

Distributed Field Reconstruction With Model-Robust Basis Pursuit

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Abstract—We study the use of distributed average consensus and compressed sensing to perform decentralized estimation of a field measured by networked sensors. We examine field reconstruction of multiple acoustic sources from isotropic magnitude measurements. Compressed projections of global network observations are spread throughout the network using consensus, after which all nodes may invert the source field using ℓ_1 recovery methods. To approximate the problem as a discrete linear system, the space of source locations is quantized, introducing model error. We propose a model-robust adaptation to basis pursuit to control for the error arising from the spatial quantization. We show conditions for stability of the robust estimator, providing bounds on the reconstruction error based on perturbation constants, source magnitudes, and mutual coherence. Experiments show that the two types of robust estimators successfully address infeasibility and consistency issues that arise in basis pursuit for spatially quantized acoustic sources.

Index Terms—distributed estimation, consensus, compressed sensing, noise-aware basis pursuit, model robust estimation

I. INTRODUCTION

A major problem in the area of distributed sensor networks is the decentralized estimation of a physical field. Networks of coordinating cheap sensors are quick to deploy and can accomplish tasks that would challenge a single powerful, specialized sensor. However, fusing information in a large sensor network in ways that conserve energy, are robust to sensor failure, and avoid data and communication bottlenecks remains a challenging problem. Our goal is to study decentralized estimation methods that scale with increasing numbers of sensors for reconstructing fields generated by sparse underlying phenomena.

We pursue a method, studied in [1], [2], [3], that uses consensus to distribute low dimensional projections of global sensor observations throughout the network. After a phase of iterative local message passing, each node has a copy of the compressed observation vector and can invert the relationship to estimate the sparse field vector. We call this approach field inversion by consensus and compressed sensing (FICCS). We test these ideas on the application of localizing multiple concurrent acoustic sources observed by distributed isotropic sensors with simplified physical models. This example is illustrative due to its low single-sensor observability, necessitating coordination among the sensors to achieve acceptable

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reconstruction and detection performance.

This approach to localizing acoustic sources differs from other decentralized approaches, e.g., [4], [5], since it handles more than one concurrent source in the field with the objective of detecting sources at quantized locations. In order to apply compressed sensing techniques to reconstruct the acoustic field, we approximate the observation model by quantizing the spatial dimensions and choose consensus projection weights for incoherence with the acoustic field.

The process of quantizing the source locations introduces error into the observation model. In practice, sparse estimators can often handle these approximations. The focus of this work is to study the effects of the state quantization error on the performance of basis pursuit. We propose a variation of basis pursuit that controls for these errors and characterizes the conditions for stable field estimation and source detection. In the related work of [6], the stability of basis pursuit under model perturbation is shown using restricted isometry. Our analysis differs by 1) proposing a new sparse estimator; and 2) analyzing stability in terms of mutual coherence, following [7]. This also differs from the work of [8], which formulates a nonconvex approach to controlling for model error.

The organization of the remaining sections is as follows. Section II defines the acoustic field inversion problem and linear approximation of observations. Section III details the FICCS approach. Section IV presents the proposed variation of basis pursuit for controlling spatial quantization error. Section V presents comparisons of sparse estimators in a numerical study and section VI concludes the paper.

II. PROBLEM FORMULATION

Our physical model for acoustic source localization consists of a finite unknown number of S point-sources and N sensors in a square two-dimensional (2D) area. Sensors measure the superposition of propagated wavefield intensities due to the multiple sources. Our propagation model assumes a simplified free space inverse power law propagation model, given below.

$$\psi_{\mathbf{p}_i}(\mathbf{l}_j) = \frac{1}{\|\mathbf{l}_j - \mathbf{p}_i\|_2^\gamma + 1} \quad (1)$$

where $\psi_{\mathbf{p}_i}(\mathbf{l}_j)$ is the measured acoustic field at sensor j due to a unit-magnitude source, \mathbf{l}_j and \mathbf{p}_i are 2D positions of sensor j and source i , respectively, and $\gamma \geq 1$ is a known parameter, taken to be 2.

