DYNAMIC SPARSE SUPPORT TRACKING WITH MULTIPLE MEASUREMENT VECTORS USING COMPRESSIVE MUSIC

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ABSTRACT

Dynamic tracking of sparse targets has been one of the important topics in array signal processing. Recently, compressed sensing (CS) approaches have been extensively investigated as a new tool for this problem using partial support information obtained by exploiting temporal redundancy. However, most of these approaches are formulated under single measurement vector compressed sensing (SMV-CS) framework, where the performance guarantees are only in a probabilistic manner. The main contribution of this paper is to allow *deterministic* tracking of time varying supports with multiple measurement vectors (MMV) by exploiting multi-sensor diversity. In particular, we show that a novel compressive MUSIC (CS-MUSIC) algorithm with optimized partial support selection not only allows removal of inaccurate portion of previous support estimation but also enables addition of newly emerged part of unknown support. Numerical results confirm the theory.

***Index Terms*—** Compressed sensing, joint sparsity, time varying signal, compressive MUSIC

1. INTRODUCTION

Dynamic target tracking has been one of the important classical topics in array signal processing with many applications. Recently, there exist renewed interests for this problem with the help of a modern mathematical tool called compressed sensing (CS) [1]. Consider the following time varying support estimation problem:

$$\min_{\mathbf{x}(t)} \|\mathbf{x}(t)\|_0, \ \text{subject to } \mathbf{b}(t) = A\mathbf{x}(t), \ \ t = 0, 1, \cdots, \quad (1)$$

where $\mathbf{b}(t) \in \mathbb{R}^m$, and $\mathbf{x}(t) \in \mathbb{R}^n$ are noiseless measurement vector, and sparse signal at time t. Assuming that the support changes slowly, theoretical results [2] indicates that we can reduce the required number of samples if we have partially known support estimated from the previous time. More specifically, let $k = |\text{supp}\mathbf{x}(t)|$, $u = |I(t) \setminus I(t-1)|$, and $e = |I(t-1) \setminus I(t)|$, where $I(t-1)$ and $I(t))$ denotes the previously estimated support and current one, respectly. Then, if the restricted isometry constant (RIP) for the sensing matrix A satisfies $\delta_{k+e+u} < 1$, then the solution $\mathbf{x}(t)$ of Eq. (1) is unique [2]. This is much weaker than $\delta_{2k} < 1$ for the original SMV-CS problem [1], in case of $u \ll k$ and $e \ll k$. They further showed an l_1 convex relaxation of Eq. (1) can provide the same l_0 solution of Eq. (1), if the following RIP condition is satisfied: $2\delta_{2u} + \delta_{3u} + \delta_{k+e-u} + \delta_{k+e}^2 + 2\delta_{k+e+u}^2 < 1$, which is also less stringent than that of original CS problem, $\delta_{2k} < \sqrt{2} - 1$ [1]. However, single measurement vector compressed sensing (SMV-CS) guarantees the support recovery only in a probabilistic sense [1]. In practise, there are many situations where we can obtain multiple measurement information for time varying objects. For example, in single-input multiple-output (SIMO) multiple access channel (MAC), multiple antenna can observe linear combination of individual codewords multiplied by the unknown channel gain from the individual user [3].

One of the main contributions of this paper is to show that a multiple measurement vector (MMV) framework provides a unique advantage of "deterministic" support tracking for slow varying support estimation. The breakthrough is based on our novel compressive MUSIC (CS-MUSIC) in MMV compressed sensing problem [4], in which part of supports are found probabilistically using CS, after which the remaining supports are determined deterministically using the generalized MUSIC criterion. In addition, CS-MUSIC allows us to find all k support as long as at least $k - r + 1$ support out of any k-support estimate are correct [5], where r denote the rank of the measurement matrix. This result can provide an important clue for deterministic and exact dynamic support tracking under MMV setup, in which the probabilistic CS support estimation step is replaced by the previous support estimation, after which the CS-MUSIC algorithm eliminates the incorrect portion of support and then add newly updated support deterministically. Other contributions of our deterministic support tracking algorithm include the support estimation error does not propagate along time due to the self-correction step and time varying sparsity level can be estimated. Furthermore, using large system model, we demonstrate that the algorithm can correctly track the time varying support even in noisy measurement cases.

