LORENTZ-POSITIVE MAPS WITH APPLICATIONS TO ROBUST MISO DOWNLINK BEAMFORMING

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ABSTRACT

Consider a unicast downlink beamforming optimization problem with robust signal-to-interference-plus-noise ratio constraints to account for non-perfect channel state information at the base station. The convexity of the robust beamforming problem remains unknown. A slightly conservative version of the robust beamforming problem is thus studied herein as a compromise. It is in the form of a semi-infinite second-order cone program (SOCP), and more importantly, it possesses an equivalent and explicit convex reformulation, due to an linear matrix inequality description of the cone of Lorentz-positive maps. Hence the robust beamforming problem can be efficiently solved by an optimization solver. The simulation results show that the conservativeness of the robust form of semi-infinite SOCP is appropriate in terms of problem feasibility rate and the average transmission power.

Index Terms— Robust MISO downlink beamforming, semi-infinite SOCP, Lorentz-positive map, SDP, imperfect CSI.

1. INTRODUCTION

In a multiuser communication system, beamforming techniques provide a powerful approach to transmit signals and yield higher spectrum efficiency and larger downlink capacity for the system. The base station (BS) is equipped with multiple antennas, and the signals for different co-channel users are weighted; and the beamforming vectors (the weights) are optimized to carry the transmissions (see [1, 2]). A basic beamforming optimization problem formulation is to minimize the transmission power while providing an acceptable quality-of-service (QoS) to each user, as well as keeping tolerable interference around some other directions. In a uni-cast downlink, the beamforming design problem can be solved by convex optimization techniques, e.g., semidefinite programming (SDP) relaxation (see [1, 2, 3, 4, 5]), assuming that the perfect channel state information (CSI) is available at the BS when optimizing the beamvectors.

In practical situations, however, the available CSI contains errors caused by estimation, limited channel state feedback quantization or delays. Thus, the design of beamforming robust to CSI errors is of great practical interest and has been recently considered in a large number of references (e.g., for a multi-input single-output (MISO) system, see [1, 6, 7, 8] and references therein). However, most of the resulting robust downlink beamforming problems are inherently non-convex and, consequently, no global optimality of an efficient solution can be guaranteed theoretically. Nonetheless, in [8], sufficient conditions are presented to constrain some design parameters so that the robust beamforming problem becomes convex, and in [7], an ellipsoid method is proposed for a restricted (conservative) version of the robust problem.

In this paper, we revisit the robust beamforming problems of [8] and [7], and provide an equivalent and explicit convex reformulation for the robust optimization problem considered in [7]. By doing so, the derived reformulation appears elegant due to a profound result of linear matrix inequality (LMI) description for a robust second-order cone (SOC) constraint [9], and the implementation is much easier since one can make use of existing optimization solvers, e.g., CVX.

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2. PROBLEM FORMULATION

Consider a single-cell communication system with an N-antenna BS serving M decentralized single-antenna receivers (users). The signal transmitted at the BS is the vector $\boldsymbol{x}(t) = \sum_{m=1}^{M} \boldsymbol{w}_m s_m(t)$, where the information signal $s_m(t) \in \mathbb{C}$ intended for receiver m is temporally white with zero mean and unit variance, and $\boldsymbol{w}_m \in \mathbb{C}^N$ is the transmit beamforming vector for receiver m. The signal received by user m is given by

$$y_m(t) = \boldsymbol{h}_m^H \boldsymbol{x}(t) + n_m(t) \tag{1}$$

where $\boldsymbol{h}_m \in \mathbb{C}^N$ is the channel vector between BS and receiver m, and $n_m(t)$ is the additive zero-mean noise with the variance of σ_m^2. The received signal-to-interference-plus-noise ratio (SINR) of user m is given by

$$\mathrm{SINR}_m = \frac{\boldsymbol{w}_m^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_m}{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2}, \tag{2}$$

which is measure for QoS. The downlink beamforming problem with perfect CSI is formulated as (cf. [6]):

