THE ESTIMATE FOR DOAS OF SIGNALS USING SPARSE RECOVERY METHOD

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ABSTRACT

This paper presents a new direction-of-arrival (DOA) estimation method using the concept of sparse representation of an array cross-correlation vector (ACCV), in which DOA estimation is achieved by finding the sparse parameter vector according to an optimization criterion. Compared with other sparse recovery algorithms the proposed method achieves a higher resolution and has a less computational complexity. The performance of our method is demonstrated and analyzed through numerical simulations.

***Index Terms*—** Direction-of-arrival, multiple measurement vectors, array cross-correlation vector, sparse recovery.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation has always been used in an active research area, playing an important role in many applications, such as microphone array systems, sonar systems, and mobile communication systems [1]. Recently, sparse-representation-based DOA estimation methods, such as [2]-[6], provide another interpretation of array data and achieve DOA estimation by finding the sparsest representation of the data.

Gorodnitsky *et al.* used a recursive weighted least-squares algorithm called FOCUSS [2] to estimate DOA in the single snapshot case only, while its multiple measurement vectors (MMV) version was M-FOCUSS [3]. Another algorithm, called the singular value decomposition (ℓ_1-SVD) [4], can work with MMV by applying singular value decomposition (SVD) to reduce the computational complexity, however, it still requires solving a joint-sparse recovery problem involving K measurement vectors (K is the number of signals). Another algorithms is JLZA-DOA [5] proposed by Hyder. The algorithm solves the MMV problem by using $\ell_{2,0}$ approximation approach, in which the ℓ_0-norm is approximated by a class of Gaussian functions. However, the parameters in JLZA-DOA are experimental values and the selection of the parameters has no guideline in arbitrary experimental situation. One of the latest algorithms is SRACV [6] proposed by Yin, in which a MMV problem is considered for DOA estimation, thus suffering from a high computational complexity.

In this paper, the scenario of far-field uncorrelated narrowband signals impinging on a uniform linear array (ULA) corrupted by additive white Gaussian noise is considered. Compared with the DOA estimation algorithms involving joint-sparse recovery mentioned above, the proposed method called ℓ_1-ACCV transforms MMV problem into a single measurement vector (SMV) model through an array cross-correlation vector (ACCV), thus decreasing the computational complexity substantially.

In this paper, the notations $(\bullet)^T$, $(\bullet)^*$, $(\bullet)^H$, $E(\bullet)$, $\lceil \bullet \rceil$, $\|\bullet\|_0$, $\|\bullet\|_1$, $\|\bullet\|_2$ denote transpose, conjugate, conjugate transpose, expectation, ceiling operation, ℓ_0-norm, ℓ_1-norm, ℓ_2-norm, respectively.

2. SIGNAL MODEL

Assume that K far-field zero-mean and uncorrelated signals $u_k(t)$, $k \in \{1,2,\ldots,K\}$ impinge on a ULA of M omnidirectional sensors corrupted by additive temporally and spatially white Gaussian noise $w_m(t)$, and the m th sensor output is $x_m(t)$, $m \in \{1,2,\ldots,M\}$, $K < M$. Let $\boldsymbol{\theta} = [\theta_1, \theta_1, \ldots, \theta_K]$ be the direction parameter collection and θ_k is DOA of signal $u_k(t)$. In the narrowband and far-field case, the array output vector can be represented as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{w}(t), \tag{1}$$

where $\mathbf{A}$ is a manifold matrix defined as $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_K)]$, $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_K(t)]^T$ is the signal vector and $\mathbf{w}(t) = [w_1(t), w_2(t), \ldots, w_M(t)]^T$ is the noise vector at time slot t. The steering vector $\mathbf{a}(\theta_k)$

This work is supported by the Science and Technology Plan Project of Anhui Province of China (No. 11010202191) and National Natural Science Foundation of China (No. 61101236).

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corresponding to the k th far-field source for the ULA is given by

$$\mathbf{a}(\theta_k) = \left[1, \nu_k, \ldots, \nu_k^{M-1}\right]^T, \quad (2)$$

where $\nu_k = e^{-j2\pi(d/\lambda)\sin(\theta_k)}$. λ is the signal wavelength and d is inter-sensor spacing.

