DEGREES OF FREEDOM REGION IN MULTI-CELL RANDOM BEAMFORMING

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ABSTRACT

Random beamforming (RBF) is a practically favorable transmission scheme for multiuser multi-antenna downlink systems. This paper studies the asymptotic rates achievable with RBF in a multi-cell system subject to the inter-cell interference, by assuming that the number of users in each cell scales in a given order with the signal-to-noise ratio (SNR). In particular, we investigate the achievable degrees of freedom (DoF) for the sum-rate in each cell when the SNR goes to infinity, and characterize the achievable DoF region for all the cells. Our results show that to achieve the DoF-optimal transmission in a multi-cell system with RBF, the numbers of transmit beams in all the cells need to be assigned in a collaborative manner based on the user densities.

Index Terms— Random beamforming (RBF), degrees of freedom (DoF), multi-cell system.

1. INTRODUCTION

In a landmark paper [1], Viswanath *et al.* introduced the single-beam Opportunistic Beamforming (OBF) scheme for a multiuser downlink system, which exploits the multiuser diversity gain and requires only partial channel knowledge at the base station (BS). One major improvement over OBF was later proposed in [2] to capture the additional spatial multiplexing gain by transmitting more than one random beams, thus referred to as Random Beamforming (RBF). The achievable sum-rate with RBF in a single cell has been shown in [2] to scale identically to that with the Dirty-Paper-Coding (DPC) scheme assuming perfect channel knowledge at the BS as the number of users goes to infinity, for any given signal-to-noise ratio (SNR).

Although substantial subsequent investigations and extensions for single-cell RBF have been pursuit, there is very few work on the performance of RBF in a more realistic multi-cell setup, where the inter-cell interference (ICI) becomes a critical issue. In [3], the authors proposed a collaborative transmission strategy whereby the BS utilizes perfect channel knowledge to serve one in-cell user with maximum ratio transmission, and employs RBF to support neighboring cells' users opportunistically. In addition, the sum-rate scaling law for the multi-cell system with RBF has been recently shown in [4] to be similar to the single-cell result given in [2], based on an approximation of the distribution of the signal-to-interference-plus-noise ratio (SINR).

However, characterization of the achievable rates for RBF at asymptotically high SNR is still missing for single- or multi-cell systems. This paper is thus motivated to study the achievable rates for RBF under the assumption that the number of users in each cell scales in a predefined order with the SNR, as the SNR goes to infinity. Thereby, we are able to investigate the achievable degrees of freedom (DoF) for the sum-rate in each cell with RBF, and furthermore characterize the achievable DoF region for all the cells. It is worth noting that the multi-cell downlink channel can be included in a more general interference channel setup due to the ICI, and there have been extensive studies on the achievable DoF of interference channels recently (see, e.g., [5] and [6]). However, the achievability of the DoF in the above work is known to rely on a transmission scheme so-called *Interference Alignment* (IA), which requires the perfect channel knowledge for all intra-cell and inter-cell links. In contrast, our study focuses on the achievable DoF of a multi-cell system with only partial channel knowledge at the BS due to the use of RBF. In addition, the IA scheme is in general applicable with a small number of users in each cell [5], while our work considers the case with large number of users per cell that even scales with the SNR.

The main results of this paper are summarized as follows: We first derive a closed-form expression of the optimal achievable DoF for RBF in a single cell setup, when the number of users scales in certain order of the SNR. It is revealed that the number of transmit beams with RBF needs to be properly assigned based on the user density in order to achieve the optimal DoF. We then seek to obtain a general characterization of the DoF region for the multi-cell downlink system with RBF. Our results suggest that a collaboration between different BSs in assigning their respective number of beams could achieve better performance in terms of sum-rate and fairness.

2. SYSTEM MODEL

We consider a multi-cell system consisting of C cells, each of which has a BS with N_T antennas to coordinate the transmission with K_c single-antenna mobile stations (MSs), $K_c \geq 1$ and $c = 1, \cdots, C$. In the c-th cell, the c-th BS transmits $M_c \leq N_T$ orthonormal beams and selects M_c from K_c users for transmission at each time. The received signal of user k in the c-th cell is given by

$$y_k^{(c)} = \boldsymbol{h}_k^{(c,c)} \sum_{m=1}^{M_c} \boldsymbol{\phi}_m^{(c)} s_m^{(c)} + \sum_{l=1,\ l\neq c}^{C} \sqrt{\gamma_{l,c}} \boldsymbol{h}_k^{(l,c)} \sum_{m=1}^{M_l} \boldsymbol{\phi}_m^{(l)} s_m^{(l)} + n_k^{(c)}, \quad (1)$$

