DERIVATION OF MONOPULSE ANGLE ACCURACY FOR PHASED ARRAY RADAR TO ACHIEVE CRAMER-RAO LOWER BOUND

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ABSTRACT

Derivation of monopulse angle accuracy for phased array radar to achieve Cramer-Rao lower bound is presented in this paper. Antenna element positions originating from antenna center are used for difference beam taper in this monopulse angle estimation. For uniform linear array, the accuracy is 1.16 times higher than conventional monopulse method. In other words, SNR can be reduced by 1.3 dB to achieve required angle accuracy. Suboptimal difference beamforming taper for the subarray-based digital beamforming radar is also derived.

Index Terms— monopulse, angle accuracy, Cramer-Rao lower bound, taper

1. INTRODUCTION

Monopulse angle estimation technique has been widely used in phased array radar systems to determine angular location of a tracking target [1]. Angle accuracy for a target by monopulse estimation can be described with normalized monopulse slope and this figure can be determined by aperture illumination design for phased array antenna with RF monopulse comparator [2]. Recent advanced phased array radar systems may employ monopulse angle estimation method by digital beamforming (DBF). In element- or subarray-level digital beamforming radar, multiple spatial channels are available for multifunctional array signal processing including digital monopulse beamforming. There are some criteria for the monopulse design such as achieving lower sidelobe level for the sum and difference beam [3]. In this paper, our criteria is to improve monopulse angle accuracy to Cramer-Rao lower Bound (CRLB).

CRLB provides a lower bound on the performance of an unbiased estimator. There are some references presenting CRLB on angle estimator for single target impinging on array antenna in array signal processing literatures [4]-[7]. In the derivation of the CRLB, a steering vector for a target angle and the derivative vector are defined. It is straightforward to notice a combination of those vectors as the sum and difference beamformers is optimal for monopulse angle estimation. However there is no reference that derives monopulse angle accuracy with those monopulse beamformers.

In this paper, a derivation of monopulse angle accuracy to achieve CRLB is presented. Specifically a difference beamforming taper using antenna element positions originating from antenna center is introduced to achieve CRLB. This is a realization of the optimal difference beamformer for the element-level DBF. It will be shown that monopulse angle accuracy by the derived monopulse is agreed with CRLB. Quantitative discussions on angle accuracy improvement for the element-level DBF radar with uniform linear array (ULA), compared with a conventional monopulse are presented. Furthermore the suboptimal difference beamforming taper for the subarray-based DBF radar is derived. Conclusions of the discussions are validated by numerical simulation.

2. OPTIMAL MONOPULSE ANGLE ACCURACY

2.1. CRLB on Angle Accuracy

Signal model for a single target arriving at a linear array antenna is considered in this paper. Received vector $\mathbf{x}(t)$ for M elements linear array is given as

$$\mathbf{x}(t)=\mathbf{a}(u)s(t)+\mathbf{n}(t) \quad (1)$$

where $t, s(t), \mathbf{n}(t)$ are snapshot index number, complex amplitude of a target signal and white Gaussian received noise vector with averaged power σ^2 respectively. $\mathbf{a}(u)$ is a steering vector for target angle u in sine space given as

$$\mathbf{a}(u)=\left[\exp\left(j\frac{2\pi d_1}{\lambda}u\right),\cdots,\exp\left(j\frac{2\pi d_M}{\lambda}u\right)\right]^T \quad (2)$$

$$u=\sin(\theta) \quad (3)$$

where d_m, λ, θ are the m-th antenna element positions, wavelength and the target angle corresponding to u respectively.

CRLB is found by solving for the diagonal elements of the inverse of the Fisher information matrix (FIM). Specifically the first step is to calculate FIM, then partition FIM into blocks corresponding to wanted parameters, unwanted parameters and cross-term respectively. The final step is to derive CRLB for wanted parameters by inversion of the FIM by using inverse matrix formula. All the steps are straightforward to describe but may waste space with long and complicated expressions.

