RANK-TWO TRANSMIT BEAMFORMED ALAMOUTI SPACE-TIME CODING FOR PHYSICAL-LAYER MULTICASTING

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ABSTRACT

In physical-layer multicasting over a multiuser MISO downlink channel, transmit beamforming using semidefinite relaxation (SDR) has been a popular approach. In this paper, we propose a rank-2 transmit beamformed Alamouti space-time code scheme, which may be seen as a generalization of the previous SDR-based beamforming framework. The beamforming problem arising from the proposed scheme is a rank-2 constrained semidefinite program (SDP). We deal with it using the SDR technique, but this time using rank-2 approximation rather than rank-1 approximation in the previous transmit beamforming. An analysis on the worst-case approximation accuracy of the rank-2 SDR approximation is provided, which reveals that the approximation accuracy degrades at a rate of $\sqrt{M}$, where M is the number of users served. This improves upon the case of transmit beamforming, where the worst-case approximation accuracy degrades at the higher rate of M. Simulation results further show that the proposed scheme performs better than the transmit beamforming scheme.

Index Terms— multicast, transmit beamforming, Alamouti space-time code, semidefinite program.

1. INTRODUCTION

This paper concentrates on common information broadcast in a multiuser multi-input single-output (MISO) downlink scenario, where the basestation has information about users' channels and uses its multi-antenna degree of freedom to serve single-antenna users in an optimized way. Particularly, we are interested in realizable and simple transmit schemes from a physical-layer point of view. A significant result in this context is multicast beamforming, first advocated by Sidiropoulos *et al.* [1]; see also [2, 3] for recent tutorial articles. In that approach, the physical-layer transmit scheme is fixed to be single-stream transmit beamforming, and the transmit beam is designed such that the users' receive SNRs are good. The latter amounts to beamformer optimization problems, specifically, the SNR-constrained problem and the max-min-fair problem, which turn out to be NP-hard in general [1,4]. This is in much contrast to its multiuser unicast beamforming counterparts, which are seemingly more difficult than multicasting but are all found to be polynomial-time solvable; see [2] and references therein. Despite this hardness, it has been recognized that the multicast beamforming problems can be efficiently approximated by a polynomial-time technique called semidefinite relaxation (SDR) [5]. Both numerical and analysis results have indicated that SDR is a powerful way to approximate the multicast beamforming solution.

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Although SDR-based multicast beamforming has many advantages, recent research has suggested that further performance gains can be obtained by modifying the physical-layer transmit scheme. Indeed, this has been hinted from an information theoretic perspective [6]. In our previous work [7], we have demonstrated that by employing an SDR-guided time-varying beamformer, coupled with channel coding, a significant improvement in bit error rates can practically be achieved. In this paper, we are interested in looking at the multicasting problem from another physical-layer viewpoint, namely, space-time coding.

In the literature, e.g., [8], one has already noticed the possibility of combining transmit beamforming and space-time coding to deliver "rank-r" beamforming. By rank-r beamforming, we mean that a transmitted symbol receives performance gains from r beamformers, enabled by space-time coding. However, exactly how that should be done is a scenario-dependent challenge. While there are many different kinds of space-time codes available, they are generally designed for performance measures, such as diversity order or diversity multiplexing tradeoff, in point-to-point channel-uninformed scenarios. It is not clear whether good space-time codes in those contexts are as promising in the multicasting scenario. Also, using space-time codes in multicasting is generally not a direct application, since the transmit beamformers need to be optimized and its tractability would largely depend on whether effective characterization of multicasting performance can be deduced from a given space-time code structure. There is however one exception— the class of orthogonal space-time block codes (OSTBCs). OSTBCs are simple to implement, and their performance can be easily characterized by an explicit SNR expression. The downside, however, is that OSTBCs do not have full rate for dimensions higher than 2 [9]. In this paper, we restrict ourselves to the dimension-2 (and full-rate) OSTBC, or the well-known Alamouti space-time code. We will develop a rank-2 transmit beamformed Alamouti scheme for physical-layer multicasting. An SDR framework will be established for this scheme, where we will provide a polynomial-time solution procedure, as well as an analysis on its SNR performance. Remarkably, both simulations and analysis will show that the proposed rank-2 beamformed Alamouti scheme can work better than the previous multicast beamforming scheme. In particular, we will show that the worst-case approximation accuracy of our proposed scheme degrades only at a rate of $\sqrt{M}$, where M is the number of users. By contrast, the previous scheme has a worst-case approximation accuracy that degrades at the higher rate of M.