The covariance matrix is obtained as the following form:

$$\mathbf{R} = E\left\{\mathbf{x}(t)\mathbf{x}^H(t)\right\} = \left[r_{p,q}\right], p,q \in \{1,2,\ldots,M\}. \quad (3)$$

Obviously, $\mathbf{R}$ is a Hermitian Toeplitz matrix, and its first column can be used to reconstruct the whole matrix.

By using the first row and the first column, an ACCV can be defined as

$$\mathbf{r} \triangleq \left[r_{1,M}, \ldots, r_{1,2}, r_{2,1}, \ldots, r_{M,1}\right]^T. \quad (4)$$

From (3) we can represent (4) as

$$\mathbf{r} = \overline{\mathbf{A}}\mathbf{p}, \quad (5)$$

where

$$\overline{\mathbf{A}} = \left[\overline{\mathbf{a}}(\theta_1), \overline{\mathbf{a}}(\theta_2), \ldots, \overline{\mathbf{a}}(\theta_K)\right],$$

$$\overline{\mathbf{a}}(\theta_k) = \left[\nu_k^{-(M-1)}, \ldots, \nu_k^{-1}, \nu_k^1 \ldots, \nu_k^{M-1}\right]^T,$$

$$\mathbf{p} = \left[p_{u_1}, p_{u_2}, \ldots, p_{u_K}\right]^T.$$

p_{u_k} represents the power of $u_k(t)$. Note that each element of $\mathbf{p}$ is positive and $\mathbf{r}$ is conjugate symmetric.

3. SPARSE SIGNAL REPRESENTATION AND RECONSTRUCTION

In the signal model (5), the DOA and the power of each signal are concerned, but we cannot derive the matrix $\overline{\mathbf{A}}$ and the vector $\mathbf{p}$ directly from $\mathbf{r}$. To solve this problem, (5) is extended to a sparse representation form with an overcomplete basis matrix $\mathbf{B}$ and a sparse power vector $\mathbf{s}$.

Let $\left\{\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_N\right\}$ ($N \gg K$) be a sampling grid of all potential of DOAs. Then we can represent (5) as

$$\mathbf{r} = \mathbf{B}\mathbf{s}, \quad (6)$$

where $\mathbf{B} = [\overline{\mathbf{a}}(\tilde{\theta}_1), \overline{\mathbf{a}}(\tilde{\theta}_2), \ldots, \overline{\mathbf{a}}(\tilde{\theta}_N)]$, $n \in \left\{1,2,\ldots,N\right\}$. The n th element $\mathbf{s}(n)$ is non-zero and equal to p_{u_k} if $\tilde{\theta}_n = \theta_k$ for some $k \in \{1,2,\ldots,K\}$ and zero otherwise. Note that $\mathbf{B}$ differs from $\overline{\mathbf{A}}$ that it contains steering vector $\overline{\mathbf{a}}(\tilde{\theta}_n)$ for all potential DOAs， rather than only the source signals DOAs. Hence, $\mathbf{B}$ is known and $\overline{\mathbf{A}}$ is assumed to be a subset of $\mathbf{B}$.

To obtain the sparse solution from (6) we should consider how to enforce sparsity for $\mathbf{s}$. The ideal measurement of sparsity is the count of nonzero entries in $\mathbf{s}$, which is denoted by $\|\mathbf{s}\|_0$. However, when we use ℓ_0-norm to enforce sparsity, it would be a hard combinatorial optimization problem. In this paper we enforce sparsity by ℓ_1-norm just like ℓ_1-SVD, which results in a convex optimization problem. The optimization criterion is given as the following form:

$$\min_{\hat{\mathbf{s}}} \|\hat{\mathbf{s}}\|_1 \quad \text{subject to} \quad \mathbf{r} = \mathbf{B}\hat{\mathbf{s}}. \quad (7)$$

By solving this convex optimization problem, DOA estimation can be achieved.

Next, we consider how many sources can be estimated by using this sparse recovery method. The following lemma [3] gives sufficient conditions for the existence of a unique solution for the MMV problem.