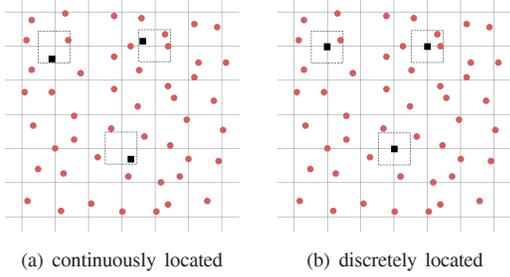


Fig. 1. Example field with three continuously located sources (a) and the approximation to a discrete lattice (b). Solid black squares are sources, red circles are sensors, and dashed squares are the regions of each source.

The continuous space of potential source locations is discretized using a finite $\sqrt{L} \times \sqrt{L}$ lattice, and a spacing of δ between lattice points, yielding a discrete set of potential source locations \mathbf{p}_l for $l = 1, 2, \dots, L$. Quantizing the spatial dimensions of the field may be equivalently thought of as sampling with an accumulator, such that each of the L discrete locations represents the sum of source magnitudes in the square region given by $\mathbf{t} \in \mathbb{R}^2$, with $p_i(1) - \frac{\delta}{2} < t_i \leq p_i(1) + \frac{\delta}{2}$, for $i = 1, 2$. The summed magnitudes of sources in each region are collected into the vector $\mathbf{v} \in \mathbb{R}^L$. We are interested in estimating this discrete sparse source vector, noting that multiple sources in the same region will be considered one source. Figure 1 shows the quantization process in an example field with 3 sources observed by randomly placed sensors. Each source is associated with the region of just one of the lattice points. The right panel shows the discrete locations used to approximate the measured fields of the continuously located sources. Modeling sources as a discrete vector \mathbf{v} , we may approximate the noisy vector of all sensor observations $\mathbf{x} \in \mathbb{R}^N$ as a linear mapping.

$$\mathbf{x} = \Psi \mathbf{v} + \mathbf{n}, \Psi \in \mathbb{R}^{N \times L} : \Psi = [\psi_{\mathbf{p}_l}(\mathbf{l}_j)]_{j,l}, \mathbf{n} \in \mathbb{R}^N \quad (2)$$

The field inversion problem considered becomes the estimation of an unknown discrete vector that can be described as a sparse linear combination of parameters, using observations collected by a network of sensors with a known linear model, $\Psi : \mathbb{R}^L \rightarrow \mathbb{R}^N$. By construction, the source vector is sparse; i.e., $\|\mathbf{v}\|_0 \leq S$, where $\|\cdot\|_0$ is called the ℓ_0 quasi-norm and counts the number of nonzero entries. The next section details the FICCS distributed network estimation algorithm, while section IV addresses model perturbation error.

III. FIELD INVERSION BY CONSENSUS AND COMPRESSED SENSING (FICCS)

The FICCS approach for distributed estimation of a sparse field over a network of sensors uses three phases; 1) Network Set-up; 2) Sensing and Collaboration; and 3) Field Inversion.

Network Set-up First, localize the N deployed sensors. An example decentralized method for this is given in [9]. Once sensors are localized, a discrete 2D lattice is imposed on the monitored area, with dimension chosen so the spacing δ is small enough for the discrete approximation to be stable. Construct the linear observation matrix, Ψ , and the set of links