Notations: Most of the notations used in this paper are standard. We use $\mathcal{CN}(\mathbf{0}, \mathbf{W})$ (resp. $\mathcal{N}(\mathbf{0}, \mathbf{W})$) to denote the circularly symmetric complex Gaussian distribution (resp. the real Gaussian distribution) with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{W}$; $\mathbb{H}^n$ to denote the set of $n \times n$ complex Hermitian matrices; $\mathbf{W} \succeq \mathbf{0}$ to denote the fact that $\mathbf{W}$ is positive semidefinite; and $\mathbf{X} \sim \mathbf{Y}$ to denote the fact that the random variables $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution.

2. PROBLEM FORMULATION AND MULTICAST BEAMFORMING

Consider a multiuser downlink scenario where users' channels are frequency-flat and MISO. The signal model is as follows:

$$y_i(t) = \mathbf{h}_i^H \mathbf{x}(t) + n_i(t), \quad t = 1, 2, \ldots \tag{1}$$

for $i = 1, \ldots, M$, where $\{y_i(t)\}_t$ is the received signal of user i; M is the number of users served; $\mathbf{x}(t) \in \mathbb{C}^N$ is the signal vector transmitted by the basestation, with N being the number of transmit antennas; $\mathbf{h}_i \in \mathbb{C}^N$ is the channel from the basestation to user i; and $n_i(t)$ is noise with distribution $\mathcal{CN}(0, 1)$.

Our problem is to transmit a common data stream to all the users, given channel state information at the basestation. As mentioned in the Introduction, a popularized physical-layer scheme for this multicasting problem is transmit beamforming, in which $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{w}s(t),$$

where $\mathbf{w} \in \mathbb{C}^N$ is a transmit beamforming vector, and $s(t) \in \mathbb{C}$ is a stream of data symbols. Assuming unit power with $s(t)$, the SNR of the received symbols at user i is $\mathsf{SNR}_i = |\mathbf{h}_i^H \mathbf{w}|^2$. We aim at optimizing the users' SNRs by considering the following max-min-fair (MMF) transmit beamforming design [1]:

$$\begin{aligned} \gamma_{\mathsf{BF}} := \max_{\mathbf{w} \in \mathbb{C}^N} \ & \min_{i=1,\ldots,M} |\mathbf{h}_i^H \mathbf{w}|^2 \\ \text{s.t.} \ & \|\mathbf{w}\|^2 \leq P, \end{aligned} \tag{2}$$

where P is the maximum allowable transmit power, and γ_{BF} denotes the best achievable worst-user receive SNR.

The MMF problem is known to be NP-hard in general [1, 4]. Here, we will focus on a suboptimal, but polynomial-time, MMF solution using semidefinite relaxation (SDR) [3–5]. To put into context, let $\mathbf{W} = \mathbf{w}\mathbf{w}^H$. By substituting this into (2) and using the equivalence

$$\mathbf{W} = \mathbf{w}\mathbf{w}^H \quad \Longleftrightarrow \quad \mathbf{W} \succeq \mathbf{0} \text{ and } \operatorname{rank}(\mathbf{W}) \leq 1,$$

we obtain the following *equivalent* formulation of (2):

$$\begin{aligned} \gamma_{\mathsf{BF}} := \max_{\mathbf{w} \in \mathbb{C}^N} \ & \min_{i=1,\ldots,M} \operatorname{Tr}(\mathbf{W}\mathbf{h}_i\mathbf{h}_i^H) \\ \text{s.t.} \ & \operatorname{Tr}(\mathbf{W}) \leq P, \quad \mathbf{W} \succeq \mathbf{0}, \quad \operatorname{rank}(\mathbf{W}) \leq 1. \end{aligned} \tag{3}$$

