# ASYMPTOTIC RESULTS FOR PASSIVE WAVEFRONT CURVATURE RANGING USING A LARGE-SCALE DISTRIBUTED ARRAY SYSTEM *

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## ABSTRACT

In this work, we present some asymptotic results on the maximum likelihood multi-rank processor for passive wavefront curvature ranging systems using large-scale distributed arrays. We assume that the operation environment for the distributed arrays is subject to a spatial coherence loss. Under an exponential coherence model, analytical expressions for the multi-rank combiners are derived. The results provide a simple guideline for choosing inter-module spacing according to the spatial coherence length for formulating multi-rank mode filters and weights used in the combiner. The general framework and the numerical procedures of designing a multi-rank combiner can be applied to other coherence models.

***Index Terms*—** Asymptotic Results, Large-Scale Array of Arrays, Passive Source Localization

## 1. INTRODUCTION

In many passive sensing systems [1, 2] used for surveillance and environmental monitoring, a large number of distributed arrays or sub-arrays are used to form a network of arrays for detection, ranging and tracking on the sources of interest. The collection of distributed arrays forms a large-aperture composite array system or network, which enables advanced signal processing techniques to deliver high resolution and robust solutions to many challenging problems. However, when a large aperture array is deployed for underwater acoustic applications, the signal wavefronts may experience different levels of coherence loss spatially [3, 4, 5, 6]. In [7], we developed a multi-rank maximum likelihood solution to passive ranging using wavefront curvature (WFC) sensed by an array of three modular arrays in environments subject to signal coherence loss. The key components in the multi-rank processor are a bank of multi-mode eigen-filters and a combiner. Built from the knowledge of spatial coherence model, such components further process the beamformed outputs from small-size modular arrays for passive ranging. In certain applications, there are a large number of distributed sub-arrays made available for WFC ranging. Under environments subject to spatial coherence losses, the sub-arrays can be defined as a set of receiving elements with high levels of spatial coherence. In operating a large-scale multi-module array system, there comes a need for a fast design guideline on how to choose the inter-module spacings, the multi-rank filters and the corresponding multi-rank combining coefficients, given the spatial coherence model, without resorting to the real-time large-size eigen-analysis. This work extends our previous research on passive WFC ranging using a towed arrays with three spatially separated modules [7] to a *large-scale* distributed sensing system (see Fig.1). We develop some interesting asymptotic results for a multi-rank processor in a general passive WFC ranging system equipped with a large number of distributed hydrophone modules.

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**Fig. 1**. A schematic picture of a large-scale passive wavefront curvature ranging system.

## 2. DATA MODEL AND SPATIAL COHERENCE

In our analysis, it is assumed that each array module contains a fixed $N_t$ number of hydrophones, while the spacing between adjacent array modules is a fixed constant $L_t$. For a large-scale array consisting of $L$ array modules, the total data from all the hydrophones can be concatenated in a long $N = N_t L$ dimensional vector. For a frequency $f$ under consideration, the received data can be modelled as,

$$\mathbf{d}(t,f) = \begin{bmatrix} \mathbf{d}_1(t,f) \\ \vdots \\ \mathbf{d}_L(t,f) \end{bmatrix} = \sigma_s \cdot \mathbf{S}_{st}(r,\theta,f) \cdot \mathbf{I}_\rho(t) + \mathbf{n}(t). \tag{1}$$

Here, the $L \times 1$ signal wavefront appearing on $L$ sub-arrays, $\mathbf{I}_\rho(t)$, is assumed to be complex Gaussian distributed with zero-mean and a spatial coherence matrix $\mathbf{R}_\rho = E\left\{\mathbf{I}_\rho(t)\mathbf{I}_\rho^H(t)\right\}$. The $L$ module sub-arrays' steering vectors ($k = 1, 2, \ldots, L$),

$$\mathbf{s}_k(r,\theta,f) = \begin{bmatrix} \exp\left\{-j2\pi f \frac{\|\mathbf{p}_{k,n} - \mathbf{p}_s(r,\theta)\|}{c}\right\} \\ \downarrow \; n = 1, 2, \ldots, N_t \end{bmatrix}_{N_t \times 1},$$

form the whole array's steering matrix as follows,

$$\mathbf{S}_{st}(r,\theta,f)=\begin{bmatrix} \mathbf{s}_1(r,\theta,f) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_2(r,\theta,f) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{s}_L(r,\theta,f) \end{bmatrix}_{N\times L}$$

In a 2-D coordinate, the source position vector, $\mathbf{p}_s(r,\theta) = [\,r\cos\theta \quad r\sin\theta\,]^T$ is parameterized by source's range $r$ and bearing $\theta$; the position vector $\mathbf{p}_{k,n}$ contains a module-array's element position; and $c$ is the sound speed.