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2. MMV-CS USING CS-MUSIC: A REVIEW

Let m, n and r be a positive integers ($m < n$) that represents the number of sensor elements, the ambient space dimension, and the number of snapshots, respectively. Suppose that we are given a multiple-measurement vector $B \in \mathbb{R}^{m \times r}$, $X = [\mathbf{x}_1, \cdots, \mathbf{x}_r] \in \mathbb{R}^{n \times r}$, and a sensing matrix $A \in \mathbb{R}^{m \times n}$. A canonical form MMV problem [4] is given by the following optimization problem:

$$\text{minimize} \quad \|X\|_0, \quad \text{subject to} \quad B = AX, \tag{2}$$

where $\|X\|_0 = |\text{supp}X| = k$, $\text{supp}X = \{1 \leq i \leq n : \mathbf{x}^i \neq 0\}$, and the measurement matrix B is full rank, i.e. $\text{rank}(B) = r \leq \|X\|_0$.

We can easily expect that the diversity due to the joint sparsity can improve the recovery performance over SMV compressed sensing. Indeed, Chen and Huo [6], and Feng and Bresler [7] showed that we can expect $\text{rank}(B)/2$ gains over SMV thanks to the MMV diversity. Furthermore, the noiseless l_0 bound is achievable using MUSIC [8] algorithm if $\text{rank}(B) = k$ [7]. However, for any $r < k$, the MUSIC condition does not hold. On the other hand, the conventional MMV-CS methods has a good recovery performance even if $r < k$, but becomes worse than MUSIC as $r \to k$. Recently, we show that this drawback of the conventional MUSIC and MMV-CS can be overcome by the following generalized MUSIC criterion [4].

Theorem 1. *[4] Assume that $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{n \times r}$, and $B \in \mathbb{R}^{m \times r}$ satisfy $AX = B$. Furthermore, we assume that $\|X\|_0 = k$ and A satisfies the RIP condition with the left RIP constant $0 \leq \delta^L_{2k-r+1} < 1$. If we are given $I_{k-r} \subset \text{supp}X$ with $|I_{k-r}| = k - r$ and $A_{I_{k-r}} \in \mathbb{R}^{m \times (k-r)}$, which consists of columns whose indices are in I_{k-r}, then for any $j \in \{1, \cdots, n\} \setminus I_{k-r}$, $\mathbf{a}_j^* \left[P_{R(Q)} - P_{R(P_{R(Q)}A_{I_{k-r}})}\right] \mathbf{a}_j = \mathbf{a}_j^* P^{\perp}_{R([A_{I_{k-r}} \ B])} \mathbf{a}_j = 0$ if and only if $j \in \text{supp}X$.*

In [4], we showed that the condition $0 \leq \delta^L_{2k-r+1}(A) < 1$ for generalized MUSIC is equivalent to l_0 bound of MMV problem, which implies that a computational expensive combinatorial optimization problem is now reduced to $|I_{k-r}|$ support estimation from the original $|I_k|$ support estimation. Furthermore, by Theorem 1, we can develop a computationally tractable relaxation algorithm called Compressive MUSIC (CS-MUSIC), which relaxed the combinatorial optimization step of findng I_{k-r} support using the conventional compressed sensing approaches [4]. More specifically, in compressive MUSIC, we determine $k - r$ indices of $\text{supp}X$ with CS-based algorithms such as 2-thresholding or S-OMP rather than l_0 optimization, where the exact identification of $k - r$ indices is a probabilistic matter. After that process, we recover remaining r indices of $\text{supp}X$ with a generalized MUSIC criterion, which is given in Theorem 1, and this reconstruction process is deterministic. This hybridization makes the compressive MUSIC applicable for all ranges of r, outperforming all the existing methods.

In the original form of the compressive MUSIC algorithm, the performance of the compressive MUSIC is very dependent on the selection of $k - r$ correct indices of the support of X. Even with significant improvement of CS-MUSIC, this is a very stringent condition. Hence, if we have a mean to identify $k - r$ correct support in any order out of any k-sparse, then we can expect that the performance of the compressive MUSIC will be improved. Indeed, the following support selection criterion can address the problem [5].