An SDR of the MMF problem in (2) can then be obtained by dropping the nonconvex rank constraint in (3):

$$\begin{aligned} \gamma_{\mathsf{BF-SDR}} := \max_{\mathbf{W} \in \mathbb{H}^N} \ & \min_{i=1,\ldots,M} \operatorname{Tr}(\mathbf{W}\mathbf{h}_i\mathbf{h}_i^H) \\ \text{s.t.} \ & \operatorname{Tr}(\mathbf{W}) \leq P, \quad \mathbf{W} \succeq \mathbf{0}. \end{aligned} \tag{4}$$

Problem (4) is a semidefinite program (SDP), which is convex and polynomial-time solvable. Besides being computationally tractable, the SDR problem (4) has been shown to possess some nice theoretical properties. Indeed, let

$$\gamma(\mathbf{W}) := \min_{i=1,\ldots,M} \operatorname{Tr}(\mathbf{W}\mathbf{h}_i\mathbf{h}_i^H) \tag{5}$$

be the worst-user receive SNR associated with the solution matrix $\mathbf{W}$, and let $\mathbf{W}^\star$ denote an optimal solution to (4).

Fact 1

(a) *([10]) When $M \leq 3$, there is a polynomial-time procedure that can generate from $\mathbf{W}^\star$ an optimal solution $\hat{\mathbf{w}}$ to* (2)*.*

(b) *([1, 4]) When $M > 3$, by using a Gaussian randomization procedure (which runs in randomized polynomial time), one can generate from $\mathbf{W}^\star$ a feasible solution $\hat{\mathbf{w}}$ to* (2) *that satisfies $\gamma(\hat{\mathbf{w}}\hat{\mathbf{w}}^H) \geq (1/8M) \cdot \gamma_{\mathsf{BF}}$. In other words, the worst-user receive SNR achieved by the beamforming vector $\hat{\mathbf{w}}$ is at least $(1/8M)$ times the best achievable worst-user SNR.*

Fact 1 reveals that the above beamforming scheme has its worst-user receive SNR degrading at a rate of M, the number of users. In the next section, we will introduce another beamforming scheme, which yields better provable results than those shown in Fact 1.

3. THE RANK-2 BEAMFORMED ALAMOUTI SCHEME

3.1. System Model

The proposed scheme, transmit beamformed Alamouti space-time coding, is described as follows. The data symbol stream $s(t)$ is parsed into blocks, specifically, by $\boldsymbol{s}(n) = [\ s(2n)\ s(2n+1)\ ]^T$. At block n, we transmit $\boldsymbol{s}(n)$ by a transmit beamformed Alamouti space-time code:

$$\boldsymbol{X}(n) \triangleq [\ \mathbf{x}(2n)\ \mathbf{x}(2n+1)\ ] = \mathbf{B}\mathbf{C}(\boldsymbol{s}(n)),$$

where $\mathbf{B} \in \mathbb{C}^{N \times 2}$ is a transmit beamforming matrix, and $\mathbf{C} : \mathbb{C}^2 \rightarrow \mathbb{C}^{2 \times 2}$ is the Alamouti space-time code. From the basic model in (1), we have

$$\boldsymbol{y}_i(n) \triangleq [\ y_i(2n)\ y_i(2n+1)\ ] = \mathbf{h}_i^H \mathbf{B}\mathbf{C}(\boldsymbol{s}(n)) + \boldsymbol{n}_i(n), \tag{6}$$

where $\boldsymbol{n}_i(n)$ is defined in the same way as $\boldsymbol{y}_i(n)$. As a key property introduced by the special structures of the Alamouti code, Eq. (6) can be turned to an equivalent SISO model, where each symbol can be independently detected and the SNR of the received symbols can be characterized by $\mathsf{SNR}_i = \mathbf{h}_i^H \mathbf{B}\mathbf{B}^H \mathbf{h}_i$.

