BIAS ANALYSIS OF SOURCE LOCALIZATION USING THE MAXIMUM LIKELIHOOD ESTIMATOR

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ABSTRACT

The nonlinear nature of the source localization problem creates bias to a location estimate. The bias could play a significant role in limiting the performance of localization and tracking when multiple measurements at different instants are available. This paper performs bias analysis of the source location estimate obtained by the maximum likelihood estimator, where the positioning measurements can be TOA, TDOA, or AOA. The effect of bias to the mean-square localization error is examined and the amounts of bias introduced by the three types of measurements are contrasted.

Index Terms— Maximum likelihood estimator, bias, time of arrival, time difference of arrival, angle of arrival

1. INTRODUCTION

The mean-square error (MSE) is a common measure on the performance of a source location estimator [1]-[4]. The MSE is the expectation of the sum of the squared differences between the coordinates of a source position estimate and the actual. Due to the nonlinearity of the source localization problem, the MSE is composed of two parts: the variance and the bias square. When the noise level is small, the variance dominates and the bias is negligible. Hence, the Cramer-Rao lower bound (CRLB), which is developed for an unbiased estimator, is often used as a reference for evaluating the performance of a location estimator.

In many applications such as tracking, multiple measurements are available at different time instants. The information of the measurements can be integrated over time and the location variance will be decreased as more measurements are collected. On the other hand, the bias will not be reduced. Bias is a known problem in tracking [2] and limits the performance. Some attempts have been made in developing localization algorithms to minimize the amount of bias, e.g. [3]-[4].

In this paper we study the bias of the maximum likelihood estimator (MLE) [5] for source localization. We choose the MLE because it is asymptotically efficient and is often served as a benchmark for performance evaluation. The measurements considered are time of arrival (TOA), time difference of arrival (TDOA) and angle of arrival (AOA). All of these measurement functions are highly nonlinear with respect to the unknown source position. The bias of the MLE of a general estimation problem has been investigated in the mathematical and statistical literature [6]-[8]. The bias result there is not easy to apply in engineering practice. In [9], Gavish and Weiss derived a bias formula for the MLE for bearing only tracking.

We derive the theoretical bias of the MLE for the three types of positioning measurements when the noise is Gaussian. The bias square is compared with the variance of the MLE to examine the effect of bias in the localization accuracy. The amounts of location bias for the three types of measurements will be contrasted, which provides insight on the type of measurements that will be less sensitive in causing bias. Simulations are included to support the theoretical developments.

The rest of the paper is organized as follows. Section 2 depicts the localization scenario and introduces the three localization measurements. In Section 3, the bias of the MLE for source localization is derived. Section 4 provides simulation results to support the theoretical analysis and Section 5 is the conclusion.

2. LOCALIZATION PROBLEMS

We consider the scenario that one target is to be localized by M receivers as shown in Fig. 1. The target is at unknown location $\mathbf{u}^o = \begin{bmatrix} x_{\mathbf{u}}^o & y_{\mathbf{u}}^o \end{bmatrix}^T$ and the receivers are at known positions $\mathbf{s}_i = \begin{bmatrix} x_{\mathbf{s}_i} & y_{\mathbf{s}_i} \end{bmatrix}^T$, $i = 1, 2, \ldots, M$. Target localization is accomplished by using TOA, TDOA, or AOA measurements.

The TOA of a signal from the target to receiver i, denoted by τ_i, obeys the relationship

$$c\tau_i = c\tau_i^o + n_{\tau_i} = ||\mathbf{u}^o - \mathbf{s}_i|| + n_{\tau_i} \tag{1}$$

where c is the signal propagation speed, τ_i^o is the true TOA, n_{τ_i} represents the additive measurement noise and $|| * ||$ denotes the Euclidean norm. The TOAs from all sensors can be expressed in vector form as $c\boldsymbol{\tau} = \begin{bmatrix} c\tau_1 & c\tau_2 & \cdots & c\tau_M \end{bmatrix}^T = c\boldsymbol{\tau}^o + \mathbf{n}_{\boldsymbol{\tau}}$, where $\boldsymbol{\tau}^o = \begin{bmatrix} \tau_1^o & \tau_2^o & \cdots & \tau_M^o \end{bmatrix}^T$ and $\mathbf{n}_{\boldsymbol{\tau}}$ is the noise vector. The TDOA measurement between receiver pair $(i, 1)$ is

$$c\tilde{\tau}_{i,1} = c\tilde{\tau}_{i,1}^o + n_{\tilde{\tau}_{i,1}} = ||\mathbf{u}^o - \mathbf{s}_i|| - ||\mathbf{u}^o - \mathbf{s}_1|| + n_{\tilde{\tau}_{i,1}}. \tag{2}$$