\mathcal{E} describing pairs of communicating nodes, determined by a uniform sensor communication radius. We require that the communication graph is connected and bidirectional. Using the graph structure, each node associates a weight for itself and each of its neighbors. Collected together in a matrix, the set of consensus weights has the structure

$$W \in \mathbb{R}^{N \times N} : w_{i,j} = 0 \text{ if } (i,j) \notin \mathcal{E} \text{ and } i \neq j \quad (3)$$

chosen such that iterations of weighted combinations of each sensor's value with its neighbors converges to the average of all N sensor values, [10]. Each node is also assigned a set of m projection weights, chosen to allow for compressed sensing recovery of sparse source vectors. The collection of all projection weights forms the matrix $\Phi \in \mathbb{R}^{m \times N}$; each column of which is assigned to a different node. A number of methods for choosing Φ exist, e.g., random Gaussian weights or random orthonormal rows, and using these leads to a required scaling of $m \sim O(S \log(L))$, [11]. Performance of Φ for compressed sensing can be measured by the mutual coherence, defined in [7], [12] for $\Phi \Psi = A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_N]$, as

$$\mu(A) = \max_{i \neq j} \frac{|\mathbf{a}_i^T \mathbf{a}_j|}{\|\mathbf{a}_i\|_2 \cdot \|\mathbf{a}_j\|_2} \quad (4)$$

Sensing Mode The sensors collect noisy measurements of a static or concurrently observed field, \mathbf{x} , defined in (2). Using distributed average consensus, they compute an m -dimensional projection of the global observations \mathbf{x} . To do this, each sensor i initializes its m dimensional local state to $\mathbf{s}_i[0] = (N x_i) \otimes \phi_i$, where ϕ_i is the i^{th} column of the projection matrix Φ and \otimes is the Kronecker product. Sensors iteratively exchange messages with the neighboring nodes using the weights of W to update m states in parallel, as

$$\mathbf{s}_i[k+1] = w_{i,i} \mathbf{s}_i[k] + \sum_{j:(i,j) \in \mathcal{E}} w_{i,j} \mathbf{s}_j[k] \quad (5)$$

Because $\lim_{k \rightarrow \infty} W^k = \frac{\mathbf{1}\mathbf{1}^T}{N}$, the recursion of (5) converges to $\Phi \mathbf{x}$, [3]. For k large enough, all nodes have approximately the same projection $\mathbf{y} = \Phi \mathbf{x}$. In practice, sensors will stop at a fixed number agreed upon during set-up. We denote the exact projection matrix of node i as $\Phi_{\text{eff}(i)}$, which is known to each node. This yields the observation equation at each node,

$$\mathbf{y}_i = \Phi_{\text{eff}(i)} \mathbf{x} = \Phi_{\text{eff}(i)} (\Psi \mathbf{v} + \mathbf{n}) = A_{\text{eff}(i)} \mathbf{v} + \mathbf{z} \quad (6)$$

Asynchronous gossip methods can also be used, as in [13], [2]. In grid or random geometric communication graphs, the lowest possible growth in the number of messages required for convergence is at least linear in the number of nodes, [13], [14]. We now bound the scaling of messages for FICCS, since the number of projections grows as $S \log(L)$, and we choose $N \sim L$, the overall growth of messages with sensors is $O(SN \log(N))$.

Field Inversion Local sensors may reconstruct estimates of the global field using the observations given by (6). We bound the norm of the additive Gaussian noise \mathbf{z} by ϵ such that $P(\|\mathbf{z}\|_2 < \epsilon) = 0.99$. We focus on two methods for compressed sensing recovery, studied in [7] and [12]. The first

is called noise-aware basis pursuit (BP), [7], which minimizes the ℓ_1 -norm, a convex sparsity metric, of the source vector subject to the noise norm bound.

$$(P_{1,\epsilon}) \quad \begin{array}{ll} \text{minimize} & \|\mathbf{v}\|_1 \\ \text{subject to} & \|\mathbf{y} - A\mathbf{v}\|_2 \leq \epsilon \end{array} \quad (7)$$

The second is a penalized least-squares estimator (L1P), studied in [12] among others, with a tradeoff parameter λ .

$$\ell_1 \text{ penalty:} \quad \text{minimize} \quad \|\mathbf{v}\|_1 + \frac{\lambda}{2} \|\mathbf{y} - A\mathbf{v}\|_2^2 \quad (8)$$

These estimators are convex programs with efficient open source solvers. For the next section, we propose two sparse estimators designed to take quantization error into account.