Now, in the same spirit as the MMF transmit beamforming problem (2), we can consider the following design for the transmit beamformed Alamouti scheme:

$$\begin{aligned} \gamma_{\mathsf{BF-ALAM}} := \max_{\mathbf{B} \in \mathbb{C}^{N \times 2}} \ & \min_{i=1,\ldots,M} \mathbf{h}_i^H \mathbf{B}\mathbf{B}^H \mathbf{h}_i \\ \text{s.t.} \ & \operatorname{Tr}(\mathbf{B}\mathbf{B}^H) \leq P, \end{aligned} \tag{7}$$

whose goal is again to find the best achievable worst-user SNR.

3.2. Rank-Constrained SDP and Approximation Bounds

Just as before, by noting that

$$\mathbf{W} = \mathbf{B}\mathbf{B}^H \quad \Longleftrightarrow \quad \mathbf{W} \succeq \mathbf{0} \text{ and } \operatorname{rank}(\mathbf{W}) \leq 2,$$

we see that (7) is equivalent to

$$\begin{aligned} \gamma_{\mathsf{BF-ALAM}} := \max_{\mathbf{W}} \ & \min_{i=1,\ldots,M} \operatorname{Tr}(\mathbf{W}\mathbf{h}_i\mathbf{h}_i^H) \\ \text{s.t.} \ & \operatorname{Tr}(\mathbf{W}) \leq P,\ \mathbf{W} \succeq \mathbf{0},\ \operatorname{rank}(\mathbf{W}) \leq 2. \end{aligned} \tag{8}$$

It should be noted that problem (3) is a special case of problem (8), which implies that our proposed design should have a performance no worse than that of the MMF beamforming design in (3).

Now, upon removing the nonconvex rank constraint, we obtain an SDR of (8), which has exactly the same form as (4). Let $\mathbf{W}^\star$ denote an optimal solution to (4). Similar to the development of the beamforming scheme, we are naturally interested in the following questions:

1. How do we generate from $\mathbf{W}^\star$ a feasible solution to (8)?
2. What is the approximation quality of the generated solution?

As it turns out, these questions can be answered using the rank reduction techniques introduced in [10, 11]. Let us begin with the following proposition:

Proposition 1 *Suppose that $M \leq 8$. Then, there is a polynomial-time procedure that can generate from $\mathbf{W}^\star$ an optimal solution $\hat{\mathbf{B}}$ to the rank-two beamformed Alamouti problem* (7).

The proof of Proposition 1 follows from a direct application of [10, Theorem 5.1]. It is interesting to contrast this result with that for the transmit beamforming scheme, which requires $M \leq 3$ (see Fact 1(a)). This already suggests that our rank-two beamformed Alamouti scheme has performance no worse than the transmit beamforming, at least in the case where M is small (i.e., $M \leq 8$).

To further understand the performance of our proposed scheme, let us turn to the case where $M > 8$. Consider the following Gaussian randomization procedure, which returns a feasible solution to the rank-two beamformed Alamouti problem (7):

Algorithm 1 Gaussian Randomization Procedure for (7)

Input: an optimal solution $\mathbf{W}^\star$ to (4), number of randomizations $L \geq 1$

for $j = 1$ to L **do**

generate two independent circularly symmetric complex Gaussian random vectors $\boldsymbol{\xi}_1^j, \boldsymbol{\xi}_2^j \sim \mathcal{CN}(\mathbf{0}, \mathbf{W}^\star)$, and define

$$\tilde{\mathbf{B}}_j = \frac{1}{\sqrt{2}} \left[\ \boldsymbol{\xi}_1^j \quad \boldsymbol{\xi}_2^j\ \right];$$

let

$$\hat{\mathbf{B}}_j = \sqrt{P \Big/ \mathrm{Tr}\left(\tilde{\mathbf{B}}_j \tilde{\mathbf{B}}_j^H\right)} \cdot \tilde{\mathbf{B}}_j$$

end for

let

$$j^* := \arg \max_{j=1,\ldots,L} \gamma(\hat{\mathbf{B}}_j \hat{\mathbf{B}}_j^H),$$

where $\gamma(\cdot)$ is given by (5)

Output: $\hat{\mathbf{B}} = \hat{\mathbf{B}}_{j^*}$

It is worth noting that Algorithm 1 is a generalization of the Gaussian randomization procedure used for the transmit beamforming scheme in [1, 4].

