STABLE SIGNAL RECOVERY IN COMPRESSED SENSING WITH A STRUCTURED MATRIX PERTURBATION

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ABSTRACT

The sparse signal recovery in standard compressed sensing (CS) requires that the sensing matrix is exactly known. The CS problem subject to perturbation in the sensing matrix is often encountered in practice and has attracted interest of researches. Unlike existing robust signal recoveries with the recovery error growing linearly with the perturbation level, this paper analyzes the CS problem subject to a structured perturbation to provide conditions for stable signal recovery under measurement noise. Under mild conditions on the perturbed sensing matrix, similar to that for the standard CS, it is shown that a sparse signal can be stably recovered by ℓ_1 minimization. A remarkable result is that the recovery is exact and independent of the perturbation if there is no measurement noise and the signal is sufficiently sparse. In the presence of noise, largest entries (in magnitude) of a compressible signal can be stably recovered. The result is demonstrated by a simulation example.

***Index Terms*—** Compressed sensing, matrix perturbation, stable signal recovery, robust signal recovery

1. INTRODUCTION

Compressed sensing (CS) has been a very active area of information theory and signal processing since the pioneering works of Candès *et al.* [1] and Donoho [2]. In CS, one seeks to recover a sparse/compressible signal from significantly reduced number of (possibly noisy) linear measurements. It has been shown that, under mild conditions, a sparse signal can be stably recovered, with the recovery error at most proportional to the measurement noise level, by an ℓ_1 minimization approach. In addition, the largest entries (in magnitude) of a compressible signal can be stably recovered. Other approaches providing similar results are also reported thereafter, e.g., IHT [3].

Note that the sensing matrix is assumed known *a priori* in the standard CS. In practical situations, such as the direction of arrival (DOA) estimation [4], the sensing matrix is often subject to perturbation which is not exactly known. There have been recent and active studies of such situations. Herman and Strohmer [5] analyzed the effect of a matrix perturbation and showed that the error of signal recovery grows linearly with the perturbation level and thus is robust to the perturbation. Similar robust recovery results were also reported in [6, 7]. These results imply that the signal recovery may suffer from large error under large perturbation. Other works dealing with sensing matrix perturbations include Zhu *et al.* [8] on a sparse total least-squares approach to alleviating effect of the perturbation and Yang *et al.* [4] on the framework of DOA estimation and its solution based on a sparse Bayesian perspective. Overall, all the existing works provide no guarantees on the signal recovery performance when incorporating the perturbation into the recovery algorithm.

This paper is on the CS problem subject to a structured matrix perturbation. In this paper, the term "stable signal recovery" means that the recovery error is at most proportional to the measurement noise level, while "robust signal recovery" means that the error grows at most linearly with the matrix perturbation level. Different from the robust signal recovery results [5, 6] that are based on the nominal but not true sensing matrix, this paper explores the stable recovery of a sparse signal from noisy measurements by incorporating the perturbation into the recovery algorithm. For this purpose, a structured matrix perturbation, with each column vector bounded within a parameterized interval, is considered in this paper. Its analysis shows that the recovery of a sparse signal is stable under similar mild conditions as for the standard CS problem by incorporating the perturbation structure into the ℓ_1 minimization. In the special noise-free case, for a sufficiently sparse signal, the recovery is exact regardless of the perturbation. Further, largest entries of a compressible signal can be stably recovered under the same conditions. A numerical simulation is carried out to demonstrate our analysis.

Notations used in this paper are as follows. Bold-case letters are reserved for vectors and matrices. Superscripts ^{o} and * refer to the original value and an optimal value regarding to an optimization problem, respectively. x_i is the ith entry of a vector $\boldsymbol{x}$. diag $(\boldsymbol{x})$ is a matrix with its diagonal entries being entries of a vector $\boldsymbol{x}$. $\odot$ is the Hadamard (elementwise) product.