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} & \sum_{m=1}^{M} \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} & \dfrac{\boldsymbol{w}_m^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_m}{\sum_{i=1, i\neq m}^{M} \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2} \geq \gamma_m, \ \forall m, \end{array} \tag{3}$$

where $\gamma_m > 0$ is the minimal acceptable SINR for user m. It is known that Problem (3) amounts to a second-order cone program

(SOCP) as follows and thus can be solved efficiently:

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} & \sum_{m=1}^M \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} & \frac{1}{\sqrt{\gamma_m}} \Re(\boldsymbol{h}_m^H \boldsymbol{w}_m) \geq \\ & \sqrt{\sum_{i=1, i\neq m}^M \boldsymbol{w}_i^H \boldsymbol{h}_m \boldsymbol{h}_m^H \boldsymbol{w}_i + \sigma_m^2},\ m = 1,\dots,M. \end{array} \tag{4}$$

In the case that the CSI is not perfectly known at the transmitter, we model the m-th user's uncertain channel as $\boldsymbol{h}_m = \bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m$ where $\bar{\boldsymbol{h}}_m$ is the nominal channel vector and $\boldsymbol{\delta}_m$ is the perturbation (channel estimation error) norm-bounded by ϵ_m (namely $\|\boldsymbol{\delta}_m\| \leq \epsilon_m$). Accordingly, the worst-case beamforming design problem is the following robust optimization problem (cf. [8]):

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} & \sum_{m=1}^M \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} & \dfrac{\boldsymbol{w}_m^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_m}{\sum_{i=1, i\neq m}^M \boldsymbol{w}_i^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_i + \sigma_m^2} \\ & \geq \gamma_m,\ \forall \|\boldsymbol{\delta}_m\| \leq \epsilon_m,\ m = 1,\dots,M. \end{array} \tag{5}$$

It is shown in [8] that when the perturbation bounds ϵ_m are small to some extend, the conventional SDP relaxation (cf. (P_ϵ) in [8]) of (5) is tight. However, it remains to be understood whether (5) has an equivalent convex reformulation in a general case, notwithstanding existing numerical simulations showing that the SDP relaxation always gives a rank-one optimal solution with some certain data sets. Another interesting beamforming problem formulation is the robust extension of (4):

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m\}}{\text{minimize}} & \sum_{m=1}^M \boldsymbol{w}_m^H \boldsymbol{w}_m \\ \text{subject to} & \frac{1}{\sqrt{\gamma_m}} \Re((\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_m) \geq \\ & \sqrt{\sum_{i=1, i\neq m}^M \boldsymbol{w}_i^H (\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)(\bar{\boldsymbol{h}}_m + \boldsymbol{\delta}_m)^H \boldsymbol{w}_i + \sigma_m^2}, \\ & \forall \|\boldsymbol{\delta}_m\| \leq \epsilon_m,\ m = 1,\dots,M. \end{array} \tag{6}$$

Note that the feasible set of (5) contains that of (6), and thus the latter one is more conservative than the former. Observe that (6) is a semi-infinite SOCP and hence convex, which does not necessarily mean that it can be solved efficiently with an existing solver.

In [7], the authors present an iterative ellipsoidal method for (6) and show its polynomial-time computational complexity (see also [10]). In contrast, we herein show that Problem (6) possesses an equivalent convex LMI reformulation by resorting to a result in [9], and thus can be solved efficiently, and in practice can be implemented easily using an optimization solver, e.g., `SeDuMi` [11].