The collection of TDOAs is $c\tilde{\boldsymbol{\tau}} = \begin{bmatrix} c\tilde{\tau}_{2,1} & c\tilde{\tau}_{3,1} & \cdots & c\tilde{\tau}_{M,1} \end{bmatrix}^T = c\tilde{\boldsymbol{\tau}}^o + \mathbf{n}_{\tilde{\boldsymbol{\tau}}}$, where $\tilde{\boldsymbol{\tau}}^o = \begin{bmatrix} \tilde{\tau}_{2,1}^o & \tilde{\tau}_{3,1}^o & \cdots & \tilde{\tau}_{M,1}^o \end{bmatrix}^T$ is the true TDOA vector and $\mathbf{n}_{\tilde{\boldsymbol{\tau}}}$ is the noise for TDOA. In the AOA case, the angle measurement of receiver i follows

$$\beta_i = \beta_i^o + n_{\beta_i} = \arctan \frac{y_{\mathbf{u}}^o - y_{\mathbf{s}_i}}{x_{\mathbf{u}}^o - x_{\mathbf{s}_i}} + n_{\beta_i}. \tag{3}$$

The AOA measurement vector is $\boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_M \end{bmatrix}^T = \boldsymbol{\beta}^o + \mathbf{n}_{\boldsymbol{\beta}}$, where $\boldsymbol{\beta}^o = \begin{bmatrix} \beta_1^o & \beta_2^o & \cdots & \beta_M^o \end{bmatrix}^T$ is the true AOA vector and $\mathbf{n}_{\boldsymbol{\beta}}$ is the AOA noise.

The three measurement types can be represented in a generic form as

$$\mathbf{m} = \mathbf{f}(\mathbf{u}^o) + \mathbf{n} \tag{4}$$

where $\mathbf{m}$ is the $N \times 1$ measurement vector with $N = M$ for TOA and AOA and $N = M - 1$ for TDOA, $\mathbf{f}(\mathbf{u}^o)$ represents the functional relationship of the noiseless measurement vector in terms of the unknown position $\mathbf{u}^o$ and $\mathbf{n}$ is the additive noise. We assume $\mathbf{n}$

is zero mean Gaussian with covariance matrix $\mathbf{Q}$. We are interested in examining the amount of bias in the source location estimate obtained by the MLE, when the three kinds of positioning measurements are used. The bias results derived below are in term of the variances of the TOA, TDOA, and AOA. The bias can be made in terms of the SNR of the raw signal measurements through the CRLB of the three positioning measurements [10]-[12].

3. THEORETICAL BIAS ANALYSIS

Since the noise is zero mean Gaussian, the MLE solution $\hat{\mathbf{u}}$ is

$$\hat{\mathbf{u}} = \arg\min(J) \tag{5a}$$

where

$$J \triangleq (\mathbf{m} - \mathbf{f}(\mathbf{u}))^T \mathbf{Q}^{-1} (\mathbf{m} - \mathbf{f}(\mathbf{u})). \tag{5b}$$

Representing the gradient of J with respect to $\mathbf{u}$ as $\mathbf{p}(\mathbf{u}) = \begin{bmatrix} p_x(\mathbf{u}) & p_y(\mathbf{u}) \end{bmatrix}^T$, $\hat{\mathbf{u}}$ satisfies the equation

$$\mathbf{p}(\hat{\mathbf{u}}) = \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\hat{\mathbf{u}}} = \mathbf{0}. \tag{6}$$

The expectation of the difference between $\hat{\mathbf{u}}$ and $\mathbf{u}^o$ gives the bias $\mathbf{b}_{\hat{\mathbf{u}}} = E\left[\hat{\mathbf{u}} - \mathbf{u}^o\right]$. We shall use (6) to obtain the bias $\mathbf{b}_{\hat{\mathbf{u}}}$, without explicitly solving $\hat{\mathbf{u}}$.