IV. MODEL ROBUST BASIS PURSUIT (MRBP)

The observation models, (2) and (6), are a linear approximation of the sensor measurements. The true model can be represented by a perturbed system,

$$\mathbf{y} = \Phi\mathbf{v} + \mathbf{z} = \Phi(\Psi_0 + E)\mathbf{v} + \mathbf{z} \quad (9)$$

where the matrix $\Psi \in \mathbb{R}^{L \times N}$ represents the actual observation system with each column corresponding to the sampled propagated field due to the true source location, whereas Ψ_0 is our assumed model using quantized source locations, and E is the matrix of perturbations to the assumed model. Accounting for model error, the basis pursuit method for reconstructing sparse solutions gives an altered fit constraint.

$$(P_{1,\epsilon}) \quad \begin{array}{ll} \text{minimize} & \|\mathbf{v}\|_1 \\ \text{subject to} & \|\mathbf{y} - \Phi\Psi_0\mathbf{v} - \Phi E\mathbf{v}\|_2 \leq \epsilon \end{array} \quad (10)$$

We do not know *a priori* the particular E that will be observed, and jointly minimizing over both E and \mathbf{v} is a nonconvex problem, as treated in [8]. Instead, we bound the multiplicative error vector, noting that \mathbf{v} is sparse. Columns of the perturbation matrix, $E = [\mathbf{e}_1 \ \cdots \ \mathbf{e}_N]$, are composed of differences between sensor measurements of a source at the lattice point as compared with the source shifted in each of the spatial dimensions by up to $\delta/2$. The error term $\Phi E\mathbf{v}$ can be decomposed as a sum over support indices T ; as in $E\mathbf{v} = \sum_{i \in T} \Phi \mathbf{e}_i v_i$. The difference vectors $\Phi \mathbf{e}_i \in \mathbb{R}^N$ are drawn from the set of allowed column perturbations, which we denote Δ_i , defined as

$$\Delta_i = \left\{ \Phi \mathbf{e} : \begin{array}{l} e_j = \psi_{\mathbf{p}_i + \mathbf{t}}(\mathbf{l}_j) - \psi_{\mathbf{p}_i}(\mathbf{l}_j), \\ 1 \leq j \leq N, \ t_1, t_2 \in (-\delta/2, \delta/2) \end{array} \right\} \quad (11)$$

where \mathbf{l}_j , \mathbf{p}_i , and $\mathbf{p}_i + \mathbf{t}$ are the 2D positions of sensor j , lattice point i , and the actual source location, respectively. These model perturbations depend both on the propagation model as well as the projection matrix Φ .

Let $g_i = \max_{\mathbf{e} \in \Delta_i} \|\Phi \mathbf{e}\|_2$ be an upper bound on the norm of any vector from Δ_i , and $\mathbf{g} = [g_i]_{i=1}^N$. These maxima are achieved at the extrema of the source regions, and so may be efficiently computed during network set-up. From these column-norm upperbounds, we can now bound the overall fit error, $\mathbf{f} = \Phi E\mathbf{v} + \mathbf{z}$, by

$$\|\mathbf{f}\|_2 \leq \sum_{i \in T} |v_i| g_i + \epsilon \quad (12)$$

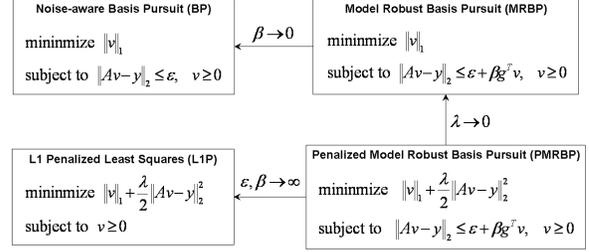


Fig. 2. Relationship of the compared sparse recovery methods

to expand the fit constraint of (10). We propose the following model-robust formulation of basis pursuit (MRBP),