where $\boldsymbol{h}_k^{(l,c)} \in \mathbb{C}^{1\times M_l}$ is the channel vector between the l-th BS and the k-th user of the c-th cell, and it is assumed that all elements of $\boldsymbol{h}_k^{(l,c)}$ are independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian random variables with zero mean and unit variance, denoted by $\mathcal{CN}(0,1)$; $\gamma_{l,c} \leq 1$ stands for the signal attenuation from the l-th BS to all the users of the c-th cell, $l \neq c$; $\boldsymbol{\phi}_m^{(c)} \in \mathbb{C}^{M_c \times 1}$ and $s_m^{(c)}$ are the m-th randomly generated unit-norm beamforming vector and transmitted data symbol from the c-th BS, respectively. It is assumed that each BS has the total sum power, P_T, i.e., $\boldsymbol{Tr}\left(\mathbb{E}[\boldsymbol{s}_c \boldsymbol{s}_c^H]\right) \leq P_T$, where $\boldsymbol{s}_c = [s_1^{(c)}, \cdots, s_{M_c}^{(c)}]^T$. It is also assumed that the background noise $n_k^{(c)} \sim \mathcal{CN}(0, \sigma^2)$, $\forall k, c$.

In this study, we apply the conventional RBF [2] in each cell. In the training phase, the c-th BS generates M_c orthonormal beams, $\boldsymbol{\phi}_1^{(c)}, \cdots, \boldsymbol{\phi}_{M_c}^{(c)}$ and uses them to broadcast the training signals to all users in the c-th cell. The total power of each BS is assumed to be distributed equally over M_c beams. Each user in the c-th cell measures the SINR values for all M_c beams, which are shown in (2) below, and feeds them back to the corresponding BS. Note that all the BSs implement the above protocol at the same time and thus result in ICI.

$$\mathrm{SINR}_{k,m}^{(c)} = \frac{P_T}{M_c} \left| \boldsymbol{h}_k^{(c,c)} \boldsymbol{\phi}_m^{(c)} \right|^2 \Bigg/ \left\{ \sigma^2 + \frac{P_T}{M_c} \sum_{i=1, i\neq m}^{M_c} \left| \boldsymbol{h}_k^{(c,c)} \boldsymbol{\phi}_i^{(c)} \right|^2 + \sum_{l=1, l\neq c}^{M_l} \gamma_{l,c} \frac{P_T}{M_l} \sum_{i=1}^{M_l} \left| \boldsymbol{h}_k^{(l,c)} \boldsymbol{\phi}_i^{(l)} \right|^2 \right\}, \quad (2)$$

where $m = 1, \cdots, M_c$. The c-th BS schedules transmission at each time by assigning its m-th beam to the user with the highest SINR, i.e.,

$$k_m^{(c)} = \arg \max_{k=1,\cdots,K_c} \mathrm{SINR}_{k,m}^{(c)}.$$

Then, the achievable sum rate in bits per complex dimension of the c-th cell is given by

$$R_{\mathrm{sum}}^{(c)} = \mathbb{E}\left[\sum_{m=1}^{M_c} \log_2 \left(1 + \mathrm{SINR}_{k_m^{(c)}, m}^{(c)} \right) \right] \overset{(a)}{\approx} M_c \mathbb{E}\left[\log_2 \left(1 + \mathrm{SINR}_{k_1^{(c)}, 1}^{(c)} \right) \right], \quad (3)$$

where (a) holds due to the negligibly small probability of assigning multiple beams to one user, and the homogeneous channel distribution assumed for each cell. Similar to [5], we define the DoF region for the multi-cell RBF scheme as

$$\mathcal{D} = \Bigg\{ (d_1, d_2, \cdots, d_C) \in \mathbb{R}_+^C : \forall (\omega_1, \omega_2, \cdots, \omega_C) \in \mathbb{R}_+^C \sum_{c=1}^{C} \omega_c d_c \leq \lim_{\rho\to\infty} \left[\max_{0\leq M_c \leq N_T} \sum_{c=1}^{C} \omega_c \frac{R_{\mathrm{sum}}^{(c)}}{\log_2 \rho} \right] \Bigg\}, \quad (4)$$

where $\rho = P_T/\sigma^2$ denotes the SNR.

In this paper, we are interested in characterizing the above DoF region for the regime with very large number of users in each cell. Specifically, we assume that the number of users in each cell scales with ρ in the order of ρ^{α_c}, with $\alpha_c \geq 0$, denoted by $K_c = \Theta(\rho^{\alpha_c})$, i.e., $K_c/\rho^{\alpha_c} \to a$ as $\rho \to \infty$ with a being a positive constant.