The Taylor-series expansion of $\mathbf{p}(\hat{\mathbf{u}})$ at $\mathbf{u}^o$ up to second order is

$$\begin{aligned} \mathbf{p}(\hat{\mathbf{u}}) \simeq & \mathbf{p}(\mathbf{u}^o) + \frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}(\hat{\mathbf{u}} - \mathbf{u}^o) \\ & + \frac{1}{2} \begin{bmatrix} (\hat{\mathbf{u}} - \mathbf{u}^o)^T \left(\frac{\partial^2 p_x(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T}\right)(\hat{\mathbf{u}} - \mathbf{u}^o) \\ (\hat{\mathbf{u}} - \mathbf{u}^o)^T \left(\frac{\partial^2 p_y(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T}\right)(\hat{\mathbf{u}} - \mathbf{u}^o) \end{bmatrix} \end{aligned} \tag{7}$$

where we have used $\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}$ to denote $\frac{\partial \mathbf{p}(\mathbf{u})}{\partial \mathbf{u}^T}$ evaluated at $\mathbf{u} = \mathbf{u}^o$, and similarly for $\frac{\partial^2 p_x(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T}$ and $\frac{\partial^2 p_y(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T}$.

Since $\mathbf{p}(\hat{\mathbf{u}}) = \mathbf{0}$, rearranging (7) gives

$$\mathbf{b}_{\hat{\mathbf{u}}} = E[\hat{\mathbf{u}} - \mathbf{u}^{\mathbf{o}}] \simeq -E\left[\left(\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^{-1}(\mathbf{p}(\mathbf{u}^o) + \mathbf{g}(\mathbf{u}^o))\right] \tag{8}$$

where $\mathbf{g}(\mathbf{u}^o)$ represents the second order terms in (7). Appendices A and B simplify (8) by evaluating the expectation and the bias is equal to

$$\mathbf{b}_{\hat{\mathbf{u}}} \simeq \mathbf{A}^{-1}\left[4\left(\sum_{i=1}^{N} \mathbf{C}_i \mathbf{e}_i\right) - \mathbf{d}\right] \tag{9}$$

where $\mathbf{e}_i$ is an $N \times 1$ zero vector except its i-th element is unity. For uncorrelated noise of equal noise power, the matrices $\mathbf{A}$ and $\mathbf{C}_i$ are given by (15a) and (18) and the vector $\mathbf{d}$ is defined in (19)-(20). For the general case of any symmetric and positive definite $\mathbf{Q}$, their values are given in Appendix B.

(9) is the generic form of the bias. When $\mathbf{f}(\mathbf{u}^o)$ takes on different measurement types, the first and second order derivatives will be different, yielding different amount of bias in the MLE solution. We shall provide below the derivatives for the three measurement types to obtain $\mathbf{A}$, $\mathbf{C}_i$ and $\mathbf{d}$.

For the TOA case, from (1), with $i = 1, 2, \cdots, M$,

$$\frac{\partial f_i(\mathbf{u}^o)}{\partial \mathbf{u}} = \boldsymbol{\rho}_i, \qquad \frac{\partial^2 f_i(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T} = r_i^{-1} \boldsymbol{\rho}_i^{\perp}, \tag{10}$$

where $f_i(\mathbf{u})$ is the i-th element of $\mathbf{f}(\mathbf{u})$, $r_i = ||\mathbf{u}^o - \mathbf{s}_i||$, $\boldsymbol{\rho}_i = r_i^{-1}(\mathbf{u}^o - \mathbf{s}_i)$, $\boldsymbol{\rho}_i^{\perp} = \mathbf{I} - \boldsymbol{\rho}_i \boldsymbol{\rho}_i^T$ and $\mathbf{I}$ is the identity matrix.

For the TDOA case, from (2), with $i = 2, 3, \cdots, M$,

$$\frac{\partial f_{i-1}(\mathbf{u}^o)}{\partial \mathbf{u}} = \boldsymbol{\rho}_i - \boldsymbol{\rho}_1 \qquad \frac{\partial^2 f_{i-1}(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T} = r_i^{-1} \boldsymbol{\rho}_i^{\perp} - r_1^{-1} \boldsymbol{\rho}_1^{\perp}. \tag{11}$$

For the AOA case, from (3), with $i = 1, 2, \cdots, M$,

$$\frac{\partial f_i(\mathbf{u}^o)}{\partial \mathbf{u}} = r_i^{-1} \mathbf{T} \boldsymbol{\rho}_i \qquad \frac{\partial^2 f_i(\mathbf{u}^o)}{\partial \mathbf{u} \partial \mathbf{u}^T} = r_i^{-2} \mathbf{T}\left(\boldsymbol{\rho}_i^{\perp} - \boldsymbol{\rho}_i \boldsymbol{\rho}_i^T\right) \tag{12}$$

where $\mathbf{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

In the next section, we support the theoretical development by computer simulation and reveal the contribution of the bias to the MSE.

