

Computationally simple DOA estimation of two resolved targets with a single snapshot

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Abstract—Direction-of-arrival (DOA) estimation of two targets with a single snapshot plays an important role in many pulsed radar array applications. We consider the case when the targets are spaced by more than the beamwidth of the array. In this case, the conventional beamformer (BF) is able to resolve them, but results in biased DOA estimation due to the leakage effect. We propose computationally simple strategies to reduce this bias. A novel method is presented, based on the analysis of the noise-free BF spectrum and a local approximation. We comment on computational cost and present simulation results.

Index Terms—Direction-of-arrival estimation, beamformer

I. INTRODUCTION

A pulsed radar system with multiple receiving antennas can be used for target localization in terms of range, relative velocity and direction-of-arrival (DOA) [1]. An accurate description of the environment is required in many application, such as automotive radar for modern driver assistance systems [2]. Using radar pre-processing, which consists of pulse compression and an FFT over the pulses, the received sensor data can be divided into processing cells corresponding to range and relative velocity, each represented by a single snapshot [1]. There are some crucial situations, in which a superposition of two targets occur in relevant processing cells [2]. Generally, these cases demand computationally expensive methods for high-resolution DOA estimation, such as subspace-based or maximum likelihood methods [3], [4]. However, when the targets are spaced by more than the beamwidth of the array, the conventional beamformer (BF) is able to resolve them [5]. The BF is computationally simple, but results in biased estimation in the two-target case, due to leakage. This paper aims at developing computationally simple strategies for reducing this bias.

We remark that the bias can also be reduced by using windows with a reduced sidelobe level, but at the cost of a degraded resolution. An alternative approach for DOA estimation of resolved targets is to iteratively subtract the targets to perform approximately single-target DOA estimation with the BF, as described in [6] for the frequency estimation problem. This approach is conceptually equivalent with the RELAX algorithm [7].

II. SIGNAL MODEL

Consider the single snapshot model with two targets

$$\mathbf{x} = s_1 \mathbf{a}(\phi_1) + s_2 \mathbf{a}(\phi_2) + \mathbf{n} \quad (1)$$

where

$$\mathbf{a}(\phi) = \frac{1}{\sqrt{M}} [1, e^{j\phi}, \dots, e^{j\phi(M-1)}]^T$$

is the steering vector of a uniform linear array (ULA), and $\phi_1 < \phi_2$ are the DOA parameters in electrical angle; s_1 and s_2 are the corresponding target response parameters, and \mathbf{n} is spatially white measurement noise. We consider the case such that $\phi_2 - \phi_1 > \text{BW}$, where $\text{BW} = 2\pi/M$ is the beamwidth of the array.

The BF spectrum is defined as

$$P(\phi) = |S(\phi)|^2 \quad (2)$$

where

$$S(\phi) = \mathbf{a}(\phi)^H \text{diag}\{\mathbf{w}\} \mathbf{x}$$

is the spatial Fourier transform, and \mathbf{w} contains the coefficients of a window function, normalized such that $\|\mathbf{w}\|^2 = M$. Given sufficiently high SNR, the two largest local maxima of $P(\phi)$ are close to the true DOA parameters [5].

A. Motivation

Due to the leakage effect, the DOA estimates, obtained using the BF spectrum, can be strongly biased. This is demonstrated by the following example.

For convenience, we use variables

$$\delta = \phi_2 - \phi_1, \quad \text{and} \quad \varphi = \angle\{s_2\} - \angle\{s_1\}$$

to denote half angular separation and relative phase, respectively. Also, ϕ_0 denotes the midpoint between ϕ_1 and ϕ_2 .

Let us consider the noise-free case with $|s_1| = |s_2|$, an array with $M = 8$ sensor elements and a rectangular window. For all combinations of $\delta/\text{BW} \in [1.5, \dots, 6.5]$ and $\varphi \in [0, 2\pi)$, we determine the two largest local maxima of the BF spectrum in (2), denoted by $\phi_{\text{BF},1} < \phi_{\text{BF},2}$. Figure 1 shows $\phi_{\text{BF},1} - \phi_1$, which corresponds to the bias of DOA estimation with the BF, due to leakage. It should be possible to exploit the observed regular structure.