2. STANDARD CS

A signal is called k-sparse if it has at most k nonzero entries and is called compressible if it can be well approximated by a sparse signal. Denote $\boldsymbol{x}^k$ the best k-sparse approximation to a signal $\boldsymbol{x}^o$ in the sense

$$\left\|\boldsymbol{x}^o-\boldsymbol{x}^k\right\|_2=\min_{\|\boldsymbol{x}\|_0\leq k}\left\|\boldsymbol{x}^o-\boldsymbol{x}\right\|_2 \tag{1}$$

with $\|\boldsymbol{x}\|_0$ counting the number of nonzero entries of a vector $\boldsymbol{x}$. In the standard CS, a compressible signal $\boldsymbol{x}^o\in\mathbb{R}^n$ is acquired via a known linear transformation that is usually described by a fat matrix $\boldsymbol{\Phi}\in\mathbb{R}^{m\times n}$ with $m<n$. The acquired data vector is given as

$$\boldsymbol{y}=\boldsymbol{\Phi}\boldsymbol{x}^o+\boldsymbol{e}, \tag{2}$$

where $\boldsymbol{e}$ denotes the vector of measurement noises, $\boldsymbol{y},\boldsymbol{e}\in\mathbb{R}^m$, and $\|\boldsymbol{e}\|_2\leq\epsilon$. Given $\boldsymbol{y}$, $\boldsymbol{\Phi}$ and ϵ, CS studies conditions for recovering the signal $\boldsymbol{x}^o$ via an efficient approach. This paper focuses on the ℓ_1 norm minimization approach. The restricted isometry property (RIP) [9] has become a dominant tool to such analysis, which is defined as follows.

Definition 1 *Define the k-restricted isometry constant (RIC) of a matrix $\boldsymbol{\Phi}$, denoted by $\delta_k(\boldsymbol{\Phi})$, as the smallest number such that*

$$\left(1-\delta_k(\boldsymbol{\Phi})\right)\|\boldsymbol{v}\|_2^2\leq\|\boldsymbol{\Phi}\boldsymbol{v}\|_2^2\leq\left(1+\delta_k(\boldsymbol{\Phi})\right)\|\boldsymbol{v}\|_2^2 \tag{3}$$

holds for all k-sparse vectors $\boldsymbol{v}$. $\boldsymbol{\Phi}$ satisfies the k-RIP if $\delta_k(\boldsymbol{\Phi})<1$.

Based on the RIP, the following theorem holds.

Theorem 1 ([10]) *Assume that $\delta_{2k}(\boldsymbol{\Phi})<\sqrt{2}-1$ and $\|\boldsymbol{e}\|_2\leq\epsilon$. Then the solution $\boldsymbol{x}^*$ to the basis pursuit denoising (BPDN) problem*

$$\min_{\boldsymbol{x}}\|\boldsymbol{x}\|_1,\ \textit{subject to}\ \|\boldsymbol{y}-\boldsymbol{\Phi}\boldsymbol{x}\|_2\leq\epsilon \tag{4}$$

satisfies

$$\|\boldsymbol{x}^*-\boldsymbol{x}^o\|_2\leq C^0k^{-1/2}\left\|\boldsymbol{x}^o-\boldsymbol{x}^k\right\|_1+C^1\epsilon, \tag{5}$$

where $C^0=\frac{2\left[1+\left(\sqrt{2}-1\right)\delta_{2k}\right]}{1-\left(\sqrt{2}+1\right)\delta_{2k}}$ and $C^1=\frac{4\sqrt{1+\delta_{2k}}}{1-\left(\sqrt{2}+1\right)\delta_{2k}}$.

Theorem 1 tells that a k-sparse signal can be stably recovered provided $\delta_{2k}(\boldsymbol{\Phi})<\sqrt{2}-1$ since $\boldsymbol{x}^o=\boldsymbol{x}^k$ in such case, and that the largest k entries of a compressible signal also can be stably recovered under the same condition. In a special noise free case, the recovery of a k-sparse signal is exact. It is also noted that the condition of the RIP has been relaxed in the literature but it is beyond the scope of this paper.