Lemma 1: Under the assumptions that any m columns of $\mathbf{B}$ are linearly independent and rank $(\mathbf{r}) = L \le m$, a solution with K_u nonzero entries, where $K_u \le \lceil (m+L)/2 \rceil - 1$, is unique.

It is obvious that the rank of $\mathbf{r}$ is 1 and the maximum number of any linearly independent columns of $\mathbf{B}$ is $2M-2$ according to the Vandermonde property. Through Lemma 1, the proposed method can resolve up to $M-1$ signals.

4. NUMERICAL SOLUTION

Herein, we outline the solution of this convex optimization problem in (7). ℓ_1-SVD [4] minimizes ℓ_1-term by constraining ℓ_2-term smaller than a threshold to solve a joint-sparse recovery problem. Here we consider constraining the ℓ_1-term instead of ℓ_2-term, which is rational according to the theory of Lagrange multiplier. The sum of elements in $\mathbf{p}$ is equal to the total power of all signals, which can be easily estimated from sample covariance matrix.

Considering the finiteness of snapshots the estimate $\hat{\mathbf{r}}$ of $\mathbf{r}$ is not accurate and also the sampling grids may not include the real angle, we evolve (7) into the following form:

$$\min_{\hat{\mathbf{s}}} \|\hat{\mathbf{r}} - \mathbf{B}\hat{\mathbf{s}}\|_2 \quad \text{subject to} \quad \|\hat{\mathbf{s}}\|_1 \le \alpha \hat{P}_{total}, \hat{\mathbf{s}} \succeq \mathbf{0}, \quad (8)$$

where $\hat{P}_{total}$ is the estimate of total power of all signals, α is a parameter which depends on the number of snapshots and signal-to-noise ratio (SNR) level. The vector inequality $\hat{\mathbf{s}} \succeq 0$ means $\hat{\mathbf{s}}(n) \ge 0$ for all n, $n \in \left\{1,2,\ldots,N\right\}$.

Firstly the sample covariance matrix can be presented as

$$\hat{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}^H(t) = [\hat{r}_{p,q}], p,q \in \{1,2,\ldots,M\}, \quad (9)$$

where T is the number of snapshots. Then, considering $\mathbf{R}$ is a Hermitian Toeplitz matrix, we can obtain $\hat{\mathbf{r}}$ by

$$\bar{\hat{r}}_{1,p} = \frac{1}{M-p+1}\sum_{q=p}^{M} \hat{r}_{q-p+1,q}, p = 1,2,\ldots,M, \quad (10)$$

$$\bar{\hat{r}}_{p,1} = \bar{\hat{r}}_{1,p}^*, p = 2,3,\ldots,M. \quad (11)$$

Next, the estimate of the total power of all signals is given by

$$\hat{P}_{total} = \bar{\hat{r}}_{1,1} - \hat{\sigma}^2, \quad (12)$$

where $\hat{\sigma}^2$ is the estimate of the noise power. If the accurate knowledge of the number of signals is known, $\hat{\sigma}^2$ can be given by the average of the $M-K$ smallest eigenvalues of $\hat{\mathbf{R}}$. Without this knowledge, in general, a somewhat underestimated $\hat{\sigma}^2$ can be given by the minimum eigenvalue of $\hat{\mathbf{R}}$.

In most scenarios, true DOAs may be not included in sampling grid, so an adaptive method can be applied to enhance the fineness of the grid [4]. This method makes the grid finer around the approximate DOAs in each iteration and gets finer estimates.