In this paper, we present two computationally simple strategies to reduce this bias. The first is a novel method, which is based on the analysis of the noise-free BF spectrum and a local approximation with quadratic and linear functions. The second method simply consists of a single iteration of the RELAX algorithm [7].

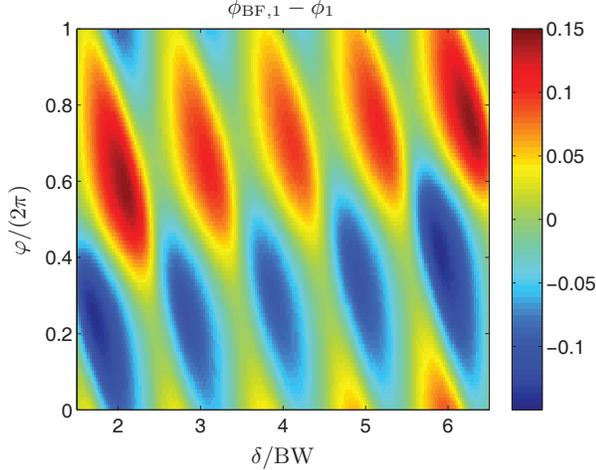


Fig. 1. Motivation example. Bias of DOA estimation with the BF caused by leakage, for several combinations of δ and φ .

III. PROPOSED ALGORITHM

Before introducing the proposed algorithm for bias reduction, we analyze the noise-free BF spectrum and the resulting bias of DOA estimation.

A. Noise-free BF spectrum

In the noise-free case, the spatial Fourier transform is

$$S(\phi) = s_1 W(\phi - \phi_1) + s_2 W(\phi - \phi_2)$$

where we define

$$W(\phi - \phi_i) = \mathbf{a}(\phi)^H \text{diag}\{\mathbf{w}\} \mathbf{a}(\phi_i), \quad i = 1, 2$$

The noise-free BF spectrum is then given in (3) at the top of the next page. Note that the first two terms are auto-terms and show peaks at exactly ϕ_1 and ϕ_2 , respectively. However, when superimposed, the sidelobes of the beampattern may slightly shift the peak locations. The last term is a cross-term and will further affect the peak locations. In the case with a single snapshot, the influence of the cross-term overshadows the effect of the sidelobes.

To analyze the bias caused by leakage, let us have a closer look at the terms $|W(\phi - \phi_i)|^2$ and $W(\phi - \phi_1)W(\phi - \phi_2)^*$. For convenience, we begin with the rectangular window.

1) *Rectangular window:* For $i = 1$ and 2, plugging in the definition, and after some simplification, we obtain

$$|W(\phi - \phi_i)|^2 = \frac{1}{M^2} \frac{\text{sinc}[(\phi - \phi_i)M/2]^2}{\text{sinc}[(\phi - \phi_i)/2]^2}$$

which is the well-known squared periodic sinc function, centered around ϕ_i . In the vicinity of ϕ_i , it can be approximated locally as a quadratic function, given by $\gamma_0 - \alpha(\phi - \phi_i)^2$ with $\alpha \approx \pi/BW^2$.

After some simplifications, we also obtain

$$W(\phi - \phi_1)W(\phi - \phi_2)^* = e^{-j\delta(M-1)/2} Q(\phi)$$

with real-valued function,

$$Q(\phi) = \frac{1}{M^2} \frac{\cos(\delta M/2) - \cos[(\phi - \phi_0)M]}{\cos(\delta/2) - \cos(\phi - \phi_0)}$$

centered around ϕ_0 . Its shape only depends on δ . Now, we are able to express the cross-term in (3) as

$$2\text{Re}\{s_1 s_2^* W(\phi - \phi_1)W(\phi - \phi_2)^*\} = 2|s_1||s_2|BQ(\phi)$$