Next, we are interested in the approximation quality of Algorithm 1. The following theorem is the main result of this paper. It shows that $\hat{\mathbf{B}}$ is an $(1/15.09\sqrt{M})$-approximate solution to the rank-two beamformed Alamouti problem (7).

Theorem 1 *With probability at least $1 - (3/4)^L$,*

$$\gamma(\hat{\mathbf{B}}\hat{\mathbf{B}}^H) \geq \frac{1}{15.09\sqrt{M}} \cdot \gamma_{\mathsf{BF-ALAM}}.$$

The proof of Theorem 1 is given in the Appendix. Note that since $\gamma_{\mathsf{BF-ALAM}} \geq \gamma_{\mathsf{BF}}$ (see the remark after (8)), it follows from Theorem 1 that $\gamma(\hat{\mathbf{B}}\hat{\mathbf{B}}^H) \geq (1/15.09\sqrt{M}) \cdot \gamma_{\mathsf{BF}}$ with high probability. In particular, the provable gap between the worst-user SNR achieved by our rank-two beamformed Alamouti scheme and the best achievable worst-user SNR scales only on the order of $\sqrt{M}$, which is substantially better than the traditional transmit beamforming case, where the provable gap scales on the order of M (see Fact 1(b)).

4. SIMULATION RESULTS

In this section, we use simulations to demonstrate the performance of the proposed rank-two transmit beamformed Alamouti scheme.

In Table 1, we list the worst-user SNRs achieved by the transmit beamforming and rank-two beamformed Alamouti schemes (cf., Sections 2 and 3, resp.). The results were obtained from $1,000$ trials of randomly generated channel realizations. We are interested in inspecting the worst-user SNRs for each rank value $r = \mathrm{rank}(\mathbf{W}^\star)$; we do so by grouping the results according to r, and then evaluating the average values of the worst-user SNRs for each group. The transmit power limit is set to $P = 1$, and the number of randomizations in the Gaussian randomized procedure in Algorithm 1 is $L = 30MN$. We can see from the table that the worst-user SNRs achieved by rank-two beamformed Alamouti are higher than that by transmit beamforming; in the case of $(N, M) = (8, 64)$, $\mathrm{rank}(\mathbf{W}^\star) = 5$, the SNR improvement is as high as two times.

Table 1: Worst-user SNRs of the transmit beamforming and rank-two beamformed Alamouti schemes.

(N, M)	$r = \mathrm{rank}(\mathbf{W}^\star)$	γ_{SDR}	Worst-user SNR γ	
			BF	BF Alamouti
$(8, 32)$	$r = 2$	1.0640	0.4150	1.0225
$(8, 32)$	$r = 3$	1.0730	0.4420	0.7896
$(8, 32)$	$r = 4$	1.0881	0.4220	0.7050
$(8, 64)$	$r = 3$	0.8072	0.2193	0.5029
$(8, 64)$	$r = 4$	0.8428	0.2176	0.4658
$(8, 64)$	$r = 5$	0.8651	0.2077	0.4393

In Figure 1, we plot the worst-user SNRs of the two beamforming schemes against the number of users M. We set $N = 8$, and $P = 10$dB. The worst-user SNRs shown are averages of those of $1,000$ independent channel realizations. Apart from seeing that the rank-two beamformed Alamouti scheme performs better than the beamforming scheme, we observe that the performance gaps of the two schemes relative to the SDR optimal value $\gamma_{\mathsf{SDR-BF}}$ ("SDR upper bound" in the legend) tend to increase with M. This phenomenon is consistent with the provable approximation accuracy results in Fact 1(b) and Theorem 1. Figure 1 also illustrates that the rank-two beamformed Alamouti scheme is more capable of handling large number of users than the transmit beamforming scheme.