Due to space limitation, the derivation process of CRLB for the target angle u is spared and the final result is provided in this paper. Note that a very comprehensible derivation process of CRLB can be found in [8]. CRLB for u, or $C_{CRLB}(u)$, is given as

$$C_{CRLB}(u) = \frac{1}{2T \cdot ESNR}\left(\frac{\lambda}{2\pi}\right)^2 \left[\mathbf{a}^H(u)\mathbf{D}\left(\mathbf{I} - \frac{\mathbf{a}(u)\mathbf{a}^H(u)}{\mathbf{a}(u)^H\mathbf{a}(u)}\right)\mathbf{D}\mathbf{a}(u)\right]^{-1} \quad (4)$$

where $T, \mathbf{I}$ are total snapshot number and identity matrix respectively. $ESNR$ is a signal to noise power ratio at antenna element and given as

$$ESNR = \frac{1}{\sigma^2 T}\sum_{t=1}^{T}\left(s^*(t)s(t)\right). \quad (5)$$

$\mathbf{D}$ is a diagonal matrix given as

$$\mathbf{D} = diag(d_1, \ldots, d_M). \quad (6)$$

Note $\mathbf{D}$ is derived from the derivation of a steering vector $\mathbf{a}(u)$ with respect to u as

$$\mathbf{a}_\Delta(u) = \frac{\partial \mathbf{a}(u)}{\partial u} = j\frac{2\pi}{\lambda}\mathbf{D}\mathbf{a}(u). \quad (7)$$

To employ monopulse angle estimation method in phased array radar, antenna elements may be arranged in a symmetric fashion around antenna center as

$$\{d_1, d_2, \ldots, d_{M-1}, d_M\} = \{d_1, d_2, \ldots, -d_2, -d_1\}. \quad (8)$$

By substituting (7) and (8) into (4), angle accuracy σ_{CRLB} for a linear array antenna with the symmetric arrangement is derived as

$$\begin{aligned}\sigma_{CRLB} &= \sqrt{C_{CRLB}(u)} \\ &= \frac{\lambda}{2\pi}\frac{1}{\sqrt{2T \cdot ESNR \cdot \sum_{m=1}^{M} d_m^2}} \\ &= \frac{\lambda}{2\pi}\frac{\sqrt{M}}{\sqrt{\sum_{m=1}^{M} d_m^2}}\frac{1}{\sqrt{2SNR}}\end{aligned} \quad (9)$$

where SNR is a signal to noise power ratio after spatial and inter-pulse coherent integration as

$$SNR = T \cdot M \cdot ESNR. \quad (10)$$

2.2. Derivation of the Monopulse Angle Accuracy

In this section, derivation of the monopulse angle accuracy to achieve CRLB is presented. Symmetric arrangement as (8) is assumed. As beam pointing angle of the linear antenna is $u = 0$, a steering vector and its derivative vector are written as

$$\mathbf{a}_0 = \mathbf{a}(0) = \mathbf{1} \quad (11)$$

$$\mathbf{a}_\Delta = \mathbf{a}_\Delta(0) = j\frac{2\pi}{\lambda}\mathbf{D}\mathbf{1} = j\frac{2\pi}{\lambda}\mathbf{d} \quad (12)$$

where $\mathbf{1}, \mathbf{d}$ are given as

$$\mathbf{1} = [1, 1, \ldots, 1, 1]^T. \quad (13)$$

$$\mathbf{d} = [d_1, d_2, \ldots, -d_2, -d_1]^T. \quad (14)$$

For sum and difference beamforming, the beam weight vectors are given respectively as

$$\mathbf{w}_\Sigma = \mathbf{a}_0 = \mathbf{1}, \quad (15)$$

$$\mathbf{w}_\Delta = -j\sqrt{\frac{\mathbf{a}_0{}^H\mathbf{a}_0}{\mathbf{a}_\Delta{}^H\mathbf{a}_\Delta}}\mathbf{a}_\Delta = \frac{\sqrt{M}}{\sqrt{\sum_{m=1}^{M} d_m^2}}\mathbf{d}. \quad (16)$$

Note that the difference beam weight vector $\mathbf{w}_\Delta$ is a scaled vector of $\mathbf{a}_\Delta$ and normalized to have an equal norm with $\mathbf{w}_\Sigma$ and to have real number elements.