with scaling term $B = \cos[\varphi + \delta(M-1)/2]$. In the vicinity of ϕ_i , $Q(\phi)$ can be approximated locally as a linear function, given by $\gamma_i + \beta_i(\phi - \phi_i)$ with

$$\beta_i = \lim_{\phi \rightarrow \phi_i} \frac{dQ(\phi)}{d\phi}$$

The limit does not exist. After applying l'Hôpital's rule twice, we obtain

$$\beta_1 = \frac{\cos(\delta/2) \sin(\delta M/2) - M \sin(\delta/2) \cos(\delta M/2)}{2M \sin(\delta/2)^2}$$

$$\beta_2 = -\beta_1$$

2) *Other window functions:* For window functions, other than the rectangular window, the local approximations are still valid, but expressions for α and β_1 are more complicated and require numerical computation. α can be determined by means of least-squares fitting, β_1 can be determined by employing the central first-order finite difference, for small Δ ,

$$\beta_1 \approx \frac{1}{2\Delta} [Q(\phi_1 + \Delta) - Q(\phi_1 - \Delta)]$$

where $Q(\phi) = e^{j\delta(M-1)/2} W(\phi - \phi_1)W(\phi - \phi_2)^*$.

Note that, for computational reasons, the values for β_1 can be stored in a 1-D lookup table as a function of δ .

3) *DOA estimation bias:* By applying the described local approximations in the vicinity of ϕ_i , the noise-free BF spectrum from (3) can be approximated as

$$\tilde{P}_i(\phi) = |s_i|^2 [\gamma_0 - \alpha(\phi - \phi_i)^2] + 2|s_1||s_2|B[\gamma_i + \beta_i(\phi - \phi_i)]$$

where we have neglected the influence of the other auto-term. We can determine the two largest local maxima of the BF spectrum by equating the first derivative to zero,

$$\frac{d\tilde{P}_i(\phi)}{d\phi} = -|s_i|^2 2\alpha(\phi - \phi_i) + 2|s_1||s_2|B\beta_i \stackrel{!}{=} 0$$

Rearranging terms and plugging in B , we obtain

$$\begin{aligned} \phi_{BF,1} &= \phi_1 + \frac{1}{\alpha} \frac{|s_2|}{|s_1|} \cos[\varphi + \delta(M-1)/2] \beta_1(\delta) \\ \phi_{BF,2} &= \phi_2 - \frac{1}{\alpha} \frac{|s_1|}{|s_2|} \cos[\varphi + \delta(M-1)/2] \beta_1(\delta) \end{aligned} \quad (4)$$

So the location of the maxima of the BF spectrum is equal to the true parameter plus a bias term, which depends on signal parameters $|s_1|$, $|s_2|$, φ and δ . Note that the bias term depends on a ratio between signal magnitudes, where the weaker target will have a larger DOA bias. We propose to obtain an enhanced estimator by subtracting this bias term.

$$P(\phi) = |s_1|^2 |W(\phi - \phi_1)|^2 + |s_2|^2 |W(\phi - \phi_2)|^2 + 2\text{Re}\{s_1 s_2^* W(\phi - \phi_1) W(\phi - \phi_2)^*\} \quad (3)$$

B. Proposed algorithm for bias reduction

Now, consider model (1) with noise. The proposed algorithm for bias reduction is summarized as follows:

- 1) Evaluate the BF spectrum, determine the two largest local maxima, $\hat{\phi}_{\text{BF},1}$ and $\hat{\phi}_{\text{BF},2}$.
- 2) Estimate target response parameters for $i = 1, 2$

$$\hat{s}_i = \mathbf{a}(\hat{\phi}_{\text{BF},i})^H \mathbf{x}$$

- 3) Calculate $\hat{\delta}$ and $\hat{\varphi}$ from $\hat{\phi}_{\text{BF},1}$, $\hat{\phi}_{\text{BF},2}$, \hat{s}_1 and \hat{s}_2 .
- 4) Determine $\beta_1(\hat{\delta})$ from a 1-D lookup table.
- 5) Estimate the corrected DOA parameters as

$$\hat{\phi}_1 = \hat{\phi}_{\text{BF},1} - \frac{1}{\alpha} \frac{|\hat{s}_2|}{|\hat{s}_1|} \cos[\hat{\varphi} + \hat{\delta}(M-1)/2] \beta_1(\hat{\delta})$$

$$\hat{\phi}_2 = \hat{\phi}_{\text{BF},2} + \frac{1}{\alpha} \frac{|\hat{s}_1|}{|\hat{s}_2|} \cos[\hat{\varphi} + \hat{\delta}(M-1)/2] \beta_1(\hat{\delta})$$