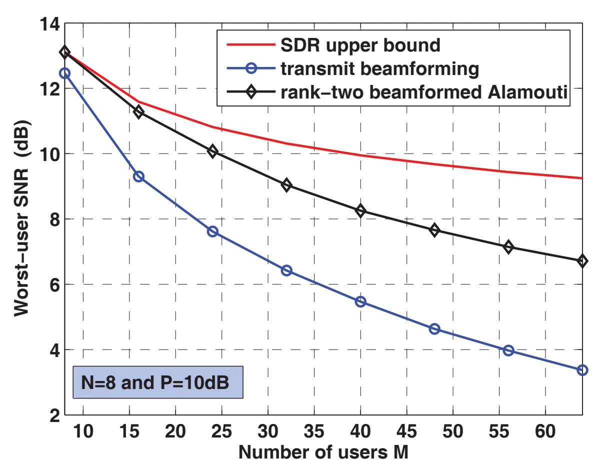

Fig. 1: The worst-user SNRs of the transmit beamforming and rank-two beamformed Alamouti schemes versus the number of users.

5. CONCLUSION

In this paper, we proposed a new beamforming scheme— called the rank-2 transmit beamformed Alamouti scheme— for physical-layer multicasting. Such a scheme combines transmit beamforming and space-time coding techniques by exploiting the properties of the Alamouti space-time code and results in "rank-r" beamforming. To solve the resulting beamforming problem, we employed the SDR technique, followed by a rank reduction procedure. We then provided a worst-case approximation accuracy analysis of our proposed scheme, which revealed that it has much better theoretical performance than the existing transmit beamforming scheme. Our simulation results also supported our theoretical findings.

6. APPENDIX: PROOF OF THEOREM 1

Without loss of generality, we may assume that $P = 1$. Now, in Algorithm 1, consider a fixed $j \in \{1, \ldots, L\}$ and let $\tilde{\mathbf{W}} = \tilde{\mathbf{B}}_j \tilde{\mathbf{B}}_j^H$. We give the proof in four steps:

Step 1*:* For any arbitrary $\boldsymbol{\mu} \in \mathbb{C}^N$, observe that

$$\mathrm{Tr}(\tilde{\mathbf{W}}\boldsymbol{\mu}\boldsymbol{\mu}^H) = \frac{1}{2}\sum_{i=1}^{2} |\boldsymbol{\mu}^H \boldsymbol{\xi}_i|^2,$$

where $\boldsymbol{\mu}^H \boldsymbol{\xi}_i \sim \mathcal{CN}(0, \boldsymbol{\mu}^H \mathbf{W}^\star \boldsymbol{\mu})$. Then, following [12, Proposition A5.5], for any $\beta \in (0, 1)$, we have

$$\Pr\left(\mathrm{Tr}(\tilde{\mathbf{W}}\boldsymbol{\mu}\boldsymbol{\mu}^H) \le \beta \cdot \mathrm{Tr}(\mathbf{W}^\star \boldsymbol{\mu}\boldsymbol{\mu}^H)\right) \le e^{2(1-\beta+\ln\beta)}. \tag{9}$$

Step 2*:* Let $\mathbf{W}^\star = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H$ be the spectral decomposition of $\mathbf{W}^\star$. Observe that

$$\mathrm{Tr}(\tilde{\mathbf{W}}) = \frac{1}{2}\sum_{i=1}^{2} ||\boldsymbol{\xi}_i||^2 \sim \frac{1}{2}\sum_{i=1}^{2} ||\boldsymbol{\eta}_i||^2,$$

where $\boldsymbol{\eta}_i \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Lambda})$ and $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ are independent. Moreover,

$$\frac{1}{2}\sum_{i=1}^{2} ||\boldsymbol{\eta}_i||^2 = \frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{2} |\eta_{ij}|^2,$$

where $\eta_{ij} \sim \mathcal{CN}(0, \boldsymbol{\Lambda}_{jj})$, and $\{\eta_{ij}\}$ are independent. It follows that for any $\alpha \in (1, \infty)$,