The DOA estimation in (8) can be efficiently worked out in the second-order cone (SOC) programming framework [7]. The canonical SOCP form of (8) can be expressed as

$$\begin{aligned} &\min_{\hat{\mathbf{s}}} g \\ &\text{subject to } \| \hat{\mathbf{r}} - \mathbf{B}\hat{\mathbf{s}} \|_2 \le g, \mathbf{1}^T \hat{\mathbf{s}} \le \beta, \hat{\mathbf{s}} \succeq \mathbf{0} \end{aligned}, \quad (13)$$

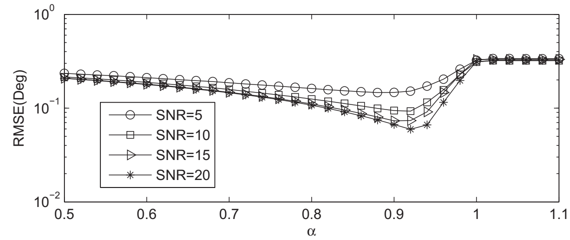
where $\mathbf{1}$ is an $N \times 1$ vector of ones, g is an auxiliary variable and $\beta = \alpha \hat{P}_{total}$. By solving (13) the DOA estimates can be obtained from the locations of dominant peaks of $\hat{\mathbf{s}}$.

Remark A: Regarding the computational complexity of ℓ_1-ACCV, the calculation of $\hat{\mathbf{R}}$ and $\hat{\mathbf{r}}$ requires $O(TM^2)+O((M-1)(M-2)/2)$ and solving (13) requires $O(N^3)$ [4]. Therefore, the computational cost of ℓ_1-ACCV is mainly in solving (13), and is obviously lower than the aforementioned sparse representation DOA estimation algorithms such as ℓ_1-SVD and JLZA, which requires $O(K^3N^3)$ and $O(N^3+MN^2+KMN)$, respectively. Although the cost of ℓ_1-ACCV is higher than some subspace-based algorithms, such as MUSIC, which requires $O(M^3)$, it has higher resolution and does not rely on knowledge of the number of signals. Also, a fact should be noted that the cost of complexity reducing is that ℓ_1-ACCV fails for coherent or correlated signals. This issue can serve as the challenging topic for further research.

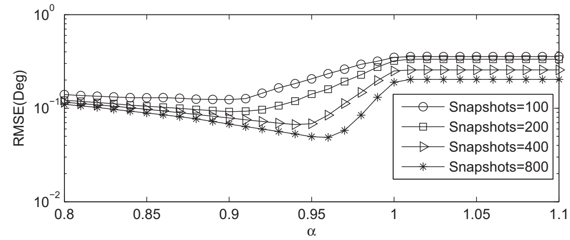
5. SIMULATIONS RESULTS

In this section we present the numerical simulation results to illustrate the performance of the proposed method. We compare the performance of ℓ_1-ACCV with some existing algorithms in which the number of signals is assumed to be known. The simulations are performed using software toolbox SeDuMi 1.1 [8]. By the way, the grid refinement procedure in [4] is used for estimation accuracy analysis.

We consider zero-mean uncorrelated narrowband signals in the far-field impinging upon a ULA of $M=8$ sensors separated by half a wavelength. The proposed method starts with a uniform grid with 1° sampling $N=180$. In all simulation examples below, SNR is defined as $\text{SNR} = \sum_{t=1}^{T} \|\mathbf{Au}(t)\|_2^2 / \sum_{t=1}^{T} \|\mathbf{n}(t)\|_2^2$, all signals' powers are identical and the number of Monte Carlo trials is 500.



(a) $T=200$, SNR=[5,10,15,20] dB.



(b) SNR=10 dB, $T=[100,200,400,800]$.

Fig.1 RMSE versus α.

Before simulations we try to analyze the selection of the parameter α. Assume that uncorrelated sources are held fixed at -10° and 10°. The root mean square error (RMSE) of the DOA estimate by using various values of α with $T=200$ at some SNR levels and SNR=10 dB at different number of snapshots are shown in Fig. 1(a) and Fig. 1(b), respectively. When the number of snapshots is given, the optimal value α_{opt} changes little with the improvement of SNR level, especially approaches a constant value in high SNR level. In the same SNR level α_{opt} approaches 1 with the increasing number of snapshots. The above results can be approximately described by the following analysis. The elements of sample covariance matrix $\hat{\mathbf{R}}$ satisfy $E\left\{\left(\hat{r}_{m,p}-r_{m,p}\right)\left(\hat{r}_{k,l}-r_{k,l}\right)\right\} = r_{m,k} r_{l,p} / T$ (see [9] for details). According to (10) and (11), we could claim that $\left|\hat{P}_{total}-P_{total}\right| \le P_{total} \cdot \gamma / \sqrt{T}$ with a high probability, where γ is associated to SNR level and the number of sensors. We choose an appropriate γ, and substitute it into the upper value of the bias to make that the probability of $\hat{P}_{total} > P_{total}(1+\gamma/\sqrt{T})$ is small. When the sparse constraint $\|\hat{\mathbf{s}}\|_1 \le \hat{P}_{total}/(1+\gamma/\sqrt{T})$ is used for this problem, i.e., $\alpha_{opt} = 1/(1+\gamma/\sqrt{T})$, we can guarantee that $\|\hat{\mathbf{s}}\|_1$ for the recovery result is close to P_{total} with a high probability. Since α_{opt} approaches a constant value in high SNR level,