3. EQUIVALENT CONVEX REFORMULATION FOR THE ROBUST BEAMFORMING PROBLEM (6)

3.1. A Standard Form of Semi-Infinite SOCP for (6)

In this section, we will present an equivalent convex reformulation of Problem (6), resorting to a result on LMI description of a robust SOC constraint in [9] (rather than S-lemma). To start with, let us rewrite (6) into a problem with real-valued design variables. We denote the real and imaginary parts of $\boldsymbol{w}_m = \Re \boldsymbol{w}_m + j \Im \boldsymbol{w}_m \in \mathbb{C}^N$ as follows:

$$\boldsymbol{w}_{m1} = \Re \boldsymbol{w}_m,\ \boldsymbol{w}_{m2} = \Im \boldsymbol{w}_m,\ m = 1,\dots,M, \tag{7}$$

and clearly $\boldsymbol{w}_{m1} \in \mathbb{R}^N$ and $\boldsymbol{w}_{m2} \in \mathbb{R}^N$. Likewise, $\bar{\boldsymbol{h}}_{m1}$, $\bar{\boldsymbol{h}}_{m2}$, $\boldsymbol{\delta}_{m1}$, $\boldsymbol{\delta}_{m2}$ are defined such that $\bar{\boldsymbol{h}}_m = \bar{\boldsymbol{h}}_{m1} + j\bar{\boldsymbol{h}}_{m2}$ and $\boldsymbol{\delta}_m = \boldsymbol{\delta}_{m1} + j\boldsymbol{\delta}_{m2}$ respectively. Denote

$$\boldsymbol{W}_{-m,1} = [\boldsymbol{w}_{11}\ \cdots\ \boldsymbol{w}_{m-1,1}\ \boldsymbol{w}_{m+1,1}\ \cdots\ \boldsymbol{w}_{M1}], \tag{8}$$

and $\boldsymbol{W}_{-m,2}$ and $\boldsymbol{W}_{-m}$ are defined analogously. Therefore, by letting

$$\boldsymbol{C}_m^T(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) = \begin{bmatrix} \frac{1}{\sqrt{\gamma_m}} \boldsymbol{w}_{m1} & \boldsymbol{W}_{-m,1} & \boldsymbol{W}_{-m,2} & \boldsymbol{0} \\ \frac{1}{\sqrt{\gamma_m}} \boldsymbol{w}_{m2} & \boldsymbol{W}_{-m,2} & -\boldsymbol{W}_{-m,1} & \boldsymbol{0} \end{bmatrix} \tag{9}$$

of size $2N \times 2M$ and

$$\boldsymbol{c}_m^T(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) = [\bar{\boldsymbol{h}}_{m1}^T\ \bar{\boldsymbol{h}}_{m2}^T] \boldsymbol{C}_m^T(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) + [0\ \cdots\ 0\ \sigma_m] \tag{10}$$

of length $2M$, we express Problem (6) into the following real-valued optimization problem.

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m\}, t}{\text{minimize}} & t \\ \text{subject to} & [t\ \boldsymbol{w}_{11}^T\ \boldsymbol{w}_{12}^T\ \cdots\ \boldsymbol{w}_{M1}^T\ \boldsymbol{w}_{M2}^T]^T \in \mathbb{L}^{2MN+1}, \\ & \boldsymbol{C}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) \begin{bmatrix} \boldsymbol{\delta}_{m1} \\ \boldsymbol{\delta}_{m2} \end{bmatrix} + \boldsymbol{c}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) \in \mathbb{L}^{2M}, \\ & \forall \left\| \begin{bmatrix} \boldsymbol{\delta}_{m1} \\ \boldsymbol{\delta}_{m2} \end{bmatrix} \right\| \leq \epsilon_m, m = 1,\dots,M, \end{array} \tag{11}$$

where $\mathbb{L}^K$ represents the K-dimensional SOC:

$$\mathbb{L}^K = \{\boldsymbol{x} \in \mathbb{R}^K \mid x_1 \geq \sqrt{x_2^2 + \cdots + x_K^2}\}. \tag{12}$$

Note that $\boldsymbol{C}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m})$ and $\boldsymbol{c}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m})$ are affine with respect to (w.r.t.) $\{\boldsymbol{w}_m\}$, and that the optimal value of (11) is the square root of that of (6).