3. SINGLE-CELL CASE

In this section, we investigate the achievable DoF of RBF in a single cell without the ICI. In this case, the DoF region in (4) collapses to a line, bounded by 0 and d^*, where d^* denotes the maximum DoF available for a single cell. For brevity, we drop the cell index c in this section. Thus, (2) reduces to

$$\mathrm{SINR}_{k,m} = \frac{\frac{P_T}{M} |\boldsymbol{h}_k \boldsymbol{\phi}_m|^2}{\sigma^2 + \frac{P_T}{M} \sum_{i=1, i\neq m}^{M} |\boldsymbol{h}_k \boldsymbol{\phi}_i|^2}. \quad (5)$$

The CDF of $\mathrm{SINR}_{k,m}, \forall k, m$ can be expressed as [2]

$$F_S(s) = 1 - \frac{e^{-s/\eta}}{(s+1)^{M-1}}, \quad (6)$$

where $\eta = P_T/(M\sigma^2)$ is the SNR per beam. We define the achievable DoF for a given pair of α and M as

$$d(\alpha, M) = \lim_{\rho\to\infty} \frac{R_{\mathrm{sum}}}{\log_2 \rho} = \lim_{\eta\to\infty} \frac{R_{\mathrm{sum}}}{\log_2 \eta}. \quad (7)$$

Therefore, for a given α, the maximum DoF is obtained as $d^*(\alpha) = \max_{M\in\{0,\cdots,N_T\}} d(\alpha, M)$.

Lemma 3.1. Assume that $K = \Theta(\rho^\alpha)$, the DoF of single-cell RBF with M transmit beams is

$$d(\alpha, M) = \begin{cases} \frac{\alpha M}{M-1}, & 0 \le \alpha \le M-1, \\ M, & \alpha > M-1. \end{cases} \tag{8}$$

Proof. Let $R_{k,1} = \log_2(1 + \mathrm{SINR}_{k,1})$. To show (8), it is sufficient to show (9) and (10) as follows:

$$\Pr\left\{ \frac{\alpha}{M-1}\log_2\eta + \log_2\log\eta \ge \max_{k=1,\cdots,K} R_{k,1} \right.$$
$$\left. \ge \frac{\alpha}{M-1}\log_2\eta - \log_2\log\eta \right\} \xrightarrow{\eta\to\infty} 1, \text{ if } 0 \le \alpha \le M-1, \tag{9}$$

$$\Pr\Bigg\{ \log_2\eta + \log_2\log\eta + \log_2\alpha \ge \max_{k=1,\cdots,K} R_{k,1}$$
$$\ge \log_2\eta + \log_2\log\eta + \log_2\beta \Bigg\} \xrightarrow{\eta\to\infty} 1, \text{ if } \alpha > M-1, \tag{10}$$

where, β is a constant, $0 < \beta < \alpha$. Please refer to [7] for a detailed proof. □

Theorem 3.1. For a single-cell RBF system where the BS has N_T antennas and the user density coefficient is α, the maximum achievable DoF and corresponding optimal number of transmit beams are[1]

$$d^*(\alpha) = \begin{cases} \lfloor\alpha\rfloor + 1, & \alpha \le N_T - 1, 1 \ge \{\alpha\}(\lfloor\alpha\rfloor + 2), \\ \frac{\alpha(\lfloor\alpha\rfloor+2)}{\lfloor\alpha\rfloor+1}, & \alpha \le N_T - 1, \{\alpha\}(\lfloor\alpha\rfloor + 2) > 1, \\ N_T, & \alpha > N_T - 1. \end{cases} \tag{11}$$

$$M^*(\alpha) = \begin{cases} \lfloor\alpha\rfloor + 1, & \alpha \le N_T - 1, 1 \ge \{\alpha\}(\lfloor\alpha\rfloor + 2), \\ \lfloor\alpha\rfloor + 2, & \alpha \le N_T - 1, \{\alpha\}(\lfloor\alpha\rfloor + 2) > 1, \\ N_T, & \alpha > N_T - 1. \end{cases} \tag{12}$$

In Fig. 1, we use Monte-Carlo simulations to confirm the validity of Lemma 3.1. It is observed that the sum-rate scaling law for single-cell RBF is quite accurate, even for small values of SNR ρ. We notice that the sum rate for $M = 2$ is higher than that of $M = 4$. This is because with $N_T = 4$ and $\alpha = 1$, the optimal number of beams to achieve $d^* = 2$ is $M^* = 2$. In Fig. 2, we demonstrate the dependence of the optimal single-cell DoF and number of transmit beams on the user density α. We remark that in a cell where the user density is small, e.g., $\alpha < 1$, we only need to transmit one or two beams to maximize the sum-rate, which is consistent with observations based on numerical results in existing literature. To obtain the maximum available DoF $d^* = 4$ with $M^* = N_T = 4$, the user density coefficient needs to be at least 3.