It is well-known that monopulse error curve can be approximated by a linear function around $u = 0$ as

$$\begin{aligned}\frac{km_{drv}}{BW_u}u &= \mathrm{Im}\left(\frac{\mathbf{w}_\Delta{}^H\mathbf{a}(u)}{\mathbf{w}_\Sigma{}^H\mathbf{a}(u)}\right) \\ &= \frac{\sqrt{M}}{\sqrt{\sum_{m=1}^{M} d_m^2}}\mathrm{Im}\left(\frac{\mathbf{d}^H\mathbf{a}(u)}{\mathbf{1}^H\mathbf{a}(u)}\right)\end{aligned} \quad (17)$$

where km_{drv}, BW_u are the normalized monopulse slope for the deriving monopulse and the beamwidth of the sum beam. By referring to (14), $\mathbf{1}^H\mathbf{a}(u)$ and $\mathbf{d}^H\mathbf{a}(u)$ in (17) can be manipulated as

$$\begin{aligned}\mathbf{1}^H\mathbf{a}(u) &= \sum_{m=1}^{M}\exp\left(j\frac{2\pi d_m}{\lambda}u\right) = 2\sum_{m=1}^{\frac{M}{2}}\cos\left(\frac{2\pi d_m}{\lambda}u\right) \\ &= M\end{aligned} \quad (18)$$

$$\begin{aligned}\mathbf{d}^H\mathbf{a}(u) &= \sum_{m=1}^{M} d_m\exp\left(j\frac{2\pi d_m}{\lambda}u\right) = j2\sum_{m=1}^{\frac{M}{2}} d_m\sin\left(\frac{2\pi d_m}{\lambda}u\right) \\ &= j\frac{2\pi}{\lambda}\left(\sum_{m=1}^{M} d_m^2\right)u\end{aligned} \quad (19)$$

In the development from the first line to the second line of the right-hand member of (18) and (19), a range of u is assumed to be small so that following approximation can be hold.

$$\cos\left(\frac{2\pi d_m}{\lambda}u\right) \approx 1 \quad (20)$$

$$\sin\left(\frac{2\pi d_m}{\lambda}u\right) \approx \frac{2\pi d_m}{\lambda}u \quad (21)$$

By substituting (18) and (19) into (17), following property can be derived.

$$\frac{km_{drv}}{BW_u} = \frac{2\pi}{\lambda} \frac{\sqrt{\sum_{m=1}^{M} d_m{}^2}}{\sqrt{M}} \tag{22}$$

Monopulse angle accuracy σ_{drv} is now given as

$$\begin{aligned}\sigma_{drv} &= \frac{BW_u}{km_{drv}} \frac{1}{\sqrt{2SNR}} \\ &= \frac{\lambda}{2\pi} \frac{\sqrt{M}}{\sqrt{\sum_{m=1}^{M} d_m{}^2}} \frac{1}{\sqrt{2SNR}}\end{aligned} \tag{23}$$

It is observed that the monopulse angle accuracy by (23) is agreed with CRLB derived by (9). Namely,

$$\sigma_{drv} = \sigma_{CRLB} \tag{24}$$

$$km_{drv} = km_{CRLB} \,. \tag{25}$$

In other words, it is proved that monopulse angle accuracy with the sum and difference beam weight vector by (15) and (16) respectively and antenna configuration with (8) can achieve CRLB.

Especially for ULA, antenna element positions are given as

$$d_m = md - \frac{(M+1)d}{2} \tag{26}$$

where d is the inter-elements spacing. From (26), following property is easily found.

$$\sum_{m=1}^{M} d_m{}^2 = \frac{d^2}{12} M\left(M^2-1\right) \tag{27}$$

By substituting (27) into (9) or (23) and with a large M where $M^2-1 \approx M^2$ can be hold, σ_{CRLB} for ULA given as

$$\begin{aligned}\sigma_{CRLB} &= \frac{1}{d}\frac{\lambda}{2\pi}\sqrt{\frac{6}{SNR\left(M^2-1\right)}} \\ &\approx \frac{1}{Md}\frac{\lambda}{2\pi}\sqrt{\frac{6}{SNR}} = \frac{\lambda}{D}\frac{\sqrt{6}}{2\pi}\sqrt{\frac{1}{SNR}} \\ &= \left(0.886\frac{\lambda}{D}\right)\cdot\left(\frac{\sqrt{6}}{0.886\cdot 2\pi}\right)\sqrt{\frac{1}{SNR}} \\ &= BW_{ULA}\cdot\left(\frac{\sqrt{6}}{0.886\cdot 2\pi}\right)\sqrt{\frac{1}{SNR}} \\ &= \frac{BW_{ULA}}{1.61\cdot\sqrt{2SNR}} = \frac{BW_{ULA}}{km_{CRLB}\cdot\sqrt{2SNR}}\end{aligned} \tag{28}$$

where D, BW_{ULA} are antenna aperture length of ULA and the beamwith of ULA.