$$\Pr\left(\mathrm{Tr}(\tilde{\mathbf{W}}) \ge \alpha \cdot \mathrm{Tr}(\mathbf{W}^\star)\right) = \Pr\left(\frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{2} |\eta_{ij}|^2 \ge \alpha \sum_{j=1}^{n} \boldsymbol{\Lambda}_{jj}\right)$$
$$= \Pr\left(\sum_{j=1}^{N} \boldsymbol{\Lambda}_{jj} \sum_{i=1}^{4} |\tilde{\eta}_{ij}|^2 \ge \alpha \sum_{j=1}^{n} \boldsymbol{\Lambda}_{jj}\right),$$

where $\tilde{\eta}_{ij} \sim \mathcal{N}(0, 1/4)$. Now, using the argument in the proof of [11, Proposition 2.1] (see the remark after the proof of [11, Proposition 2.2]), we see that for $\alpha \ge 4/3$,

$$\Pr\left(\mathrm{Tr}(\tilde{\mathbf{W}}) \ge \alpha \cdot \mathrm{Tr}(\mathbf{W}^\star)\right) \le e^{-\frac{1}{2}(\alpha + 4\ln\frac{3}{4})}. \tag{10}$$

Step 3*:* By setting $\beta = (e\sqrt{2M})^{-1}$ and $\alpha = 2\ln 4 - 4\ln(3/4) \approx 3.92$ in (9) and (10), respectively, we obtain

$$\Pr\left(\mathrm{Tr}(\tilde{\mathbf{W}}\mathbf{h}_i\mathbf{h}_i^H) \le \beta \cdot \mathrm{Tr}(\mathbf{W}^\star \mathbf{h}_i\mathbf{h}_i^H)\right) \le e^{2(1+\ln\beta)} = \frac{1}{2M},$$
$$\Pr\left(\mathrm{Tr}(\tilde{\mathbf{W}}) \ge \alpha \cdot \mathrm{Tr}(\mathbf{W}^\star)\right) \le e^{-\frac{1}{2}(\alpha+4\ln\frac{3}{4})} = \frac{1}{4}.$$

It follows from the union bound that

$$\Pr\Big(\left\{\mathrm{Tr}(\tilde{\mathbf{W}}\mathbf{h}_i\mathbf{h}_i^H) \le \beta \cdot \mathrm{Tr}(\mathbf{W}^\star \mathbf{h}_i\mathbf{h}_i^H) \ \ \forall i = 1, \ldots, M\right\}$$
$$\wedge \left\{\mathrm{Tr}(\tilde{\mathbf{W}}) \ge \alpha \cdot \mathrm{Tr}(\mathbf{W}^\star)\right\}\Big) \ge \frac{1}{4}.$$

In particular, with probability at least $1/4$, we have

$$\begin{aligned} \frac{\mathrm{Tr}(\tilde{\mathbf{W}}\mathbf{h}_i\mathbf{h}_i^H)}{\mathrm{Tr}(\tilde{\mathbf{W}})} &\ge \frac{\beta}{\alpha} \cdot \frac{\mathrm{Tr}(\mathbf{W}^\star \mathbf{h}_i\mathbf{h}_i^H)}{\mathrm{Tr}(\mathbf{W}^\star)} \\ &= \frac{1}{15.09\sqrt{M}} \cdot \mathrm{Tr}(\mathbf{W}^\star \mathbf{h}_i\mathbf{h}_i^H) \end{aligned}$$

for $i = 1, \ldots, M$ (recall that $\mathrm{Tr}(\mathbf{W}^\star) = P = 1$).

Step 4*:* By the result in Step 3 and the union bound, it follows that

$$\Pr\left(\left\{\exists j : \frac{\mathrm{Tr}(\mathbf{h}_i^H \tilde{\mathbf{B}}_j \tilde{\mathbf{B}}_j^H \mathbf{h}_i)}{\mathrm{Tr}(\tilde{\mathbf{B}}_j \tilde{\mathbf{B}}_j^H)} \ge \frac{1}{15.09\sqrt{M}} \cdot \mathrm{Tr}(\mathbf{W}^\star \mathbf{h}_i\mathbf{h}_i^H) \ \ \forall i\right\}\right)$$
$$\ge 1 - (3/4)^L.$$

This, together with the construction of $\hat{\mathbf{B}}$ in Algorithm 1, implies the desired result.

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