A KHATRI-RAO BASED METHOD FOR DOA ESTIMATION IN THE PRESENCE OF MUTUAL COUPLING

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ABSTRACT

A Khatri-Rao (KR) product based method for direction-of-arrival (DOA) estimation using uniform linear array (ULA) in the presence of mutual coupling is presented. Based on the fact that mutual coupling matrices of ULAs can be modeled as banded complex symmetric Toeplitz matrices, a cost function with the form of KR product is derived. An alternating minimization procedure is employed to estimate the DOAs of all the radiating signals as well as the mutual coupling coefficients and the power of signals. Simulation results that demonstrate the validity of the proposed method are included.

Index Terms**—** Array Signal Processing, Mutual Coupling, Direction of Arrival Estimation

1. INTRODUCTION

Direction-of-arrival (DOA) estimation of multiple narrowband signals has been widely investigated by the signal processing community [1]. Many high-resolution methods have been proposed [2-3]. However, the performance of these methods may be seriously degraded by the unknown manifold errors, such as the mutual coupling between neighboring array elements [4-5].

In the last decades, many calibration methods have been proposed. The most likely way for mutual coupling calibration is to make use of extra sources with known locations, namely, calibration sources. B. C. Ng *et al*. [6] proposed a maximum likelihood calibration method to compensate the mutual coupling as well as gain, phase and position errors. However, the procedure of mutual coupling calibration with calibration sources has the drawback of being time consuming. Alternatively, another kind of array calibration methods, named auto-calibration, is more preferable. In [4], Friedlander *et al*. proposed an iterative method that provides estimates of the DOAs of all the radiating sources as well as calibration of gain/phase of each array element and mutual coupling of the receiving array. Sellone *et al*. proposed another iterative method, which alternatively minimizes a cost function with respect to two complex symmetric Toeplitz matrices and a complex Hermitian Toeplitz matrix [7]. Ye *et al*. proposed a mutual coupling calibration method by setting a group of auxiliary sensors in a uniform linear array (ULA) [8].

In this paper, a new DOA estimation method is proposed for ULAs in the presence of mutual coupling. It is based on the observation that the mutual coupling matrix for a ULA can be modeled as a banded complex symmetric Toeplitz matrix [4]. A cost function derived from Khatri-Rao (KR) product is minimized alternatively with respect to a complex symmetric Toeplitz matrix with complex symmetric Toeplitz sub-matrices, the KR product of the array manifold matrix, and the powers of signals.

The rest of the paper is organized as follows. In Section 2, the signal model is introduced, while the DOA estimation method is described in detail in Section 3. In Section 4 simulation results are given. Finally, some conclusions are drawn in Section 5.

2. MUTUAL COUPLING MODEL

Consider K far-field, narrowband signals $s_k(t)$, $k=1,2,\cdots,K$ impinging on a ULA of M omnidirectional sensors with inter-sensor spacing d in the presence of additive Gaussian noise. Ignoring the mutual coupling between the array elements, the output of the mth sensor can be described as

$$x_m(t)=\sum_{k=1}^{K}s_k(t)e^{-j2\pi(d/\lambda)(m-1)\sin(\theta_k)}+n_m(t), \qquad (1)$$

where λ is the wavelength of the narrowband signal, θ_k is the DOA of the kth signal, and $n_m(t)$ is the additive noise of the mth sensor.

Let $\mathbf{x}(t)=[x_1(t),x_2(t),\cdots,x_M(t)]^T$ and similarly define $\mathbf{s}(t)$ and $\mathbf{n}(t)$. The matrix formulation of (1) can be written as

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$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (2)$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_K)]$ is the so-called array manifold matrix. The steering vector $\mathbf{a}(\theta)$ is defined as

$$\mathbf{a}(\theta) = [1, e^{-j2\pi(d/\lambda)\sin(\theta)}, \cdots, e^{-j2\pi(M-1)(d/\lambda)\sin(\theta)}]^T, \quad (3)$$

Now we consider the mutual coupling between the array elements. In practice, interactions between array elements will result in mutual coupling. The mutual coupling coefficient is dependent on the distance between the elements and the magnitude decreases quite fast as the distance increases. As a result the outputs of the array should be modified as

$$\mathbf{x}(t) = \mathbf{C}\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (4)$$

where $\mathbf{C}$ is an $M \times M$ complex mutual coupling matrix (MCM), which can be modeled as a banded symmetric Toeplitz matrix [4] as follows:

$$\mathbf{C} = Stoep(\mathbf{c}) = \begin{bmatrix} 1 & c_1 & \cdots & c_{P-1} & \cdots & 0 \\ c_1 & 1 & & & & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ c_{P-1} & & & 1 & & \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & & & & 1 \end{bmatrix}_{M\times M}, \quad (5)$$

where $\mathbf{c} = [1, c_1, \cdots, c_{P-1}]^T$, $1 > |c_1| > \cdots > |c_{P-1}| > 0$, $P < M$.

Our work is based on the assumptions that:

a) The signals are zero-mean and stationary, mutually independently with each other;
b) The noises are zero-mean and spatially white, statistically independent with the signals, and have the same variance;
c) All signals come from different directions.