We remark that it is also possible to follow a 2-D lookup table approach, in which the full correction term from (4), but without $|s_1|$ and $|s_2|$, is computed numerically and stored in a 2-D lookup table of δ and φ , similar to Figure 1.

C. Computational aspects

Let ϕ_k , $k = 0, \dots, K-1$ be the discrete grid, required for the spectral search; the step size is $\Delta\phi = 2\pi/K$. The main part of the computational cost constitutes the evaluation of the BF spectrum in Step 1) in Section III-B, which can be efficiently computed by using an FFT of $\text{diag}\{\mathbf{w}\}\mathbf{x}$, padded with $K-M$ zeros.

Note that without increasing K , we can improve the DOA estimation accuracy, using a quadratic interpolation [2], [6],

$$\hat{\phi}_i = \phi_{n[i]} + 0.5\Delta\phi \frac{P_{n[i]-1} - P_{n[i]+1}}{P_{n[i]-1} + 2P_{n[i]} - P_{n[i]+1}} \quad (5)$$

where $n[i]$ and $P_{n[i]} = P(\phi_{n[i]})$ for $i = 1, 2$ are the index and spectral value, respectively, of the two largest local maxima.

The calculations required in Steps 3)-5) in Section III-B, which involve finding entries from lookup tables and simple scalar multiply-add operations, can be implemented efficiently and do not significantly contribute to the overall cost.

IV. RELAX FOR TWO TARGETS

The RELAX algorithm is an iterative technique for DOA estimation of multiple targets [7]. It aims at finding the maximum likelihood estimate (MLE), or non-linear least squares solution. For the case with two targets, this is to minimize

$$\|\mathbf{x} - s_1 \mathbf{a}(\phi_1) - s_2 \mathbf{a}(\phi_2)\|^2 \quad (6)$$

w.r.t. ϕ_1 , ϕ_2 , s_1 and s_2 . The principle of the RELAX algorithm is to iteratively subtract the present targets in order to perform approximately single-target DOA estimation with the

BF, which corresponds to the MLE and does not suffer from leakage. Towards this end, let

$$\mathbf{x}_i = \mathbf{x} - \hat{s}_{2-i+1} \mathbf{a}(\hat{\phi}_{2-i+1}), \quad i = 1, 2$$

where \hat{s}_{2-i+1} and $\hat{\phi}_{2-i+1}$ are assumed to be given. Then, the minimization of (6) w.r.t. ϕ_i and s_i can be simplified to

$$\hat{\phi}_i = \arg \max_{\phi_i} |\mathbf{a}(\phi_i)^H \mathbf{x}_i|^2,$$

$$\hat{s}_i = \mathbf{a}(\hat{\phi}_i)^H \mathbf{x}_i, \quad (7)$$

respectively. Note that $\|\mathbf{a}(\phi)\|^2 = 1$. Again, the cost function is simply the location of the dominant peak of the BF spectrum, with a rectangular window. Hence, it can also be efficiently computed by using an FFT. We have observed that, when the DOA estimates are obtained using a quadratic interpolation from (5), zero padding such that $K = 4M$ produces satisfactory results. This will be demonstrated with simulations.

As described in [7], the RELAX algorithm, for the case with two targets, is summarized as follows:

- 1) Assume a single target present and estimate $\hat{\phi}_1$ and \hat{s}_1 from \mathbf{x} , as in (7).
- 2) Assume two targets present, compute

$$\mathbf{x}_2 = \mathbf{x} - \hat{s}_1 \mathbf{a}(\hat{\phi}_1)$$

using the previous estimates. Determine $\hat{\phi}_2$ and \hat{s}_2 from \mathbf{x}_2 , as in (7).