Theorem 2. *[5] Assume that we have a canonical MMV model $AX = B$ where $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{n \times r}$, $\|X\|_0 = k$ and $r < k < m < n$. If there is an index set $I_k \subset \{1, \cdots, n\}$ such that $|I_k| = \min\{k, \text{spark}(A) - r\}$ and $|I_k \cap \text{supp}X| \geq k - r + 1$, then for any $j \in I_k$, $j \in \text{supp}X$ if and only if*

$$\mathbf{a}_j^* P^{\perp}_{R([A_{I_k \setminus \{j\}} \ B])} \mathbf{a}_j = 0,$$

where $A_{I_k \setminus \{j\}}$ consists of columns of A whose index belongs to $I_k \setminus \{j\}$.

Theorem 2 informs us that we only require the success of partial support recover out of k-sparse estimate, rather than $k - r$ consecutive correct CS step [4]. In particular, if the columns of A are in general position, then we can take index set I_k with $|I_k| = \min\{k, m - r + 1\}$. Also, if A has an RIP condition with $0 \leq \delta_{2k}(A) < 1$, then we can take $|I_k| = k$ since $r \leq k$. Accordingly, the compressive MUSIC with optimized partial support is then performed by following procedure [5].

- **[Step 1: compressed sensing]** Estimate k indices of $\text{supp}X$ by any MMV compressive sensing algorithm. Let I_k be the set of indices which are taken in step 1.
- **[Step 2: support deletion]** For $j \in I_k$, calculate the quantities $\zeta(j) = \|P^{\perp}_{R([A_{I_k \setminus \{j\}} \ B])} \mathbf{a}_j\|^2$. Make an ascending ordering of $\zeta(j)$, $j \in I_k$ and choose indices that corresponds the first $k - r$ elements and put these indices into S and remove the remaining ones.
- **[Step 3: support addition]** For $j \in \{1, \cdots, n\} \setminus S$, calculate the quantities $\eta(j) = \mathbf{a}_j^* P^{\perp}_{R([A_{I_{k-r}} \ B])} \mathbf{a}_j$. Make an asending ordering of $\eta(j)$, $j \notin S$ and choose indices that correspond to the first r elements and put these indices into S.

3. DETERMINISTIC SUPPORT TRACKING USING COMPRESSIVE MUSIC

A time varying form noiseless MMV problem is given by set of MMV problem with time varying k-sparse vectors $X(t) \in \mathbb{R}^{n \times r}$ that satisfies $Y(t) = AX(t)$ as follows:

$$\min_{X(t)} \|X(t)\|_0, \quad \text{subject to} \quad Y(t) = AX(t), \quad t = 0, 1, \cdots. \tag{3}$$

Here, for the time varying cases, we assume $\|X(t)\|_0 = k(t)$, $\mathrm{supp}X(t) = \{1 \le i \le n : \mathbf{x}(t)^i \ne 0\}$, and the measurement matrix $Y(t)$ is full rank, with a fixed rank, i.e. $\mathrm{rank}(Y(t)) = r \le k(t)$. Furthermore, the nonzero rows of X are in general position [4]. Finally, we assume the following slowly varying sparsity condition for time varying support.

$$|\mathrm{supp}X(t) \setminus \mathrm{supp}X(t-1)| \le r-1, \qquad (4)$$

for all $t = 1, 2, \cdots$. Then, Theorem 3 shows that if we have a correct estimation for the initial support $I(0)$ of $X(0)$, then we can recursively identify the support of time-varying input signals in a *deterministic* manner.

Theorem 3. *Assume that we have a noiseless MMV problem for slowly time varying pattern which satisfies* (4) *and let* $k(t) = \|X(t)\|_0$ *for* $t = 0, 1, \cdots, N$. *Furthermore we assume that* $r \le k(t) \le k_{\max}$ *and* $0 \le \delta_{2k_{\max}}(A) < 1$. *Then, if we have a correct initial support estimation for* $X(0)$*, then we can identify the correct support for all* $t > 0$ *by applying the following procedure recursively:*

- **[Initial support estimation]** *Let* $I(t-1)$ *be the support estimation of* $X(t-1)$*;*
- **[Support deletion]** *Find an index set* $I(t)^a \subset I(t-1)$ *such that* $I(t)^a := \{j \in I(t-1) :$ $\mathbf{a}_j^* P^{\perp}_{R([Y(t)\ A_{I(t-1)\setminus\{j\}}])}\mathbf{a}_j = 0\}$;
- **[Support addition]** *Find an index set* $I(t) \supset I(t)^a$ *such that* $I(t) = \{j : \mathbf{a}_j^* P^{\perp}_{R([A_{I(t)^a}\ Y(t)])}\mathbf{a}_j = 0\}$.
- *Set* $\hat{k}(t) := |I(t)|$ *be the sparsity estimate for* $X(t)$ *and* $I(t)$ *be the support estimate for* $X(t)$.