3. CS SUBJECT TO A STRUCTURED PERTURBATION

3.1. Problem Formulation

In the standard CS, the sensing matrix is assumed to be exactly known. Such an ideal assumption is not always the case in practice. Consider that the true sensing matrix is $\boldsymbol{\Phi}=\boldsymbol{A}+\boldsymbol{E}\in\mathbb{R}^{m\times n}$, where $\boldsymbol{A}$ is the nominal sensing matrix and $\boldsymbol{E}\in\mathbb{R}^{m\times n}$ represents the unknown perturbation. In this paper we consider a structured perturbation in the form $\boldsymbol{E}=\boldsymbol{B}\boldsymbol{\Delta}^o$, where $\boldsymbol{B}\in\mathbb{R}^{m\times n}$ is known *a priori*, $\boldsymbol{\Delta}^o=\text{diag}(\boldsymbol{\beta}^o)$ is a bounded uncertain term with $\boldsymbol{\beta}^o\in[-r,r]^n$ and $r>0$. As a result, the observation model in (2) becomes

$$\boldsymbol{y}=\boldsymbol{\Phi}\boldsymbol{x}^o+\boldsymbol{e},\quad \boldsymbol{\Phi}=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{\Delta}^o \tag{6}$$

with $\boldsymbol{\Delta}^o=\text{diag}(\boldsymbol{\beta}^o)$, $\boldsymbol{\beta}^o\in[-r,r]^n$ and $\|\boldsymbol{e}\|_2\leq\epsilon$. Given $\boldsymbol{y}$, $\boldsymbol{A}$, $\boldsymbol{B}$, r and ϵ, the CS task is to recover $\boldsymbol{x}^o$, possibly, as well as $\boldsymbol{\beta}^o$.

3.2. Main Results of This Paper

A vector $\boldsymbol{v}$ is called $2k$-duplicately (D-) sparse if $\boldsymbol{v}=\left[\boldsymbol{v}_1^T,\boldsymbol{v}_2^T\right]^T$ with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ being of the same dimension and share the same support. We introduce the concept of duplicate (D-) RIP as follows.

Definition 2 *Define the $2k$-duplicate (D-) RIC of a matrix $\boldsymbol{\Phi}$, denoted by $\bar{\delta}_{2k}(\boldsymbol{\Phi})$, as the smallest number such that*

$$\left(1-\bar{\delta}_{2k}(\boldsymbol{\Phi})\right)\|\boldsymbol{v}\|_2^2\leq\|\boldsymbol{\Phi}\boldsymbol{v}\|_2^2\leq\left(1+\bar{\delta}_{2k}(\boldsymbol{\Phi})\right)\|\boldsymbol{v}\|_2^2 \tag{7}$$

holds for all $2k$-D-sparse vectors $\boldsymbol{v}$. $\boldsymbol{\Phi}$ satisfies the $2k$-D-RIP if $\bar{\delta}_{2k}(\boldsymbol{\Phi})<1$.

With respect to the perturbed observation model in (6), let $\boldsymbol{\Psi}=[\boldsymbol{A},\boldsymbol{B}]$. The main results of this paper can be stated in the following theorems. Readers are referred to [11] for their proofs.

Theorem 2 *Assume that $\bar{\delta}_{4k}(\boldsymbol{\Psi})<\left(\sqrt{2\left(1+r^2\right)}+1\right)^{-1}$, $\|\boldsymbol{x}^o\|_0\leq k$ and $\|\boldsymbol{e}\|_2\leq\epsilon$. Then the solution $(\boldsymbol{x}^*,\boldsymbol{\beta}^*)$ to the perturbed (P-) BPDN problem*

$$\min_{\boldsymbol{x}\in\mathbb{R}^n,\boldsymbol{\beta}\in[-r,r]^n}\|\boldsymbol{x}\|_1,\ \textit{subject to}\ \|\boldsymbol{y}-(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{\Delta})\boldsymbol{x}\|_2\leq\epsilon \tag{8}$$