To simplify the notations, we write (9)-(10) respectively into $\boldsymbol{C}_m$ and $\boldsymbol{c}_m$ in what follows. Let us denote

$$\boldsymbol{B}_m = [\boldsymbol{c}_m\ \epsilon_m \boldsymbol{C}_m], \tag{13}$$

keeping in mind that $\boldsymbol{B}_m$ is indeed $\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m})$ affine w.r.t. the design variables. By the notation (13) and letting $\boldsymbol{\delta}'_m = [\alpha_m, \boldsymbol{\delta}_{m1}^T, \boldsymbol{\delta}_{m2}^T]^T \in \mathbb{R}^{2N+1}$, we have another equivalent reformulation of (11) (or (6)):

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m\}, t}{\text{minimize}} & t \\ \text{subject to} & [t\ \boldsymbol{w}_{11}^T\ \boldsymbol{w}_{12}^T\ \cdots\ \boldsymbol{w}_{M1}^T\ \boldsymbol{w}_{M2}^T]^T \in \mathbb{L}^{2MN+1}, \\ & \boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) \boldsymbol{\delta}'_m \in \mathbb{L}^{2M},\ \forall \boldsymbol{\delta}'_m \in \mathbb{L}^{2N+1}, \\ & m = 1,\dots,M. \end{array} \tag{14}$$

In order to solve (14), let us consider the second set of constraints, i.e., the robust SOC constraints. Define the set

$$\left\{ \boldsymbol{B}_m \in \mathbb{R}^{2M \times (2N+1)} \;\middle|\; \boldsymbol{B}_m \boldsymbol{y}_m \in \mathbb{L}^{2M},\ \forall \boldsymbol{y}_m \in \mathbb{L}^{2N+1} \right\}. \tag{15}$$

The set (15) contains linear maps (or matrices) that take $\mathbb{L}^{2N+1}$ to $\mathbb{L}^{2M}$.

Lemma 3.1 *The set* (15) *is equivalent to the matrices* $\boldsymbol{B}_m$ *being Lorentz-positive, i.e.,*

$$\boldsymbol{x}_m^T \boldsymbol{B}_m \boldsymbol{y}_m \geq 0,\ \forall \boldsymbol{x}_m \in \mathbb{L}^{2M},\ \forall \boldsymbol{y}_m \in \mathbb{L}^{2N+1}.$$

The set (15) of all Lorentz-positive matrices forms a closed convex cone, and the cone has an LMI description, as shown in [9, Theorem 5.6]. Having such a description of (15), we can claim that (14) has an equivalent convex (or linear conic program) reformulation. In order to present the theorem and reformulate it in an implementable way, we need some basic notions and facts to be introduced.

3.2. An LMI Characterization of the Cone of Lorentz-Positive Maps

Let $\mathcal{S}^N$ and $\mathcal{A}^N$ be the sets of all $N \times N$ symmetric matrices and the set of all $N \times N$ skew-symmetric matrices, respectively, and $\mathcal{L}_{L,K}$ stand for the $KL(K+1)(L+1)/4$-dimensional linear space of biquadratic forms (cf. [12, p. 1148]) $\mathcal{L}_{L,K} =$

$$\left\{ \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \cdots & \boldsymbol{M}_{1L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{L1} & \cdots & \boldsymbol{M}_{LL} \end{bmatrix} \in \mathcal{S}^{KL} \,\Big|\, \boldsymbol{M}_{ln} = \boldsymbol{M}_{ln}^T \right\}. \tag{16}$$

It is a subspace of $\mathcal{S}^{KL}$, and the orthogonal complement of it within $\mathcal{S}^{KL}$ is clearly the $KL(K-1)(L-1)/4$-dimensional space $\mathcal{L}_{L,K}^{\perp} =$

$$\left\{ \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \cdots & \boldsymbol{M}_{1L} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{L1} & \cdots & \boldsymbol{M}_{LL} \end{bmatrix} \in \mathcal{S}^{KL} \,\Big|\, \boldsymbol{M}_{ln} = -\boldsymbol{M}_{ln}^T \right\}. \tag{17}$$