[1] The notations $\lfloor\alpha\rfloor$ and $\{\alpha\}$ denote the integer and fractional parts of α, respectively.

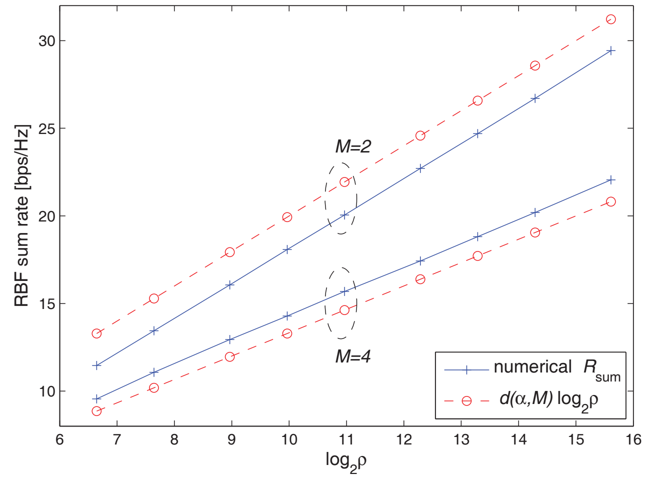

Fig. 1. Comparison of the numerical sum rate and the scaling law $d(\alpha, M)\log_2\rho$, with $N_T = 4$, $\alpha = 1$, and $K = \lfloor\rho^\alpha\rfloor$.

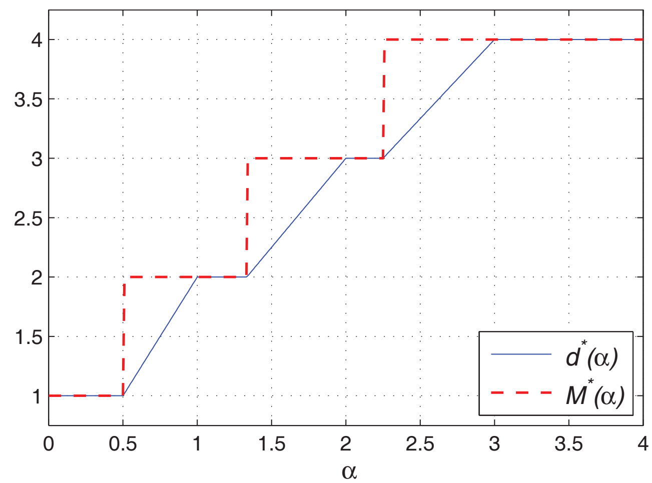

Fig. 2. The maximum DoF $d^*(\alpha)$ and optimal number of beams $M^*(\alpha)$ with $N_T = 4$.

4. MULTI-CELL CASE

In this section, we study the DoF region (4) for the general multi-cell case. First, we need to derive the CDF of $\mathrm{SINR}_{k,m}^{(c)}$ given in (2). However, extending the derivations for the SINR CDF from single-cell to multi-cell possesses some challenges. To the best of our knowledge, no closed-form expression is available, while some approximated expressions have been obtained [4]. Thus, we first show the following lemma on the SINR CDF in the multi-cell case

Lemma 4.1. The CDF of $\mathrm{SINR}_{k,m}^{(c)}$, $\forall k, m$, is

$$F_S^{(c)}(s) = 1 - \frac{e^{-s/\eta_c}}{(s+1)^{M_c-1}\prod_{l=1,l\ne c}^{C}\left(\frac{\mu_{l,c}}{\eta_c}s + 1\right)^{M_l}}. \tag{13}$$

where $\eta_c = P_T/(M_c\sigma^2)$ and $\mu_{l,c} = \gamma_{l,c}P_T/(M_l\sigma^2)$.