$$(P_{1,\epsilon,\mathbf{g}}) \quad \begin{array}{ll} \text{minimize} & \|\mathbf{v}\|_1 \\ \text{subject to} & \|\mathbf{y} - \Phi\Psi_0(\mathbf{v}_+ - \mathbf{v}_-)\|_2 \leq \epsilon + \sum_{i \in T} |v_i| g_i \end{array} \quad (13)$$

with a version for nonnegative source magnitudes,

$$(P_{1,\epsilon,\mathbf{g}}^+) \quad \begin{array}{ll} \text{minimize} & \mathbf{1}^T \mathbf{v} \\ \text{subject to} & \|\mathbf{y} - A_0 \mathbf{v}\|_2 \leq \epsilon + \mathbf{g}^T \mathbf{v}, \mathbf{v} \geq \mathbf{0} \end{array} \quad (14)$$

We recognize $(P_{1,\epsilon,\mathbf{g}})$ and $(P_{1,\epsilon,\mathbf{g}}^+)$ as second order cone programs, where the cone constraint has also a linear term.

Theorem 1 For a true vector, \mathbf{v}_0 satisfying $\|\mathbf{v}\|_0 \leq S$ and $\max_i |v_i| \leq v_{\max}$; and system $\mathbf{y} = \Phi(\Psi_0 + E)\mathbf{v}_0 + \mathbf{z}$, with coherence $\mu(\Phi\Psi_0)$ given by (4), E satisfying $\max_i \|\Phi \mathbf{e}_i\|_2 \leq g_{\max}$, and $\|\mathbf{z}\|_2 \leq \epsilon$, then for $S < \frac{1}{4} \left(\frac{1}{\mu} + 1 \right)$, the estimation errors of (13) and (14), are upperbounded by

$$\|\hat{\mathbf{v}} - \mathbf{v}_0\|_2 \leq \frac{2(\epsilon + S \cdot g_{\max} v_{\max})}{\sqrt{1 - \mu(4S - 1)}} \quad (15)$$

The proof, which follows from a modification of the stability analysis of BP given by Donoho, Elad, Temlyakov in [7], is in an extended companion to this paper [15]. The enlarged cone constraint of (14) can significantly reduce the estimation performance of MRBP, so we also add an ℓ_2 fit penalty to the objective function, calling this penalized model-robust basis pursuit (PMRBP). Figure 2 shows the comparison of sparse estimators, with parameter $\beta \in [0, 1]$ relating the four methods.

V. NUMERICAL EXPERIMENTS

To test and compare the four estimators of fig. 2, we simulate a network of 150 sensors placed randomly, ensuring a minimum intersensor distance of $1/3$, to observe the square area covered by a 10-by-10 lattice of discrete source positions with regular spacing of $\delta = 2$. We also compare the same deployment with a 30-by-30 lattice, upsampled by 3, spaced by $\delta = 2/3$. Each of 50 trials place 3 sources of magnitude 15 uniformly at random in the area. Measurement noise is modeled as independent, Gaussian with variance 0.001. Projection weights for Φ of (6), are chosen as rows of a random orthonormal matrix. The system coherence and maximum perturbation constants are shown as a function of the number of network projections m in fig. 3.

We study the ℓ_2 estimation error and detection performance of estimates computed using the `cvx` software package, [16]. When computing error and detections for estimates on the