As will become clear later in section 4, the spatial coherence matrix $\mathbf{R}_\rho$ plays an important role in analytically formulating the multi-rank solutions (coherent or non-coherent) to passive ranging problem. Let us now focus on the eigen-analysis of $\mathbf{R}_\rho$. Using a commonly adopted exponential model for spatial coherence in underwater environment [3, 4, 6], the coherence matrix among all $L$ array modules ($L_t$-spaced) can be found having the Toeplitz form of,

$$\begin{aligned} \mathbf{R}_\rho &= \begin{bmatrix} 1 & \rho & \cdots & \rho^{(L-1)^2} \\ \rho & 1 & \cdots & \rho^{(L-2)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{(L-1)^2} & \rho^{(L-2)^2} & \cdots & 1 \end{bmatrix} \qquad (2) \\ &= Toeplitz([1,\rho,\rho^4,\ldots,\rho^{(L-1)^2}]), \end{aligned}$$

where the parameter $\rho = \exp\{-2(L_t/L_{coh})^2\}$ is a function of $L_{coh}$ the coherence length of the wavefield, and $L_t$ the spacing between centers of adjacent array modules. Recalling the Wiener Khichine Theorem that connects a random sequence's auto-correlation sequence $r_\rho(k)$ to its power spectral density (PSD) $P_\rho(\nu)$ through a Fourier transform pair,

$$r_\rho(k) = \int_{-1/2}^{+1/2} P_\rho(\nu) e^{j2\pi\nu k} d\nu,$$

we can rewrite the coherence matrix in eq. (2) precisely as,

$$\mathbf{R}_\rho = \int_{-1/2}^{+1/2} \mathbf{e}(\nu) P_\rho(\nu) \mathbf{e}^H(\nu) d\nu.$$

Here the vector $\mathbf{e}(\nu) = \begin{bmatrix} 1 & e^{j2\pi\nu} & \cdots & e^{j2\pi(L-1)\nu} \end{bmatrix}^T$ is simply the well known discrete-time Fourier transform vector.

In a large-scale distributed array system, where the number of module $L$ is large enough, we can approximate the eigen-decomposition of the coherence matrix using the following asymptotic spectral decomposition,

$$\mathbf{R}_\rho = \sum_{i=1}^{L} \lambda_i \mathbf{v}_i\, \mathbf{v}_i^H \approx \frac{1}{L} \sum_{i=1}^{L} P_\rho(\nu_i) \mathbf{e}(\nu_i)\, \mathbf{e}^H(\nu_i). \qquad (3)$$

For large $L$, the eigen-mode $\mathbf{v}_i \approx \frac{1}{\sqrt{L}} \mathbf{e}(\nu_i)$ becomes a normalized DFT vector and the eigenvalue $\lambda_i \approx P_\rho(\nu_i)$ becomes the spatial PSD, each being evaluated at the DFT bin $\nu_i = (i-1)/L, (i = 1, 2, \ldots, L)$. The importance of the results is that given a pre-selected spatial coherence model and the system parameters of a large-scale array network, the asymptotic eigen-analysis can be pre-calculated either analytically or numerically, and used for building the multi-rank processor for passive WFC ranging application. It can be seen in the sequel, the proposed general framework of using these simple asymptotic eigen-results in designing a multi-rank processor for a large-scale array network applies to any spatial coherence model of one's choice.

## 3. EFFECTIVENESS OF ASYMPTOTIC RESULTS

To illustrate the idea and the effectiveness of resulting solution, let us focus back on to the commonly adopted spatial coherence model, the exponential module used in the matrix $\mathbf{R}_\rho$ of eq. (2). In this case, the spatial PSD takes a Gaussian shape. Specifically, defining $\sigma_\rho^2 = L_{coh}^2/(2L_t^2)$, we can come up with the analytical result for the spatial PSD,

$$P_\rho(\nu) = \sqrt{\pi}\sigma_\rho \exp -(\pi\sigma_\rho\nu)^2,$$

under the condition of no significant spectral aliasing (due to $L_t$-spacing of $L$ module arrays). Using the $3\sigma$ rule (a $99.7\%$ CI), this translates into a requirement on the inter-module spacing, which is $L_t \le \frac{\pi}{6} L_{coh} \approx 0.5236 L_{coh}$. Alternatively, a $2\sigma$ (a $95.5\%$ CI) rule translates into a requirement on the inter-module spacing, which is $L_t \le \frac{\pi}{4} L_{coh} \approx 0.7854 L_{coh}$.