Next, compute

$$\mathbf{x}_1 = \mathbf{x} - \hat{s}_2 \mathbf{a}(\hat{\phi}_2)$$

and redetermine $\hat{\phi}_1$ and \hat{s}_1 from \mathbf{x}_1 .

- 3) Convergence check; if the relative change of (6) between two iterations is smaller than $\epsilon = 0.01$, stop. Otherwise, continue with Step 2).

Let J denote the number of iterations until convergence. The RELAX algorithm for two targets roughly requires $2J+1$ times the computational cost of a standard DOA estimation with the BF, as described in Section III-C. Note that generally, the RELAX algorithm does not need the two targets to be resolved by the conventional BF. However, in this case, many iterations are required until convergence.

We remark that RELAX can also be implemented in the beamspace domain [8]. Instead of computing \mathbf{x}_2 , \mathbf{x}_1 , and their corresponding BF spectra explicitly in Step 2), we have

$$S_i(\phi) = S(\phi) - \hat{s}_{2-i+1} W(\phi - \hat{\phi}_{2-i+1}), \quad i = 1, 2$$

where $S(\phi)$ has been calculated in Step 1) and $W(\phi)$, which is stored in a 1-D lookup table, only has to be scaled and shifted in each iteration. Note that this approach has a reduced computational cost.

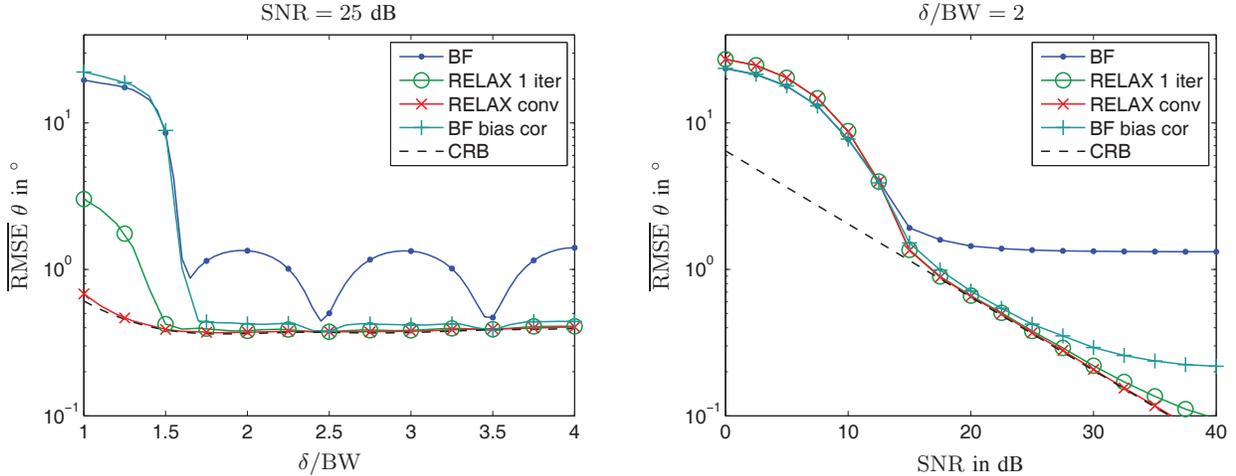


Fig. 2. Simulation results. Averaged RMSE versus δ at SNR = 25 dB (left) and versus SNR at $\delta/BW = 2$ (right), $M = 8$ elements with 20 dB Chebyshev window, log-normal target magnitudes and arbitrary phases.