4. SIMULATION

The simulation used 8 receivers to localize one target. The target position is randomly generated with uniform distribution in an area of 500×500 centered at the origin. The receivers are randomly deployed in an area of 50×50 with uniform distribution also centered at the origin. For a given geometry, the bias square and MSE of the MLE solution are computed over $L = 5000$ ensemble runs, where they are obtained by $\left\| \frac{1}{L} \sum_{l=1}^{L} (\hat{\mathbf{u}}^{(l)} - \mathbf{u}^o) \right\|^2$ and $\frac{1}{L} \sum_{l=1}^{L} (\hat{\mathbf{u}}^{(l)} - \mathbf{u}^o)^T (\hat{\mathbf{u}}^{(l)} - \mathbf{u}^o)$, and $\hat{\mathbf{u}}^{(l)}$ is the target position estimate at ensemble l. The MLE is implemented using gradient search initialized at the true target position. To eliminate the dependency of a particular geometry, the bias square and MSE presented are the average of 300 different geometries.

Fig. 2 shows the bias square and MSE of the TOA localization case as the noise power increases, where the covariance matrix is $\mathbf{Q} = \sigma_\tau^2 \mathbf{I}$. The bias values from the developed formula match very well with the simulation bias results before the threshold effect occurs. When σ_τ^2 is very small, 10^{-2}, the bias square of the MLE is much smaller than the CRLB. However, the contribution of the bias to the MSE increases as the noise level increases.

Fig. 3 illustrates the localization performance using TDOAs in the presence of correlated noise. The noise covariance matrix is $\mathbf{Q} = \frac{1}{2}\sigma_\tau^2(\mathbf{I} + \mathbf{E})$ and $\mathbf{E}$ is the all one element matrix. The theoretical bias follows very well with the simulation results . The simulation of the AOA case is shown in Fig. 4, where $\mathbf{Q} = \sigma_\beta^2 \mathbf{I}$ and the unit of σ_β is radians. The observations are very similar to the other two cases. Comparing Figs. 2-4 reveals that TOA creates least bias in the source position estimate, followed by TDOA and then AOA. At the same noise power, TOA has much smaller bias than TDOA. Note that the better performance of TOA is on the expense of time synchronization between the target and receivers which is possible for cooperative localization only.

When receivers are able to have multiple measurements at different time instants, say K, the contribution of bias to the MSE for the TOA case as K increases is shown in Fig. 5. The noise power σ_τ^2 is set at 10 and the results are also the average of 300 random geometries. At each ensemble run, K estimates are averaged before obtaining the bias square and MSE. The number of ensemble averages is 500. Also shown in Fig. 5 is the theoretical MSE that is obtained by the sum of the trace of the CRLB and the theoretical bias square from (9). As K increases, the variance drops, the bias square stays constant, and the MSE approaches the bias square which limits the MSE. The bias therefore needs to be carefully considered in localization with multiple measurements and tracking.

5. CONCLUSION

We have derived the theoretical expression that can predict accurately the bias of the source location estimate from the MLE when the SNR of the received signals is not small. The bias is insignificant compared to variance when the noise level of the positioning measurements is small and it becomes non-negligible as the noise level increases. The bias values for TOA, TDOA and AOA localization are contrasted and TOA yields the least amount of bias, followed by TDOA and AOA. The significant effect of bias in limiting performance when multiple measurements are available is also demonstrated.