In the Fig.2 and Fig.3, we illustrate the effectiveness of the asymptotic results, for different choices of inter-module spacings ($L_t = 0.25 L_{coh}$ and $L_t = 0.5 L_{coh}$), in approximating the eigen-modes of a distributed array system with $L = 20$ and $L = 10$ modular arrays, respectively. One can see that for a smooth Gaussian-shaped PSD, the approximation in eq.(3) holds well even for a reasonable value of $L = 10$.

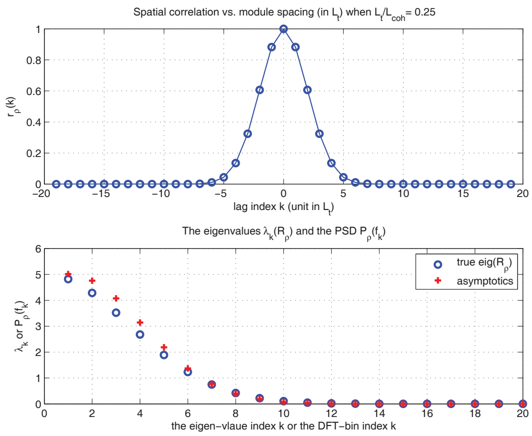
## 4. MULTI-RANK PROCESSORS

In this section, we show that incorporating the asymptotic results on the spatial coherence matrix leads to a conceptually simple multi-stage procedure for passive WFC ranging. Under Gaussian assumption on the signal wavefront impinging on array in an additive white Gaussian noise field, $\mathbf{n}(t) \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2 \mathbf{I}_{N\times N})$, all the information about the source of interest (bearing and range) are contained in the second-order data statistics. The correlation matrix for data in (1) becomes,

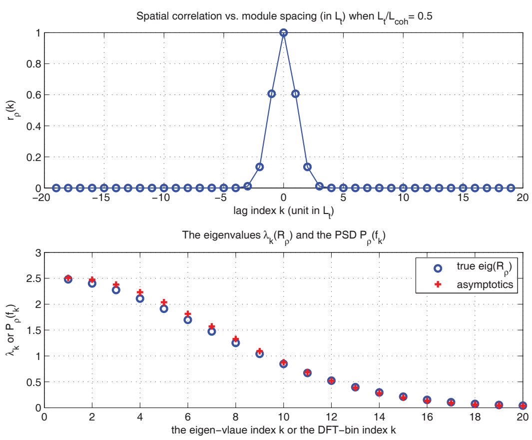
$$\mathbf{R}_d \;= \sigma_s^2 \cdot \mathbf{S}_{st}(r,\theta,f) \cdot \mathbf{R}_\rho \cdot \mathbf{S}_{st}^H(r,\theta,f) + \sigma_n^2 \mathbf{I}_{N\times N}$$

Extending the results in [7] to an array system with $L$ identical modular arrays, the maximum likelihood estimate for source's range and bearing can be found by scanning for the maximum of the objective function $J_{ML}(r,\theta)$ and its asymptotic approximation, as follows,

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \arg\max_{r,\theta} J_{ML}(r,\theta), \qquad (4)$$

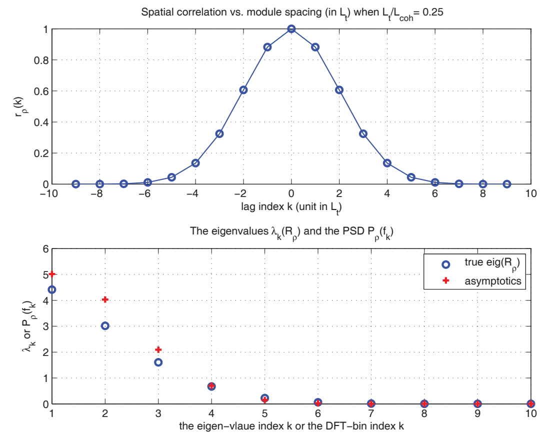



(a) inter-module spacing $L_t = 0.25L_{coh}$.