Proof. Please refer to [7]. □

We then define the DoF for the sum-rate of the c-th cell as $d_c(\alpha_c, \boldsymbol{M}) = \lim_{\rho\to\infty} \frac{R^{(c)}_{\text{sum}}}{\log_2 \rho}$, where $\boldsymbol{M}$=$[M_1,\cdots,M_C]^T$, and obtain the following lemma

Lemma 4.2. Assume that $K_c = \Theta(\rho^{\alpha_c})$, $c = 1,\cdots,C$, $d_c(\alpha_c, \boldsymbol{M})$ is given by

$$d_c(\alpha_c, \boldsymbol{M}) = \begin{cases} \frac{\alpha_c M_c}{\sum_{c=1}^{C} M_c - 1}, & 0 \leq \alpha_c \leq \sum_{c=1}^{C} M_c - 1, \\ M_c, & \alpha_c > \sum_{c=1}^{C} M_c - 1. \end{cases} \tag{14}$$

Proof. The proof uses Lemma 4.1 and similar arguments in the proof of Lemma 3.1 , and is thus omitted. □

Theorem 4.1. Let $\boldsymbol{\alpha}$=$[\alpha_1,\cdots,\alpha_C]^T$ and $\boldsymbol{d}(\boldsymbol{\alpha}, \boldsymbol{M})$= $\left[d_1(\alpha_1, \boldsymbol{M}),\cdots, d_C(\alpha_C, \boldsymbol{M})\right]^T$. The DoF region of a C-cell RBF system is characterized as

$$\mathcal{D}(\boldsymbol{\alpha}) = \boldsymbol{conv}\Big\{\boldsymbol{d}(\boldsymbol{\alpha}, \boldsymbol{M}), M_c \in \{0,\cdots,N_T\}, \\ c = 1,\cdots,C\Big\}. \tag{15}$$

Theorem 4.1 comes directly from Lemma 4.2, and the definition of the DoF region in (4). Esentially, we can obtain the DoF region $\mathcal{D}(\boldsymbol{\alpha})$ by taking a convex hull operation over all points $\boldsymbol{d}(\boldsymbol{\alpha}, \boldsymbol{M})$ with different values of $\boldsymbol{M}$.

In Fig. 3, we depict the DoF regions of a two-cell RBF system with different user density coefficients α_1 and α_2. The vertices of those regions can be obtained with appropriate numbers of beams M_1 and M_2, while time-sharing between these vertices yields the entire boundary. Only when $\alpha_1 \geq 7$ and $\alpha_2 \geq 7$, the maximum DoF pair, shown as point P_1, is achievable with $(M_1, M_2) = (4, 4)$. When the user densities in two cells are both small, e.g., $\alpha_1 = 2$ and $\alpha_2 = 2.5$, time-sharing between (M_1, M_2)=(3,0) and (0,4), i.e., two BSs transmitting alternately, is the optimal strategy. However, with moderate user density coefficients, e.g., $\alpha_1 = 5.7$ and $\alpha_2 = 4.7$, orthogonal transmission of the two BSs is suboptimal, as demonstrated by the dash-line. In general, the DoF trade-off between the two cells indicates that a cooperation between the two BSs is needed.

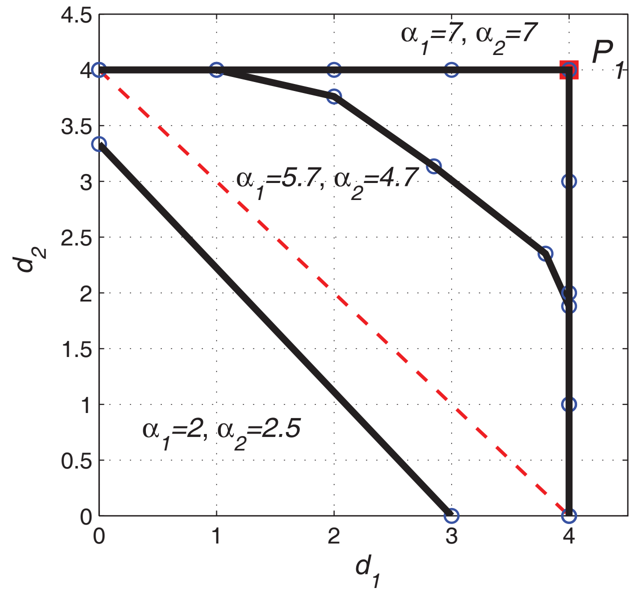

Fig. 3. Achievable DoF region of a two-cell RBF system, with $N_T = 4$.

5. CONCLUSION

This paper studies the asymptotic sum-rates with RBF in a multi-cell system for the regime of high SNR and large number of users per cell. The optimal DoF is derived for the single cell case in closed-form expressions, while a characterization for the achievable DoF region in the general multi-cell case is presented. Insights for cooperative BS transmission with RBF are drawn.

6. REFERENCES

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