APPENDIX A. UNCORRELATED NOISE OF EQUAL NOISE POWER CASE

Let us consider $\mathbf{Q} = \sigma^2\mathbf{I}$, where σ^2 is the noise power.

$$\mathbf{p}(\mathbf{u}) = \frac{\partial J}{\partial \mathbf{u}} = -2\sigma^{-2}\left(\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}^T}\right)^T (\mathbf{m} - \mathbf{f}(\mathbf{u})) \tag{13}$$

and $\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}$ is equal to

$$\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T} = \mathbf{A} - \mathbf{B} \tag{14}$$

where

$$\mathbf{A} = 2\sigma^{-2}\left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^T \left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right) \tag{15a}$$

$$\mathbf{B} = 2\sigma^{-2}\sum_{i=1}^{N}(m_i - f_i(\mathbf{u}^o))\frac{\partial^2 f_i(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}. \tag{15b}$$

The inverse of (14) can be approximated by

$$\left(\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^{-1} = (\mathbf{I}-\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1} \simeq \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} \tag{16}$$

when the measurement noise is small, so that the higher order terms of $\mathbf{B}$ can be ignored.

From (13) and (16), since the matrix $\mathbf{A}^{-1}$ is independent of noise,

$$E\left[-\left(\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^{-1}\mathbf{p}(\mathbf{u}^o)\right] \simeq E\left[2\sigma^{-2}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^T \mathbf{n}\right]$$

where $\mathbf{n} = \mathbf{m} - \mathbf{f}(\mathbf{u}^o)$. Substituting the definition of $\mathbf{B}$ from (15b) and noting that $E[\mathbf{n}\mathbf{n}^T] = \sigma^2\mathbf{I}$, the first component of bias is

$$E\left[-\left(\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^{-1}\mathbf{p}(\mathbf{u}^o)\right] \simeq 4\mathbf{A}^{-1}\sum_{i=1}^{N}\mathbf{C}_i\mathbf{e}_i \tag{17}$$

where

$$\mathbf{C}_i = \sigma^{-2}\left(\frac{\partial^2 f_i(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}\right)\mathbf{A}^{-1}\left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^T. \tag{18}$$

The second bias component $-E\left[\left(\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^{-1}\mathbf{g}(\mathbf{u}^o)\right]$ is quite tedious to evaluate and we will make some approximation. When the noise level is small, we have from (14), $\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T} \simeq \mathbf{A}$. In addition,

$$\begin{aligned}\mathbf{d} &\triangleq E[\mathbf{g}(\mathbf{u}^o)] \\ &= \frac{1}{2}E\begin{bmatrix} tr\left(\frac{\partial^2 p_x(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T} \times (\hat{\mathbf{u}} - \mathbf{u}^o)(\hat{\mathbf{u}} - \mathbf{u}^o)^T\right) \\ tr\left(\frac{\partial^2 p_y(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T} \times (\hat{\mathbf{u}} - \mathbf{u}^o)(\hat{\mathbf{u}} - \mathbf{u}^o)^T\right)\end{bmatrix} \\ &\simeq \frac{1}{2}\begin{bmatrix} tr\left(E\left[\frac{\partial^2 p_x(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}\right] \times \mathbf{CRLB}(\mathbf{u}^o)\right) \\ tr\left(E\left[\frac{\partial^2 p_y(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}\right] \times \mathbf{CRLB}(\mathbf{u}^o)\right)\end{bmatrix}\end{aligned} \tag{19}$$

where $tr(*)$ represents the trace operation. The approximation is valid for small measurement noise and the fact that MLE is asymptotically efficient. $\mathbf{CRLB}(\mathbf{u}^o)$ is the CRLB of $\mathbf{u}^o$ when its bias is neglected.

After some algebraic manipulation, it can be verified that, starting from (14),

$$E\left[\frac{\partial^2 p_x(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}\right] = 2\sum_{i=1}^{N}\{\mathbf{h}_i^T\mathbf{e}_x\mathbf{G}_i + \mathbf{G}_i\mathbf{e}_x\mathbf{h}_i^T + (\mathbf{G}_i\mathbf{e}_x\mathbf{h}_i^T)^T\}, \tag{20a}$$

$$E\left[\frac{\partial^2 p_y(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}\right] = 2\sum_{i=1}^{N}\{\mathbf{h}_i^T\mathbf{e}_y\mathbf{G}_i + \mathbf{G}_i\mathbf{e}_y\mathbf{h}_i^T + (\mathbf{G}_i\mathbf{e}_y\mathbf{h}_i^T)^T\}. \tag{20b}$$

where $\mathbf{h}_i = \sigma^{-1}\frac{\partial f_i(\mathbf{u}^o)}{\partial \mathbf{u}}$, $\mathbf{G}_i = \sigma^{-1}\frac{\partial^2 f_i(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}$, $\mathbf{e}_x = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $\mathbf{e}_y = \begin{bmatrix}0 & 1\end{bmatrix}^T$.