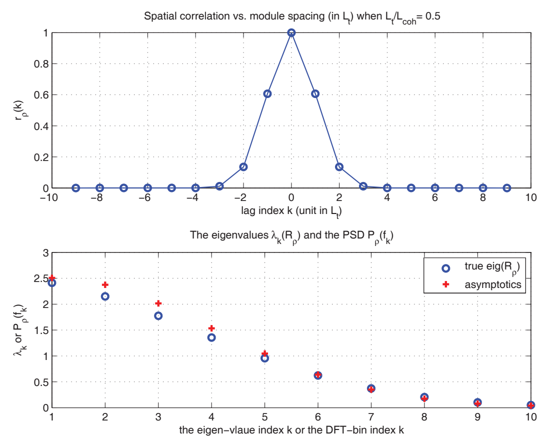



(b) inter-module spacing $L_t = 0.5L_{coh}$.

**Fig. 2**. Effectiveness of the asymptotic results in approximating the eigen-results in a 20-module array system. Parameters in use: number of modular arrays $L = 20$, in (a) inter-module spacing $L_t = 0.25L_{coh}$, in (b) inter-module spacing $L_t = 0.5L_{coh}$.




(a) inter-module spacing $L_t = 0.25L_{coh}$.




(b) inter-module spacing $L_t = 0.5L_{coh}$.

**Fig. 3**. Effectiveness of the asymptotic results in approximating the eigen-results in a 10-module array system. Parameters in use: number of modular arrays $L = 10$, in (a) inter-module spacing $L_t = 0.25L_{coh}$, in (b) inter-module spacing $L_t = 0.5L_{coh}$.

with

$$\begin{aligned} J_{ML}(r,\theta) &= \sum_{i=1}^{L} \frac{\lambda_i \, \eta_{SNR}}{1+\lambda_i \, \eta_{SNR}} \left|\mathbf{v}_i^H \mathbf{y}(t)\right|^2 \\ &\approx \sum_{i=1}^{L} \frac{P_\rho(\nu_i) \, \eta_{SNR}}{1+P_\rho(\nu_i) \, \eta_{SNR}} \left|\mathbf{e}^H(\nu_i)\mathbf{y}(t)\right|^2 . \end{aligned} \tag{5}$$

Here, SNR is defined as $\eta_{SNR} = \sigma_s^2/\sigma_n^2$; the $L \times 1$ vector $\mathbf{y}(t) = \mathbf{y}(t; r, \theta, f)$ is a collection of beamformed data from all modular arrays. Specifically, we have

$$\mathbf{y}(t; r,\theta, f) = \mathbf{S}_{st}^H(r,\theta,f)\mathbf{d}(t) = \begin{bmatrix} \mathbf{s}_1^H(r,\theta,f)\cdot \mathbf{d}_1(t) \\ \vdots \\ \mathbf{s}_L^H(r,\theta,f)\cdot \mathbf{d}_L(t) \end{bmatrix},$$

with $y_k(t) = y_k(t; r, \theta, f) = \mathbf{s}_k^H(r,\theta,f)\mathbf{d}_k(t,f)$ being the output from each of the sub-array processing module.