Proof. See [9]. □

In the noisy case, when the sparsity are known *a priori* and does not change along time, we can apply the procedure which is given in [9]. If the sparsity changes along time, in the noisy cases, some of the steps in Theorem 3 should be modified as follows:

- **[Support deletion]** Set $\epsilon_1 > 0$ and find an index set $I(t)^a$ such that $I(t)^a = \{j \in I(t-1) :$ $\mathbf{a}_j^* P^{\perp}_{R([Y(t)\ A_{I_1(t)\setminus\{j\}}])}\mathbf{a}_j < \epsilon_1\}$, where $I_1(t) \subset I(t-1)$ such that $\mathrm{nrank}[Y(t)\ A_{I_1(t)}]$ is numerically full column rank.
- **[Support addition]** Set $\epsilon_2 > 0$ and find an index set $I(t)^b$ such that $I(t)^b = \{j \notin I(t)^a : \mathbf{a}_j^* P^{\perp}_{R([Y(t)\ A_{I_2(t)}])}\mathbf{a}_j$ $< \epsilon_2\}$, where an index set $I_2(t) \subset I(t)^a$ such that $[Y(t)\ A_{I_2(t)}]$ is numerically full column rank.

The following theorem gives us a sufficient condition for threshold values in the large system limit. In a large system noisy canonical MMV model [4], we assume that the additional conditions such that $A \in \mathbb{R}^{m\times n}$ is a random matrix with i.i.d. $\mathcal{N}(0, 1/m)$ entries and the noise $N(t)$ is independent from $X(t)$. Furthermore, we assume the following quantities exist: $\rho := \lim_{n\to\infty} m(n)/n > 0$, $\gamma := \lim_{n\to\infty} k_{\max}(n)/m(n) > 0$, $\alpha := \lim_{n\to\infty} r(n)/k_{\max}(n) \ge 0$, and $\alpha \le 1-\epsilon$ for some $0 < \epsilon < 1$.

Theorem 4. *Suppose a minimum* SNR *satisfies*

$$\mathsf{SNR}_{\min}(Y(t)) := \frac{\sigma_{\min}(B(t))}{\|N\|} > 1 + \frac{4(\kappa(B(t)) + 1)}{1-\gamma(1+\alpha)}, \qquad (5)$$

where $\sigma_{\min}(B(t))$ *is the minimum singular value for* $B(t)$, $\|N\|$ *is the spectral norm of* $N \in \mathbb{R}^{m\times r}$ *and* $B(t)$ *is the noiseless measurements. Furthermore, we assume that numerical rank estimations are correct. Then, for the noisy MMV problem for slowly time varying pattern such that* $|\mathrm{supp}X(t) \setminus \mathrm{supp}X(t-1)| \le r-1$*, the threshold values for support selection criterion and support addition are given by* $\epsilon_1 := (1-\gamma(1+\alpha))/2$ *and* $\epsilon_2 := (1-\gamma)/2$.

Proof. See [9]. □

4. NUMERICAL RESULTS

The first simulation is to demonstrate the performance of the proposed method to solve the time varying MMV problem in Eq. (3) for different number of supports changes at each time. We declared the algorithm as a success if the estimated support is the same as the true $\mathrm{supp}X$, and the success rates were averaged for 5000 experiments. The simulation parameters were as follows: $m = 40$, $n = 100$, $r = 9$, and $k \in \{1, 2, \cdots, 30\}$, respectively. Elements of sensing matrix A were generated by i.i.d. Gaussian random variable $\frac{1}{\sqrt{m}}\mathcal{N}(0, 1)$, and Gaussian noise of $\mathrm{SNR} = 40dB$ was added to each measurement vectors. At each time point, the nonzero part of $X(t)$ is generated by $\mathcal{N}(0, 1)$. Fig. 1 shows the recovery rates of time varying MMV problem using support tracking method for $t = 1, 2, \cdots, 5$ when the number of changed supports are 4 and 8. We used CS-MUSIC algorithm with S-OMP and then applied optimized partial support selection at $t = 1$, and time varying supports are estimated by support tracking method recursively from $t = 2$ to $t = 5$. An interesting observation is that the performance of the proposed method rather improves over time in Fig. 1(a). For maximally allowed sparsity change rate (in Fig. 1(b)), the recovery rate is deteriorated as time goes on, but the error does not propagate beyond certain time period.