That is, $\mathcal{S}^{KL} = \mathcal{L}_{L,K} \oplus \mathcal{L}_{L,K}^{\perp}$. By the notations, it is immediately seen that $\mathcal{S}_+^L \otimes \mathcal{S}_+^K \subseteq \mathcal{L}_{L,K}$ and

$$\mathcal{A}^L \otimes \mathcal{A}^K \subseteq \mathcal{L}_{L,K}^{\perp}. \tag{18}$$

Let $\boldsymbol{G} \in \mathbb{R}^{L \times K}$ and $\boldsymbol{G}^T = [\boldsymbol{g}_1 \ \cdots \ \boldsymbol{g}_L]$ (namely, $\boldsymbol{g}_l^T \in \mathbb{R}^K$ is the l-th row of $\boldsymbol{G}$). Given a vector $\boldsymbol{a} = [a_1, \ldots, a_K]^T$ (with $K \geq 3$), we denote by $\boldsymbol{A}(\boldsymbol{a})$ the arrow matrix generated by $\boldsymbol{a}$:

$$\boldsymbol{A}(\boldsymbol{a}) = \begin{bmatrix} a_1 + a_2 & a_3 & a_4 & \cdots & a_K \\ a_3 & a_1 - a_2 & 0 & \cdots & 0 \\ a_4 & 0 & a_1 - a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_K & 0 & 0 & \cdots & a_1 - a_2 \end{bmatrix} \in \mathcal{S}^{K-1} \tag{19}$$

and by $\widehat{\boldsymbol{A}}(\boldsymbol{G})$ the arrow matrix generated by the L arrow matrices $[\boldsymbol{A}(\boldsymbol{g}_1), \ldots, \boldsymbol{A}(\boldsymbol{g}_L)]$ (with $L \geq 3$), namely,

$$\widehat{\boldsymbol{A}}(\boldsymbol{G}) = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{g}_0) & \boldsymbol{A}(\boldsymbol{g}_3) & \boldsymbol{A}(\boldsymbol{g}_4) & \cdots & \boldsymbol{A}(\boldsymbol{g}_L) \\ \boldsymbol{A}(\boldsymbol{g}_3) & \boldsymbol{A}(\boldsymbol{g}_{-1}) & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{A}(\boldsymbol{g}_4) & \boldsymbol{0} & \boldsymbol{A}(\boldsymbol{g}_{-1}) & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}(\boldsymbol{g}_L) & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{A}(\boldsymbol{g}_{-1}) \end{bmatrix} \tag{20}$$

in $\mathcal{S}^{(K-1)(L-1)}$, where $\boldsymbol{g}_0 = \boldsymbol{g}_1 + \boldsymbol{g}_2$ and $\boldsymbol{g}_{-1} = \boldsymbol{g}_1 - \boldsymbol{g}_2$. It is clear that $\widehat{\boldsymbol{A}}(\boldsymbol{G})$ is matrix $\widehat{\boldsymbol{A}}(\boldsymbol{a}\boldsymbol{b}^T)$ with element $a_l b_k$ substituted by G_{lk}, where $\boldsymbol{b} \in \mathbb{R}^L$ is given (cf. [10, page 95]).

With the above notations in hand, we cite the theorem ([9, Theorem 5.6]) as the following lemma, albeit written in a different way.

Lemma 3.2 *Suppose that* $\min\{L, K\} \geq 3$. *Then a matrix* $\boldsymbol{G} \in \mathbb{R}^{L \times K}$ *is Lorentz-positive, if and only if there is* $\boldsymbol{X} \in \mathcal{A}^{L-1} \otimes \mathcal{A}^{K-1}$ *such that*

$$\widehat{\boldsymbol{A}}(\boldsymbol{G}) + \boldsymbol{X} \succeq \boldsymbol{0} \, (\in \mathcal{S}_+^{(K-1)(L-1)}), \tag{21}$$

where $\widehat{\boldsymbol{A}}(\cdot)$ *is defined in* (20).

By exploiting the notation (17), relationship (18), and Theorem 3.1 in [9], one can easily reformulate Lemma 3.2 into an easily implementable proposition as follows.