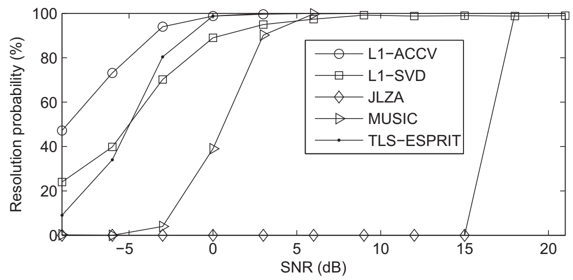


Fig. 2 Resolution probability for L1-ACCV, L1-SVD, JLZA, MUSIC, and TLS-ESPRIT.

γ also approaches a constant value in high SNR level. In Fig. 1(b), the estimate of γ is 1.23, which is also used in the following two simulations.

Next, in Fig. 2 we compare the resolution ability of our proposed method with those of ℓ_1-SVD, JLZA, MUSIC [10] and TLS-ESPRIT [11] using two closely spaced signals at $\boldsymbol{\theta} = [-3^\circ, 3^\circ]$ and $T = 200$. When $|\hat{\theta}_1 - \theta|$ and $|\hat{\theta}_2 - \theta_2|$ are both smaller than $|\theta_1 - \theta_2|/2$, we consider the two signals are resolved successfully. Fig. 2 shows that our method achieves a higher resolution than other algorithms.

Fig. 3 shows the RMSE results versus SNR for ℓ_1-ACCV, some existing algorithms and the CRB [12] in scenario of three signals. Three uncorrelated signals are held fixed at $\boldsymbol{\theta} = [-10.2^\circ, 9.1^\circ, 19.8^\circ]$ and $T = 200$. It is shown that ℓ_1-ACCV always outperforms ℓ_1-SVD and has a good performance at low SNR. However, the RMSE curve of ℓ_1-ACCV degrades slowly for high SNR for that ℓ_1-ACCV is sensitive to the number of snapshots. In other words, if a better accuracy is pursued, more snapshots are needed.

6. CONCLUSIONS

In this paper, we proposed a novel method for DOA estimation for uncorrelated signals. By transforming the joint-sparse recovery problem into a SMV model we greatly decreased computational complexity and eliminated the noise component. Moreover, our method provides a novel way of DOA estimation without the knowledge of the number of signals.

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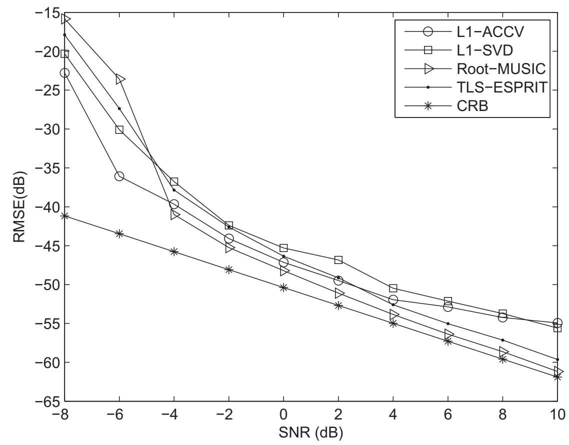

Fig. 3 RMSE of L1-ACCV, L1-SVD, Root-MUSIC and TLS-ESPRIT