n}_{\text{qc}}$, uncorrelated with $\mathbf{x}$. The covariance matrix of $\mathbf{n}_{\text{qc}}$ is $\Gamma_{\text{qc}} = \sigma^2_{\text{qc}} \mathbf{I}_{m\times m}$. One has $\sigma^2_{\text{qc}} = \sigma^2_{\text{q}} + \sigma^2_{\text{c}}$, where $\sigma_q^2 = \Delta^2/12$ accounts for the quantization noise and σ_c^2 depends on the quantization and modulation schemes, as well as of the channel noise, see [16] for more details. The LMMSE estimate of $\mathbf{x}$ is then:

$$\widehat{\mathbf{x}}_{\text{LMMSE}} = \Phi^T \left(\left(\Phi\Phi^T + \sigma_{\text{qc}}^2/\sigma_x^2 \mathbf{I}_{m\times m} \right)^{-1} \right)^T \widehat{\mathbf{z}}. \qquad (10)$$

5.1. Performances of the proposed estimators

The BP algorithm described in Section 4 is implemented by considering probability mass functions approximating the continuous distributions. The range that has been considered for the input variables x_j is from -10 to 10. The number of points on which the probability distribution functions are evaluated has been set to 1024. The considered resolution is then of $20/1024$. The total number of iterations of the BP algorithm is equal to 20. At each iteration, the messages $p^x_{i\leftarrow j}(k, x_j)$ and $p^u_{i\rightarrow j}(k, u_i)$ are vectors of 1024 entries where the probability distribution is evaluated.

The experimental results that have been obtained are presented in Figure 6. One can see that the gain brought by the MAP and

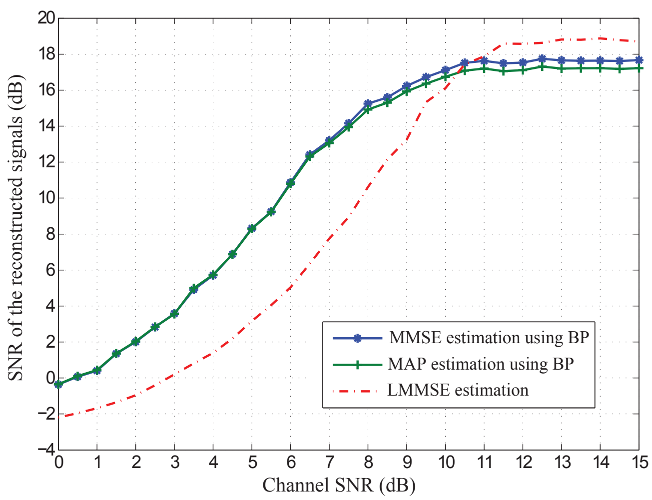

Fig. 6. The reconstruction SNR as a function of the channel SNR.

MMSE estimators using BP reaches more than 5 dB in terms of the reconstruction SNR for a channel SNR equal to 6 dB. For a channel SNR greater than 11 dB the LMMSE estimator performs better, its gain is about 1.5 dB in reconstruction SNR compared to the MMSE estimator and 2 dB compared to the MAP estimator. This is due to the fact that the proposed estimators involve marginal PPDs instead of the joint PPD of $\mathbf{x}$. Moreover, the BP may not converge to the exact marginal PPDs due to the presence of cycles in the graph representing the relation between the components of the OFB input signal and the noisy measurements of its subbands. Both effects do not appear when the impact of the noise due to the channel is significant.

6. CONCLUSION

In this work we have presented approximate implementations of a MAP estimator and a MMSE estimator involving marginal PPDs evaluated using a BP to recover the input signal of an OFB from noisy subbands. The experimental results show that when the channel is noisy, this approach performs better in terms of reconstruction SNR than classical reconstruction provided by a LMMSE estimator.

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