with $\boldsymbol{\Delta}=$ diag $(\boldsymbol{\beta})$ satisfies that

$$\|\boldsymbol{x}^*-\boldsymbol{x}^o\|_2\leq C\epsilon, \tag{9}$$

$$\|(\boldsymbol{\beta}^*-\boldsymbol{\beta}^o)\odot\boldsymbol{x}^o\|_2\leq\mathcal{C}\epsilon, \tag{10}$$

where

$$C=\frac{4\sqrt{1+\bar{\delta}_{4k}(\boldsymbol{\Psi})}}{1-\left(\sqrt{2\left(1+r^2\right)}+1\right)\bar{\delta}_{4k}(\boldsymbol{\Psi})},$$

$$\mathcal{C}=\frac{\left[2+\sqrt{1+r^2}\,\|\boldsymbol{\Psi}\|_2\,C\right]}{\sqrt{1-\bar{\delta}_{4k}(\boldsymbol{\Psi})}}.$$

Theorem 3 *Assume that* $\bar{\delta}_{4k}(\mathbf{\Psi}) < \left(\sqrt{2(1+r^2)}+1\right)^{-1}$ *and* $\|\boldsymbol{e}\|_2 \leq \epsilon$. *Then the solution* $(\boldsymbol{x}^*, \boldsymbol{\beta}^*)$ *to the P-BPDN problem in (8) satisfies that*

$$\|\boldsymbol{x}^* - \boldsymbol{x}^o\|_2 \leq \left(C_0 k^{-1/2} + C_1\right)\left\|\boldsymbol{x}^o - \boldsymbol{x}^k\right\|_1 + C_2\epsilon, \quad (11)$$

$$\left\|(\boldsymbol{\beta}^* - \boldsymbol{\beta}^o) \odot \boldsymbol{x}^k\right\|_2 \leq \left(\mathcal{C}_0 k^{-1/2} + \mathcal{C}_1\right)\left\|\boldsymbol{x}^o - \boldsymbol{x}^k\right\|_1 + \mathcal{C}_2\epsilon, \quad (12)$$

where

$$\begin{aligned} C_0 &= 2\left[1 + \left(\sqrt{2(1+r^2)} - 1\right)\bar{\delta}_{4k}(\mathbf{\Psi})\right]/a, \\ C_1 &= 2\sqrt{2}r\bar{\delta}_{4k}(\mathbf{\Psi})/a, \\ \mathcal{C}_0 &= \sqrt{1+r^2}\,\|\mathbf{\Psi}\|_2\, C_0/b, \\ \mathcal{C}_1 &= \left[\sqrt{1+r^2}C_1 + 2r\right]\|\mathbf{\Psi}\|_2/b \end{aligned}$$

with $a = 1 - \left(\sqrt{2(1+r^2)}+1\right)\bar{\delta}_{4k}(\mathbf{\Psi})$, $b = \sqrt{1-\bar{\delta}_{4k}(\mathbf{\Psi})}$ *and* $C_2 = C$, $\mathcal{C}_2 = \mathcal{C}$ *with* $C, \mathcal{C}$ *as defined in Theorem 2.*

3.3. Some Discussions on the Main Results

Theorem 2 states that, in the CS problem subject to the structured perturbation considered in this paper, the original sparse signal can be stably recovered via an ℓ_1 minimization approach incorporated with the perturbation structure, provided that the D-RIP is sufficiently small with respect to the perturbation level in terms of r. Meanwhile, the perturbation parameter $\boldsymbol{\beta}^o$ can be stably recovered on the support of $\boldsymbol{x}^o$. Note that it is impossible to recover $\boldsymbol{\beta}^o$ out of the support of $\boldsymbol{x}^o$ since it has no contributions to the observation $\boldsymbol{y}$.