Next, we applied the proposed algorithm to target tracking problem in $2D$ image and compared it to MUSIC algorithm. The first row of Fig. 2 indicates the original targets moving toward the direction of red arrows over time. The second and third row indicate the results of support tracking method and MUSIC algorithm, respectively. Each column (from left to right) indicates the sampled image at $t = 1, 13, 27$, and $t =$

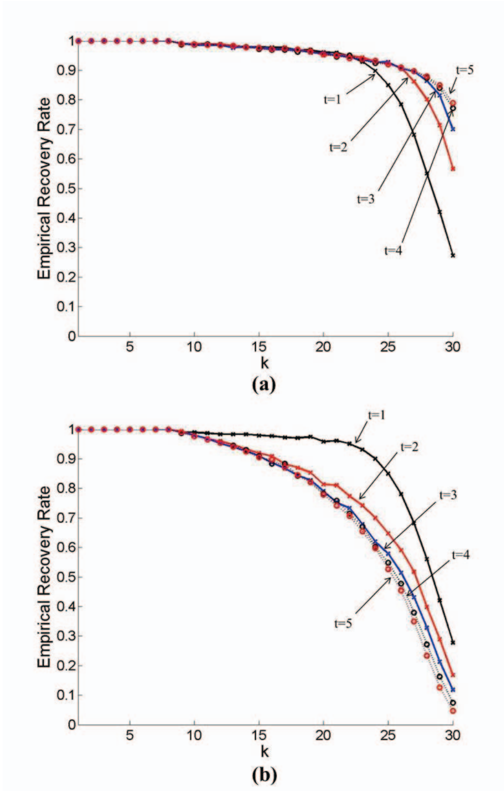

Fig. 1. Recovery rates of time varying MMV problem using support tracking method when $m = 40$, $n = 100$, $r = 9$, SNR$= 40$dB, and $t = 1, 2, \cdots, 5$. The number of changes in supports at each time point is (a) 4, (b) 8.

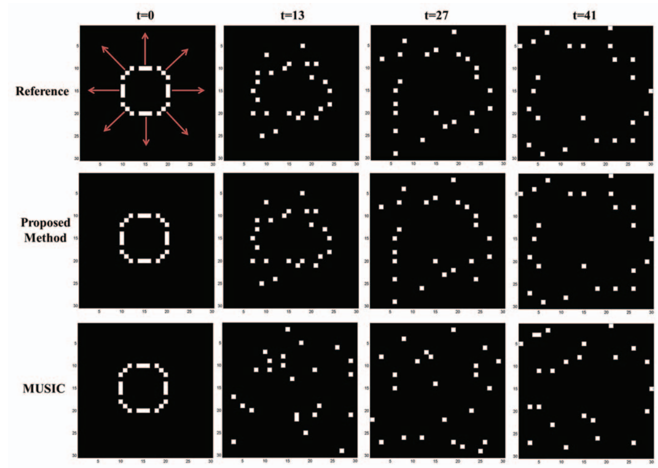

Fig. 2. Results of the target tracking problem in $2D$ image when $m = 50$, $n = 900$, $k = 24$, $r = 9$, and SNR$= 40$dB.

41, respectively. Note that the proposed method successfully follows the movement of original targets, whereas MUSIC fails.

5. CONCLUSION

This paper proposed a support tracking algorithm to recover the slowly time varying supports deterministically using multiple measurement vectors using CS-MUSIC algorithm. The incorrectly estimated part of supports at previous time can be removed using a support deletion criterion, after which a newly update part of support were estimated using support addition criterion by exploiting the generalized MUSIC criterion. Numerical results demonstrated that the proposed algorithm reliably reconstructs the time varying supports for various level of changes .

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