$$D = Md \tag{29}$$

$$BW_{ULA} = 0.886\frac{\lambda}{D} \tag{30}$$

And by referring to (23), km_{CRLB} is specifically given as

$$km_{CRLB} = 1.61 \,. \tag{31}$$

2.3. Accuracy Improvement

In conventional monopulse estimation used in phased array radar, the sum and the difference beam are generated by summation and subtraction of beam outputs from symmetrically-divided subarray. For ULA case, following property is hold

$$\frac{km_{Mono}}{BW_u} = \frac{\pi D}{2\lambda} \,. \tag{32}$$

From (30) and (32), km_{Mono} for conventional monopulse in phased array radars can be easily found as

$$km_{Mono} = \frac{\pi D}{2\lambda} BW_u = \frac{\pi}{2} 0.886 = 1.39 \tag{33}$$

As a result, angle accuracy for conventional monopulse with ULA is given as

$$\sigma_{Mono} = \frac{BW_u}{km_{Mono}\sqrt{2SNR}} \tag{34}$$

By comaparing (28) with (34), angle accuracy improvement of the derived monopulse in the earlier section to conventional monopulse is given as

$$\frac{\sigma_{Mono}}{\sigma_{CRLB}} = \frac{km_{CRLB}}{km_{Mono}} = 1.16 \tag{35}$$

From (35), it is found that angle accuracy of the derived monopulse is 1.16 times higher than conventional monopulse due to the difference of the normalized monopulse slope. By this improvement, radar system engineer can allocate 1.3 dB lower SNR for achieving required angle accuracy for phased array radars.

2.3. Difference Beamformer for Subarray-based DBF

Subarray-based digital beamforming with several ten channels is a practical solution for phased array radars with several hundred or thousand antenna elements [9]. For this case, the suboptimal difference beamforming taper $\mathbf{w}_\Delta^{(M_{SA})}$ to $\mathbf{w}_\Delta$ where M_{SA} is a number of subarray is briefly derived. The suboptimal beamformer $\mathbf{w}_\Delta^{(M_{SA})}$ can be given by solving following a linearly constrained optimization problem.

$$\mathbf{w}_\Delta^{(M_{SA})} = \arg\min_{\mathbf{r}} \left\| \mathbf{Tr} - \mathbf{w}_\Delta \right\|^2 \quad \text{subject to } \mathbf{r}^H \mathbf{T}^H \mathbf{w}_\Sigma = 0 \tag{36}$$

where $\mathbf{T}$ is the subarray transformation matrix with $\mathbf{TT}^H = \mathbf{I}$. By using Lagrange multipliers, $\mathbf{w}_\Delta^{(M_{SA})}$ for (36) can be solved as

$$\mathbf{w}_\Delta^{(M_{SA})} = \left(\mathbf{T}^H\mathbf{T}\right)^{-1}\mathbf{T}^H\left(\mathbf{I} - \frac{\mathbf{w}_\Sigma \mathbf{w}_\Sigma{}^H \mathbf{T}\left(\mathbf{T}^H\mathbf{T}\right)^{-1}\mathbf{T}^H}{\mathbf{w}_\Sigma{}^H\mathbf{T}\left(\mathbf{T}^H\mathbf{T}\right)^{-1}\mathbf{T}^H\mathbf{w}_\Sigma}\right)\mathbf{w}_\Delta \,. \tag{37}$$

3. NUMERICAL SIMULATIONS

Numerical simulations are carried out to validate angle accuracy improvement of the derived monopulse over conventional monopulse. Firstly a scenario that single target signal is impinging on the element-level DBF radar with ULA consisting of 64 elements with half-wavelength element spacing is considered and RMSE of estimated angles by 1000 trials is validated for SNR ranging from 10 to 35 dB. Detail condition for the simulation is summarized in the Table 1.