V. SIMULATIONS

Consider an array with $M = 8$ elements and a Chebyshev window with 20 dB sidelobe attenuation. Two targets from ϕ_1 and ϕ_2 are simulated according to (1). For a fair assessment of the grid search DOA estimation, in each simulation run, we add a random jitter, distributed between $\pm\Delta\phi/2$, to the DOA parameters. We use $s_1 = 1$; target magnitude and phase of s_2 are distributed as

$$|s_2| \sim \mathcal{N}_{\log}(0 \text{ dB}, 3 \text{ dB}), \quad \text{and} \quad \angle\{s_2\} \sim \mathcal{U}[0, 2\pi)$$

which denote log-normal distribution and uniform distribution, respectively. The ratio of target magnitudes is chosen random to simulate realistic radar scenarios, with fluctuating radar cross-section [1]. \mathbf{n} is spatially white, circular complex Gaussian distributed with variance σ^2 . We define the SNR w.r.t. the first target as $1/\sigma^2$.

For calculating estimation errors, we use physical angle $\theta = \sin^{-1}(\phi/\pi)$ in degrees. As a performance metric, the averaged root mean square error (RMSE) is defined as

$$\overline{\text{RMSE}}^2 = \frac{1}{2} \left[\frac{1}{\text{MC}} \sum_{m=1}^{\text{MC}} (\hat{\theta}_{1,m} - \theta_1)^2 + (\hat{\theta}_{2,m} - \theta_2)^2 \right]$$

where $\hat{\theta}_{1,m}$ and $\hat{\theta}_{2,m}$ denote the DOA estimates in run m , and $\text{MC} = 5000$ is the number of Monte-Carlo runs. The proposed method for bias correction is compared with the conventional BF, and the RELAX algorithm, for a single iteration and until convergence. We also add the Cramér-Rao bound (CRB) [9]. For the spectral search, we use $K = 4M$.

In Figure 2, we show the simulation results versus angular separation δ (left) at SNR = 25 dB and versus SNR (right) at $\delta/BW = 2$. Due to leakage, the conventional BF results in an average error larger than 1° . Using the proposed method for bias correction, we are able to reduce the average error below 0.5° without significantly more computations. Note also that for $\delta > 1.5BW$, a single iteration of RELAX is very close to

convergence, and almost achieves the CRB; the computational cost is roughly three times the computational cost of a standard DOA estimation with the BF.

VI. CONCLUSIONS

In this paper, we have considered the problem of DOA estimation of two resolved targets with a single snapshot. We have presented a novel method for bias reduction, which is based on the analysis of the noise-free BF spectrum and a local approximation. The presented method is computationally simple. The simulation results suggest that it achieves almost equivalent results with the RELAX algorithm in the moderate SNR case.

REFERENCES

- [1] M. Richards, *Fundamentals of Radar Signal Processing*, McGraw-Hill, 2005.
- [2] P. Heidenreich and A. Zoubir, "Computational aspects of maximum likelihood DOA estimation of two targets with applications to automotive radar," in *Proceedings of the 5th Biennial on DSP for In-Vehicle Systems*, Kiel, Germany, Sep 2011.
- [3] H. Krim and M. Viberg, "Two decades of array signal processing research," *IEEE Signal Processing Magazine*, vol. 13, no. 4, pp. 67–94, Jul 1996.
- [4] T. Tuncer and B. Friedlander, *Classical and Modern Direction-of-Arrival Estimation*, Academic Press, 2009.
- [5] P. Stoica and R. Moses, *Introduction to Spectral Analysis*, Prentice-Hall, 1997.
- [6] M. Macleod, "Fast nearly ML estimation of the parameters of real or complex single tones or resolved multiple tones," *IEEE Trans. Signal Processing*, vol. 46, no. 1, pp. 141–148, Jan 1998.
- [7] J. Li and P. Stoica, "Efficient mixed-spectrum estimation with applications to target feature extraction," *IEEE Trans. Signal Processing*, vol. 44, no. 2, pp. 281–295, Feb 1996.
- [8] F. Engels, "Target shape estimation using an automotive radar," in *Proceedings of the 5th Biennial on DSP for In-Vehicle Systems*, Kiel, Germany, Sep 2011.
- [9] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and the Cramér-Rao bound," *IEEE Trans. Acoustics, Speech and Sig. Proc.*, vol. 37, no. 5, pp. 720–741, May 1989.