Proposition 3.3 *Suppose that* $\min\{L, K\} \geq 3$. *Then a matrix* $\boldsymbol{G} \in \mathbb{R}^{L \times K}$ *is Lorentz-positive, if and only if*

$$\widehat{\boldsymbol{A}}(\boldsymbol{G}) \in \mathcal{S}_+^{(K-1)(L-1)} + \mathcal{L}_{L-1,K-1}^{\perp}. \tag{22}$$

3.3. Equivalent Convex Reformulation of Problem (6)

Capitalizing on Proposition 3.3, we can derive an equivalent condition for $\boldsymbol{B}_m$ complying with (15). In other words, $\boldsymbol{B}_m$ belonging to set (15) is tantamount to the condition that there is $\boldsymbol{X}_m \in \mathcal{L}_{2M-1,2N}^{\perp}$ such that

$$\widehat{\boldsymbol{A}}(\boldsymbol{B}_m) + \boldsymbol{X}_m \in \mathcal{S}_+^{2(2M-1)N}.$$

Considering that the robust beamforming problem (6) amounts to (14), we obtain an identical form of linear conic program (cf. [10]) for (6) and summarize it as follows.

Proposition 3.4 *The robust MISO downlink beamforming problem* (6) *is equivalent to the following linear conic program:*

$$\begin{array}{ll} \underset{\{\boldsymbol{w}_m, \boldsymbol{X}_m\}, t}{\text{minimize}} & t \\ \text{subject to} & [t \ \boldsymbol{w}_{11}^T \ \boldsymbol{w}_{12}^T \ \cdots \ \boldsymbol{w}_{M1}^T \ \boldsymbol{w}_{M2}^T]^T \in \mathbb{L}^{2MN+1}, \\ & \widehat{\boldsymbol{A}}(\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m})) + \boldsymbol{X}_m \succeq \boldsymbol{0}, \ m = 1, \ldots, M, \\ & \boldsymbol{X}_m \in \mathcal{L}_{2M-1,2N}^{\perp}. \end{array} \tag{23}$$

We remark that $\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}) = [\boldsymbol{c}_m \ \epsilon_m \boldsymbol{C}_m] \in \mathbb{R}^{2M \times (2N+1)}$ is affine w.r.t. the design variables $\{\boldsymbol{w}_m\}$, and hence so is $\widehat{\boldsymbol{A}}(\boldsymbol{B}_m(\boldsymbol{w}_m, \boldsymbol{W}_{-m}))$.

4. NUMERICAL EXAMPLES

We consider a simulated scenario with an N-antenna BS serving three single-antenna users ($M = 3$). The elements of the channel vectors ($\bar{\boldsymbol{h}}_1, \bar{\boldsymbol{h}}_2, \bar{\boldsymbol{h}}_3$) are the i.i.d. standard complex Gaussian variables. The noise variance is set $\sigma_m^2 = 0.1$ for each user, and the SINR threshold value for the users is set to a common 12 dB. The bound of the error norm is assigned $\epsilon_m = \epsilon \|\bar{\boldsymbol{h}}_m\|$ for user m (namely, $\epsilon \geq \dfrac{\|\boldsymbol{\delta}_m\|}{\|\bar{\boldsymbol{h}}_m\|}$ is the percentage of the maximal norm of CSI error out of the norm of the channel). We compare the performance of the SDP relaxation problem (cf. (P_ϵ) in [8]) of (5), i.e., a benchmark, and the performance of the convex equivalent reformulation (23) of semi-infinite SOCP (6). The two convex problems (i.e., (P_ϵ) in [8] and (23)) are termed "SDR" and "Robust-SOCP" respectively in the following figures. We run simulations for the scenarios with different $N \in \{3, 4, 5\}$, and for a given N, 2000 sets of channel realizations are generated, and for each set of channel realization, both the convex problems are solved respectively for $\epsilon \in \{0.02, 0.04, 0.06, 0.08, 0.10, 0.12\}$, and all results are averaged over the 2000 simulation runs.