As the D-RIP condition is satisfied in Theorem 2, the sparse signal recovery error of the perturbed CS is constrained by the noise level ϵ, and the influence of the perturbation is limited to the coefficient before ϵ. For example, if $\bar{\delta}_{4k}(\mathbf{\Psi}) = 0.2$, then $\|\boldsymbol{x}^* - \boldsymbol{x}^o\|_2 \leq 8.48\epsilon, 8.50\epsilon, 11.0\epsilon$ corresponding to $r = 0.01, 0.1, 1$, respectively. It shows that the influence of the perturbation can be arbitrarily small, even for a large perturbation, provided that the noise is sufficiently small. As a special case, the recovery is exact in the noise free case. This is in contrast to the robust recovery results [5, 6] where the recovery error exists once a matrix perturbation appears. Theorem 3 is a general form of Theorem 2 and it shows that largest entries of a compressible signal, as well as the perturbation parameter $\boldsymbol{\beta}^o$ on the support of $\boldsymbol{x}^k$, can be stably recovered under the same D-RIP condition.

The result in this paper is parallel to that in the standard CS. Omitting the difference between $\delta_{2k}(\mathbf{\Phi})$ and $\bar{\delta}_{4k}(\mathbf{\Psi})$, as the perturbation vanishes or equivalently $r \to 0$, the conditions in Theorems 1 and 3 coincide, as well as the upper bounds in (5) and (11) on the recovery errors.

Existing works studying the RIP mainly focus on random matrices. In the standard CS, $\mathbf{\Phi}$ has the k-RIP with constant δ with a large probability provided that $m \geq C_\delta k \log(n/k)$ and $\mathbf{\Phi}$ has properly scaled i.i.d. subgaussian distributed entries with constant C_δ depending on δ and the distribution. The D-RIP can be considered as a model-based RIP introduced in [12]. Suppose that $\boldsymbol{A}$, $\boldsymbol{B}$ are mutually independent and both are i.i.d. subgaussian distributed (the true sensing matrix $\mathbf{\Phi} = \boldsymbol{A} + \boldsymbol{B}\mathbf{\Delta}^o$ is also i.i.d. subgaussian distributed if $\boldsymbol{\beta}^o$ is independent of $\boldsymbol{A}$ and $\boldsymbol{B}$). The model-based RIP is determined by the number of subspaces of the structured sparse signals that are referred to as the D-sparse ones in the present paper. For $\mathbf{\Psi} = [\boldsymbol{A}, \boldsymbol{B}]$, the number of $2k$-dimensional subspaces for $2k$-D-sparse signals is $\binom{n}{k}$. Consequently, $\mathbf{\Psi}$ has the $2k$-D-RIP with constant δ with a large probability also provided that $m \geq C_\delta k \log(n/k)$ by [12, Theorem 1]. So, in the case of high dimensional system, the D-RIP condition on $\mathbf{\Psi}$, as $r \to 0$, in Theorem 2 or 3 can be satisfied once the RIP condition on $\mathbf{\Phi}$ (after proper scaling of columns) in the standard CS is met. It means that the structured perturbation in the perturbed CS gradually strengthens the D-RIP condition for stable signal recovery but there exists no gap between our considered perturbed CS and the standard CS in the case of high dimensional systems.

As mentioned before, the RIP condition for guaranteed stable recovery in the standard CS has been relaxed. Similar techniques may be adopted to possibly relax the D-RIP condition in the perturbed CS. While this paper is focused on the ℓ_1 minimization approach, it is also possible to modify other algorithms in the standard CS, e.g., IHT, and apply them to the perturbed case to provide similar recovery guarantees.