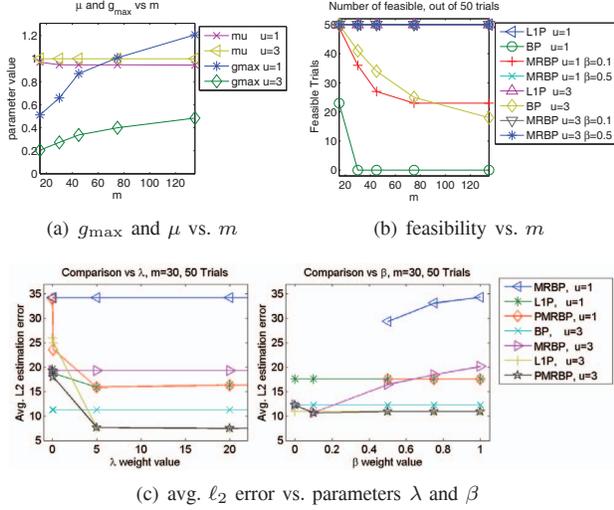


Fig. 3. FICCS experiments. (a) shows coherence μ and maximum perturbation constant g_{\max} versus number of projections m and two lattice sampling rates, $u = 1, 3$ corresponding to $\delta = 2, 2/3$. (b) the number of feasible trials for each estimator versus m . (c) shows average ℓ_2 estimation error versus λ (on the left) and versus β (on the right) for fixed $m = 30$.

upsampled lattice, we quantize the upsampled $\hat{v} \in \mathbb{R}^{900}$ to a coarse $\hat{v} \in \mathbb{R}^{100}$ to make the error metrics comparable. We study average ℓ_2 error versus the number of network projections m and parameters λ and β . The number of feasible trials is shown for $\lambda = 1$ and small β in fig. 3 panel (b), showing that there are significant problems with infeasibility using BP, whereas MRBP is feasible for $\beta \geq 0.5$. The left side of panel (c) shows the average error of estimates, for a fixed m and $\beta = 1$, as a function of the fit tradeoff parameter λ . In the coarse lattice, denoted by $u = 1$, the ℓ_1 penalty (L1P) estimator performed the best over a range of λ , while BP had no feasible trials. In the fine lattice, for small λ , PMRBP performs a little better than L1P. For moderate λ , both outperform the BP estimates. The errors of MRBP are consistently high motivating the use of penalized estimators. Additionally, as β increases MRBP error grows, while L1P and PMRBP programs maintain consistently low error. Each of these studies supports the use of the fine lattice, which has much lower constants g , but higher coherence μ .

Figure 4 shows that detection rates increased with m for all estimators except BP, which performed poorly due to increasing model error with m . This inconsistent behavior is not displayed by the robust estimators, MRBP and PRMBP. The low value of $\lambda = 0.1$ led to poor detection rates using L1P compared with PRMBP, indicating better robustness of PMRBP to the choice of λ . For moderate values of λ and $m \geq 30$, the estimates of PRMBP and L1P performed well and matched each other, achieving almost 100% detection rates with the average number of false alarms below 1.5.

VI. CONCLUSIONS

We described a distributed algorithm to invert a sparse field of acoustic sources discretized to a finite lattice. The algorithm

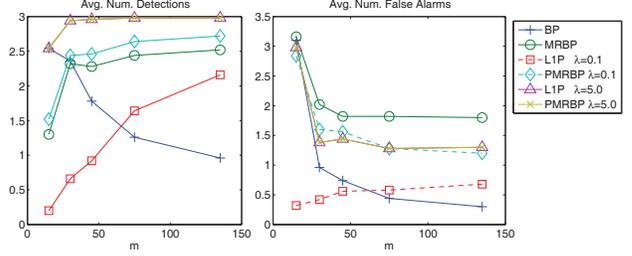


Fig. 4. Average number of sources detected (left), out of 3, and false alarms (right) with threshold 1, $\beta = 1$, for the sparse estimators, using quantized estimates from the $u = 3$ lattice vs. m . Penalty methods compare $\lambda = 0.1, 5$.

combines ideas from compressed sensing and consensus to achieve scalable decentralized network estimation. We propose a method for controlling for the induced modeling errors, called model-robust basis pursuit and derived stability conditions for the estimates. We found that the robust estimators addressed feasibility and consistency issues of basis pursuit, and noted the need for an ℓ_2 penalty in the objective of MRBP to overcome the expansion of the cone constraint.

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