RMSE versus SNR is shown in Fig.1. Note RMSE is normalized by a beamwidth of the sum beam. It is observed that plots of the derived monopulse are agreed with CRLB and angle accuracy is 1.16 times higher than conventional monopulse. In other words, SNR to allocate to meet required angle accuracy can be reduced by 1.3 dB.

Second scenario is for the subarray-based DBF where the antenna elements are uniformly grouped into $M_{SA} = 2,4,8,16,32,64$ non-overlapping subarrays and the suboptimal difference beamforming taper $\mathbf{w}_{\Delta}^{(M_{SA})}$ is used. Note that the cases for $M_{SA} = 2$ and 64 are equivalent to conventional monopulse and the derived monopulse with element-level DBF respectively. RMSE for various number of subarray M_{SA} at SNR = 20 dB is shown in Fig. 2. It is observed that RMSE of the derived monopulse with the suboptimal beamformer $\mathbf{w}_{\Delta}^{(M_{SA})}$ is nonlinearly degraded from CRLB by reducing the number of subarray M_{SA}.

TABLE I. SIMULATION CONDITION

Array type	ULA
Number of antenna elements	64 elements positioned by 0.5λ step
DOA of a signal	0 deg
Number of snapshot	16
SNR	10 dB to 35 dB
Number of independent trial	1000 Trial

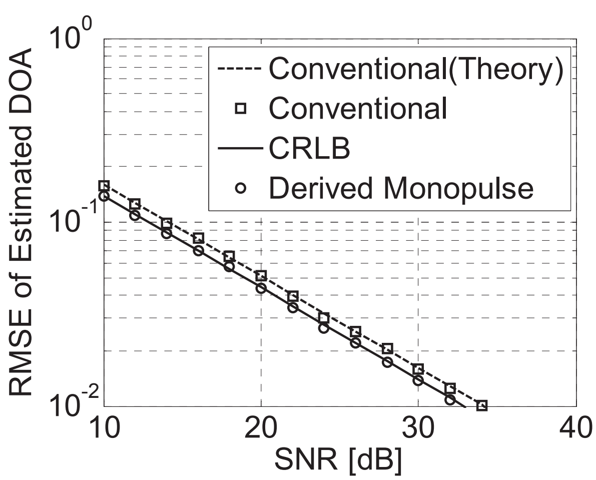

Figure 1. RMSE of angle estimation by the derived and conventional monopulse vs. SNR. RSME is normalized by a beamwidth of the sum beam. SNR is defined at output of spatial and inter-pulse coherent integration.

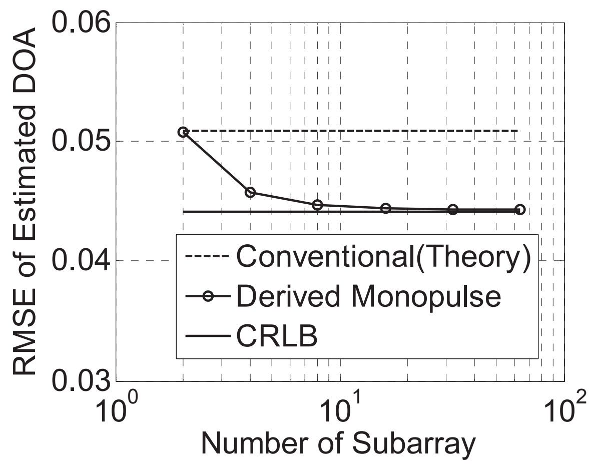

Figure 2. RMSE of angle estimation vs. number of subarray for SNR = 20 dB. The antenna elements are uniformly grouped into 2, 4, 8, 16, 32 and 64 subarrays. Cases of 2 and 64 subarrays are equivalent to conventional and the derived monopulse with element-level DBF.

4. CLOSING REMARKS

Derivation of monopulse angle accuracy to achieve CRLB was presented in this paper. Antenna element positions originating from antenna center were used for difference beam taper. Suboptimal difference beamforming taper for the subarray-based DBF radar was also derived. For ULA, the theoretical accuracy improvement was 1.16 times higher than conventional monopulse method. This figure was validated by numerical simulations.

5. REFERENCES

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