### 4.1. Interpretation on the multi-rank processor based on asymptotic results

From eqs.(4) and (5), one can see that the multi-rank MLE solution to passive WFC ranging is essentially a two stage processing. In *the first stage of processing* done at the distributed modular array level, each array conducts a far-field beamforming operation using its own steering vector $\mathbf{s}_k(r,\theta,f)$ on their available data $\mathbf{d}_k(t)$, respectively, in an effort to search and focus on the source of interest. This operation results in $L$ parallel channels of beamformed data streams $y_k(t) = \mathbf{s}_k^H(r,\theta,f)\mathbf{d}_k(t), k = 1, 2, \ldots, L$. During the second stage of processing, we first translate the $L$-channel beamformed data vector $\mathbf{y}(t) = \begin{bmatrix} y_1(t) & \cdots & y_L(t) \end{bmatrix}^T$ into its L-point *spatial-spectral* domain representations, $y(\nu_k) = \mathbf{v}_k^H \mathbf{y}(t)$, $(k = 1, 2, \ldots, L)$. We then carry out the multi-rank combining using a weighted combination of all $L$ spatial-spectral modes to build up the ML objective function, $J_{ML}(r, \theta)$, for passive ranging. Therefore, the peak-track from the above objective function $J_{ML}(r, \theta)$ over a running time duration results in the range-track for the source of interest. Note an interesting fact that the mode vectors, $\mathbf{v}_k$'s , which are used to combine all distributed beamformers' outputs in $L$ different ways, are fixed once the total module number $L$ is known. However, the weighting coefficient, $P_\rho(\nu_k)\eta_{SNR}/(1 + P_\rho(\nu_k)\eta_{SNR})$, depends on the spatial correlation model in use. For applications where a prior spatial PSD of coherence model, $P_\rho(\nu_k)$, is approximately known, one can easily adopt a reduce-rank scheme that only utilize certain significant spatial-spectral modes of the spatial correlation function to combine beamformed data, without much performance loss. For adaptive applications, where $P_\rho(\nu_k)$ has to be estimated from the cross-spectral density matrix (CSDM) of modular array's beamformed outputs. Given very limited amount of stationary available data, reduced-rank scheme provides a feasible solution for balancing the performance and solution stability in WFC ranging.

As pointed out earlier, the general *numerical procedures* of pre-calculating the parameters used in the multi-rank processor applies to any coherence model of one's choice, no matter what are the choice of $L_t$ and $L_{coh}$, if or not aliasing is present (due to the choice of $L_t$ with respect to the $L_{coh}$). However, the performance of the passive ranging system may vary for different levels of spatial coherence.

For an ideal environment with perfect coherence, $L_t \ll L_{coh}$ always holds. We have $\mathbf{R}_\rho = \mathbf{1} \cdot \mathbf{1}^T$ and $P(\nu) = \delta(\nu)$. Hence, the only non-zero coefficient in the PSD is $P_\rho(\nu_1) = 1$, resulting in a rank-1 solution,

$$\begin{aligned} J_{ML}(r,\theta) &= \left|\mathbf{e}^H(\nu_1)\mathbf{y}(t)\right|^2 \\ &= \left|\sum_{i=1}^{L} y_k(t,r,\theta)\right|^2 = \left|\sum_{i=1}^{L} \mathbf{s}_k^H(r,\theta)\mathbf{d}_k(t)\right|^2 . \end{aligned}$$

In this case, the solution is simply reduced to a coherent beamforming solution across the whole array (a rank-1 matched filter solution).

On the other extreme, when there is no coherence existing, equivalently $L_t \gg L_{coh}$, we have $\mathbf{R}_\rho = \mathbf{I}$ and $P_\rho(\nu) = 1$. Hence, $P_\rho(\nu_i) = 1, i = 1, 2, \ldots, L$, resulting in

$$\begin{aligned} J_{ML}(r,\theta) &\propto \sum_{i=1}^{L} \left|\, \mathbf{e}^H(\nu_i)\mathbf{y}(t)\,\right|^2 \\ &= \left\| \begin{bmatrix} \mathbf{e}^H(\nu_1) \\ \vdots \\ \mathbf{e}^H(\nu_L) \end{bmatrix} \mathbf{y}(t,r,\theta) \right\|^2 \\ &\propto \left\|\mathbf{y}(t,r,\theta)\right\|^2 = \sum_{i=1}^{L} \left|\mathbf{s}_k^H(r,\theta)\mathbf{d}_k(t)\right|^2 , \end{aligned}$$

which is simply a *non-coherent solution* by combining all modular arrays' beamforming output powers.

In practice, for effective and unambiguous passive WFC ranging, we need to use a broadband approach, where many frequency bins of reasonable coherence levels should be utilized. Therefore, the framework developed here can be further extended to include frequency parameter into the spatial coherence matrix.

## 5. CONCLUDING REMARKS

We develop in this work some asymptotic results suitable for multi-rank processing using a *large-scale distributed arrays* for passive WFC ranging system operating in environments subject to spatial coherence loss. Under a Gaussian assumption on data model, our solution yields a two-stage processing scheme, a module level beamforming followed by a spatial combining utilizing the spatial coherence existing in the distributed beamformers' outputs. Using an exponential coherence model, our analytical solution provides a simple way of choosing inter-module spacing, mode filters and combination coefficients used in the multi-rank processor to harvest possible spatial coherence. The general framework based on asymptotic relation should apply to different coherence models encountered in practice, as long as the distributed system is large enough.

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