Fig. 1 shows the problem feasibility rate versus the relative perturbation bound ϵ for different values of N. As we can see, the feasibility rate of the SDP relaxation problem is only slightly higher than that of (23), and this behavior coincides with the fact that (6) is a more conservative (but convex) form, but not excessively so as demonstrated. It is observed that the feasibility rate increases when the number of transmit antenna N increases, and that the feasibility rate decreases when the perturbation bound ϵ increases.

Figs. 2 and 3 display the average transmission power versus the error norm bound ϵ for the scenarios of the various N. Particularly, in Fig. 2, the transmission power by the SDP relaxation method is averaged over all channel realizations where only the SDP (P_ϵ) in [8] is feasible, while in Fig. 3 the transmission power is averaged over all channels where both the two convex problems ((P_ϵ) in [8] and (23)) are feasible. The average transmission power by (23) in the

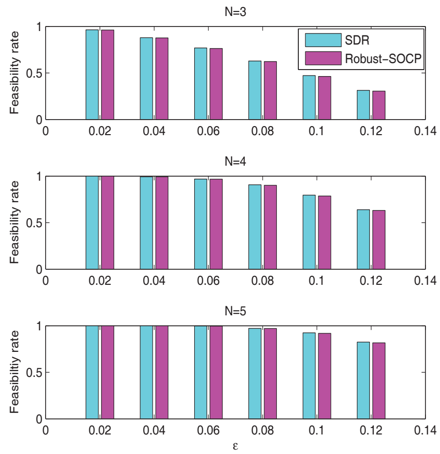

Fig. 1. Feasibility rate versus the perturbation bound ϵ for different values of transmit antennas N ($M = 3$ users).

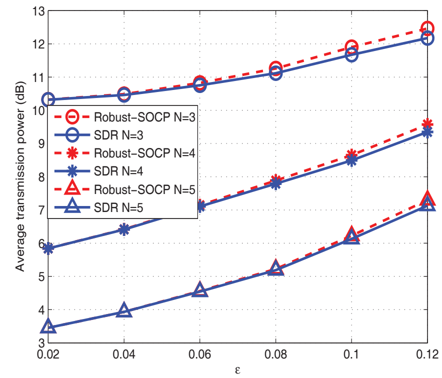

Fig. 3. Average transmission power versus the perturbation bound ϵ for different values of N ($M = 3$ users). (Average over the channels where both the SDP and (23) are feasible.)

both figures is taken over the channels where it is feasible. As expected, the higher transmission power is required to meet the robust QoS constraints for the larger bound ϵ of uncertainty, as well as for less transmit antennas. From the simulations results, it is clear that the more conservative model in (6) is sufficiently tight in practice.

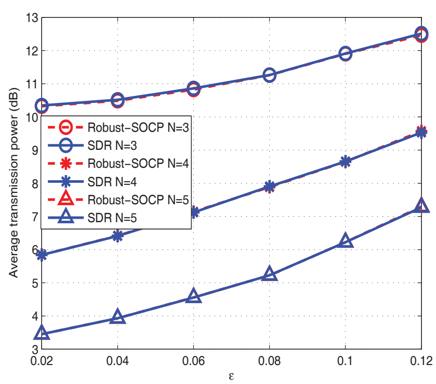

Fig. 2. Average transmission power versus the perturbation bound ϵ for different values of N ($M = 3$ users). (Average over the channels where the SDP is feasible.)

5. CONCLUSIONS

In a uni-cast MISO transmission system, we have considered the robust downlink beamforming problem, which minimizes the total transmission power subject to the worst-case SINR constraints. Given that the convexity of the robust beamforming problem remains unknown, we have presented an equivalent and explicit convex reformulation for the more conservative robust problem in the form of a semi-infinite SOCP. The optimization tool we utilized is the exact LMI description of the cone of Lorentz-positive matrices, and the derived problem reformulation is a standard form of linear conic program and thus can be numerically implemented in a convenient fashion. The numerical performance shows the conservativeness of the semi-infinite SOCP formulation is present, but not overly conservative, comparing the SDP relaxation of the original robust MISO downlink beamforming problem.

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