With the use of the $\mathbf{CRLB}(\mathbf{u}^o)$ for a given positioning measurement type, $E[\mathbf{g}(\mathbf{u}^o)]$ can be evaluated and the second component of the bias can be obtained:

$$-E\left[\left(\frac{\partial \mathbf{p}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^{-1}\mathbf{g}(\mathbf{u}^o)\right] \simeq -\mathbf{A}^{-1}\mathbf{d}. \tag{21}$$

APPENDIX B. GENERAL CASE

Let $\check{\mathbf{m}} = \mathbf{Q}^{-\frac{1}{2}}\mathbf{m}$ and $\check{\mathbf{f}}(\mathbf{u}) = \mathbf{Q}^{-\frac{1}{2}}\mathbf{f}(\mathbf{u})$, where $\mathbf{Q}^{\frac{1}{2}}$ represents the square root of $\mathbf{Q}$ so that $\mathbf{Q}^{\frac{1}{2}}\mathbf{Q}^{\frac{1}{2}} = \mathbf{Q}$. The (i, j) element of $\mathbf{Q}^{-\frac{1}{2}}$ is denoted by $\alpha_{i,j}$. Following the same derivations as in Appendix A, the bias is (9), where the matrices $\mathbf{A}$ and $\mathbf{C}_i$ become

$$\mathbf{A} = 2\left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^T \mathbf{Q}^{-1}\left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right) \tag{22a}$$

$$\mathbf{C}_i = \left(\sum_{j=1}^{N}\alpha_{i,j}\frac{\partial^2 f_j(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}\right)\mathbf{A}^{-1}\left(\frac{\partial \mathbf{f}(\mathbf{u}^o)}{\partial \mathbf{u}^T}\right)^T \mathbf{Q}^{-\frac{1}{2}} \tag{22b}$$

The vector $\mathbf{d}$ is given by (19) and the second derivations are (20a) and (20b), where $\mathbf{h}_i = \sum_{j=1}^{N}\alpha_{i,j}\frac{\partial f_j(\mathbf{u}^o)}{\partial \mathbf{u}}$, $\mathbf{G}_i = \sum_{j=1}^{N}\alpha_{i,j}\frac{\partial^2 f_j(\mathbf{u}^o)}{\partial \mathbf{u}\partial \mathbf{u}^T}$.

6. REFERENCES

[1] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Trans. Signal Process.*, vol. 42, pp. 1905-1915, Aug. 1994.

[2] K. Dogancay, "Bias compensation for the bearings-only pseudolinear target track estimator," *IEEE Trans. Signal Process.*, vol. 54, pp. 59-68, Jan. 2006.

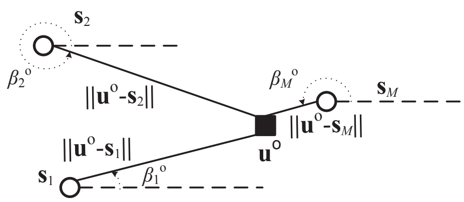

Fig. 1. Localization scenario. The solid rectangle denotes the source to be localized. The circle represent the sensors.

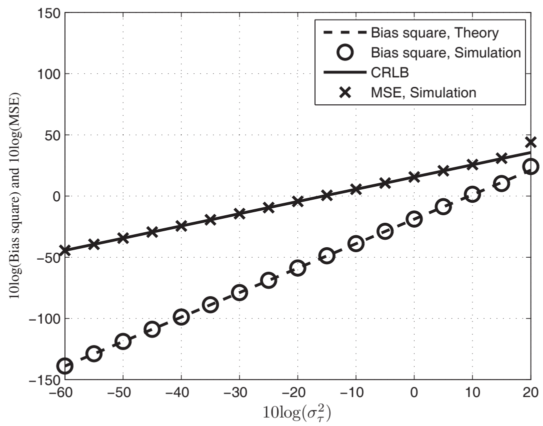

Fig. 2. Bias square and MSE in TOA localization using MLE.