3.4. Alternating Algorithm for P-BPDN

The P-BPDN problem in (8) is nonconvex and the global minimum cannot be easily obtained. A simple method is to solve a series of BPDN problems with

$$\begin{aligned} \boldsymbol{x}^{(j+1)} &= \arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1, \text{ subject to} \\ &\qquad \left\|\boldsymbol{y} - \left(\boldsymbol{A} + \boldsymbol{B}\mathbf{\Delta}^{(j)}\right)\boldsymbol{x}\right\|_2 \leq \epsilon, \quad (13) \\ \boldsymbol{\beta}^{(j+1)} &= \arg\min_{\boldsymbol{\beta} \in [-r,r]^n} \left\|\boldsymbol{y} - (\boldsymbol{A} + \boldsymbol{B}\mathbf{\Delta})\boldsymbol{x}^{(j+1)}\right\|_2 \quad (14) \end{aligned}$$

starting from $\boldsymbol{\beta}^{(0)} = \mathbf{0}$, where the superscript $^{(j)}$ indicates the jth iteration and $\mathbf{\Delta}^{(j)} = \operatorname{diag}\left(\boldsymbol{\beta}^{(j)}\right)$. The alternating algorithm defined by (13) and (14) converges to a stationary point. It also can be shown that an optimal solution to (8) is a stationary point of this alternating algorithm. Readers are referred to [11] for the details.

A numerical result is presented in Fig. 1, where a sparse signal of length $n = 100$, composed of $k = 5$ unit spikes, is sought to be recovered from $m = 30$ noisy measurements. In the simulation, $\boldsymbol{\beta}^o$ is uniformly distributed in $[-r, r]$ with $r = 0.1$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are mutually independent and i.i.d. zero

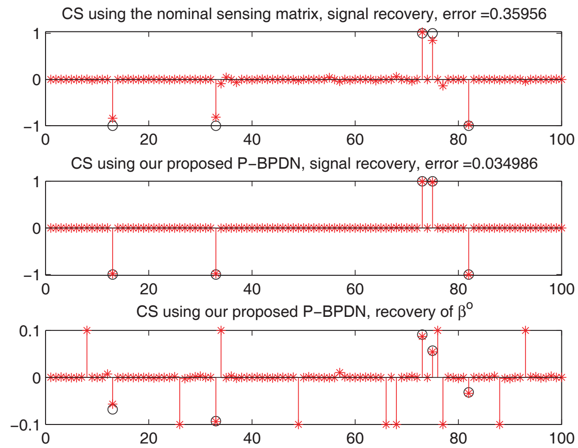

Fig. 1. Sparse signal recovery from noisy measurements in CS subject to structured matrix perturbation. Black circles refer to the original signal and perturbation parameter; red stars refer to the recoveries. Errors refer to the Euclidean distances between the recovered signals and the original one.

mean Gaussian distributed. The Euclidean distances from the recovered signals (red stars) of CS, using the nominal sensing matrix (corresponding to the robust signal recoveries [5]) and the P-BPDN approach (the proposed stable signal recovery), to the original one (black circles) are measured, respectively. As shown in Fig. 1, a large recovery error exhibits in CS using the nominal sensing matrix $\boldsymbol{A}$, while a good recovery of the original signal, as well as that of the perturbation parameter $\boldsymbol{\beta}^o$ on the support of $\boldsymbol{x}^o$, is obtained using our proposed P-BPDN. Note that $\boldsymbol{\beta}^o$ out of the support of $\boldsymbol{x}^o$ cannot be recovered as illustrated in Subsection 3.3.

4. CONCLUSION

This paper studied the CS problem subject to a structured matrix perturbation and measurement noise. It was shown that, as in the standard CS, sparse signals can be stably recovered in the perturbed CS by an ℓ_1 minimization approach incorporated with the perturbation structure. A general result for compressible signals was also reported. An alternating algorithm was proposed for the perturbed ℓ_1 minimization problem and a numerical simulation was presented to confirm our analysis.

In this paper we showed that the RIP condition for guaranteed stable recovery can be satisfied for a random perturbation and nominal sensing matrix that are mutually independent. One future work is to study practical situations where the perturbation may depend on the nominal sensing matrix.

5. REFERENCES

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