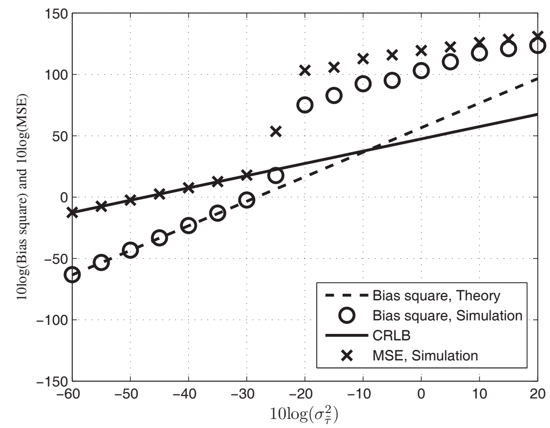

Fig. 3. Bias square and MSE in TDOA localization using MLE.

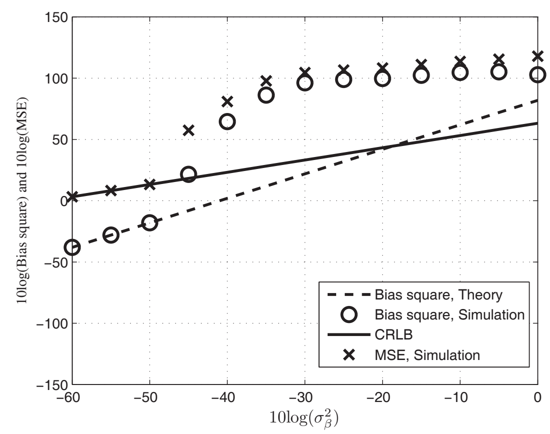

Fig. 4. Bias square and MSE in AOA localization using MLE.

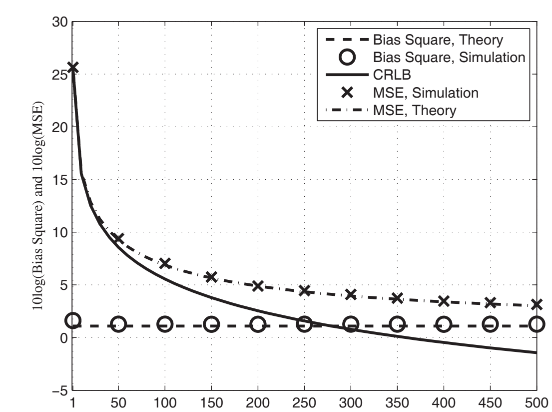

Fig. 5. Behavior of the bias square and MSE of the MLE for TOA localization with multiple measurements at different time instants.

[3] Y. Huang, J. Benesty, G. Elko, and R. Mersereau, "Real-time passive source localization: a practical linear-correction least-squares approach," *IEEE Trans. Speech Audio Process.*, vol. 9, pp. 943-956, Nov. 2001.

[4] N. Liu, Z. Xu, and B. M. Sadler, "Low-complexity hyperbolic source localization with a linear sensor array," *IEEE Signal Process. Lett.*, vol. 15, pp. 865-868, 2008.

[5] S. M. Kay, *Fundamental of Statistical Signal Process., Estimation Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[6] M. S. Bartlett, "Approximate confidence intervals.II. more than one unknown parameter," *Biometrika*, vol. 40, pp. 306-317, 1952.

[7] J. B. Haldane, "The estimation of two parameters from a sample," *Sankhya*, vol. 12, pp. 313-320, 1953.

[8] D. R. Cox and E. J. Snell, "A general definition of residuals," *Journal of the Royal Statistical Society, Series B (Methodological)*, vol. 30, pp. 248-275, 1968.

[9] M. Gavish and A. J. Weiss, "Performance analysis of bearing-only target location algorithms," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-28, pp. 817-828, Jul. 1992.

[10] S. Gezici, T. Zhi, G. B. Giannakis, H. Kobayashi, A. F. Molisch, H. V. Poor, and Z. Sahinoglu, "Localization via ultra-wideband radios: a look at positioning aspects for future sensor networks ," *IEEE Signal Process. Mag.*, vol. 22, pp. 70-84, Jul. 2005.

[11] G. C. Carter, "Coherence and time delay estimation," *Proc. IEEE*, vol. 75, pp. 236-255, Feb. 1987.

[12] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood and Cramer-Rao bound", *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-37, pp. 720-741, May 1989.