Based on the above assumptions, the covariance matrices of the noise vector, signal vector and array output vector are

$$E\{\mathbf{n}(t)\mathbf{n}^H(t)\} = \sigma_n^2 \mathbf{I}_M, \quad (6)$$

$$\mathbf{R}_s = E\{\mathbf{s}(t)\mathbf{s}^H(t)\} = diag\{[\sigma_1^2, \sigma_2^2, \cdots, \sigma_K^2]^T\}, \quad (7)$$

$$\mathbf{R}_x = E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{C}\mathbf{A}\mathbf{R}_s\mathbf{A}^H\mathbf{C}^H + \sigma_n^2\mathbf{I}_M, \quad (8)$$

where $E\{\bullet\}$ denotes statistical expectation and the superscript ^{H} denotes conjugate transpose, σ_n^2 is the power of noise of each array element, σ_k^2 is the power of signal $s_k(t)$, and $\mathbf{I}_M$ is the $M \times M$ identity matrix.

3. DOA ESTIMATION IN THE PRESENCE OF MUTUAL COUPLING

3.1 Introductory theorems

We use the symbol $\odot$ to denote the KR product. For two matrices: $\mathbf{A} \in \mathbb{C}^{m\times k}$ and $\mathbf{B} \in \mathbb{C}^{n\times k}$ with the same number of columns, the KR product of them is defined as [9]

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1, \mathbf{a}_2 \otimes \mathbf{b}_2, \cdots, \mathbf{a}_k \otimes \mathbf{b}_k], \quad (9)$$

where $\otimes$ denotes the Kronecker product and $\mathbf{a}_i$, $\mathbf{b}_i$ are the ith column of $\mathbf{A}$ and $\mathbf{B}$, respectively.

Here we incorporate a useful property of KR product. We just give the conclusion and the proof can be found in [9].

Property 1: Let $\mathbf{A} \in \mathbb{C}^{m\times k}, \mathbf{B} \in \mathbb{C}^{n\times k}, \mathbf{d} \in \mathbb{C}^k$ and $\mathbf{D} = diag(\mathbf{d})$. Then

$$vec(\mathbf{A}\mathbf{D}\mathbf{B}^H) = (\mathbf{B}^* \odot \mathbf{A})\mathbf{d}, \quad (10)$$

In order to develop the proposed method, it is convenient to introduce two useful lemmas.

Lemma 1 [4]: If $\mathbf{c} = [1, c_1, \cdots, c_{P-1}]^T$ and $\mathbf{C} = Stoep(\mathbf{c})$, then for any $M \times 1$ complex vector $\mathbf{x}$, we have

$$\mathbf{C}\mathbf{x} = T_1[\mathbf{x}]\mathbf{c}, \quad (11)$$

where $T_1[\mathbf{x}]$ is given by the sum of the two following $M \times P$ matrices

$$[\mathbf{W}_1]_{p,q} = \begin{cases} x_{p+q-1}, & p+q \le M+1 \\ 0, & otherwise \end{cases}, \quad (12)$$

$$[\mathbf{W}_2]_{p,q} = \begin{cases} x_{p-q+1}, & p \ge q \ge 2 \\ 0, & otherwise \end{cases}, \quad (13)$$

Lemma 2: If $\mathbf{c} = [1, c_1, \cdots, c_{P-1}]^T$ and $\mathbf{C} = Stoep(\mathbf{c})$, then for any $M^2 \times 1$ complex vector $\tilde{\mathbf{x}} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_M^T]^T$, $\mathbf{x}_m^T \in \mathbb{C}^M$, we have

$$(\mathbf{C}^* \otimes \mathbf{C})\tilde{\mathbf{x}} = T_2[\tilde{\mathbf{x}}]\tilde{\mathbf{c}} \quad (14)$$

where $\tilde{\mathbf{c}} = vec(\mathbf{c}\mathbf{c}^H)$ and $T_2[\tilde{\mathbf{x}}]$ is given by the sum of the two following $M^2 \times P^2$ matrices

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{1,1} & \mathbf{F}_{1,2} & \cdots & \mathbf{F}_{1,P} \\ \mathbf{F}_{2,1} & \mathbf{F}_{2,2} & \cdots & \mathbf{F}_{2,P} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{M,1} & \mathbf{F}_{M,2} & \cdots & \mathbf{F}_{M,P} \end{bmatrix}, \quad (15)$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} & \cdots & \mathbf{G}_{1,P} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \cdots & \mathbf{G}_{2,P} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{M,1} & \mathbf{G}_{M,2} & \cdots & \mathbf{G}_{M,P} \end{bmatrix}, \quad (16)$$

$$\mathbf{F}_{p,q} = \begin{cases} T_1[\mathbf{x}_{p+q-1}], & p+q \le M+1 \\ 0, & otherwise \end{cases}, \quad (17)$$

$$\mathbf{G}_{p,q} = \begin{cases} T_1[\mathbf{x}_{p-q+1}], & p \ge q \ge 2 \\ 0, & otherwise \end{cases}, \quad (18)$$

where $T_1[\bullet]$ is defined in Lemma 1.

Proof: The proof of lemma 1 is given in [4], and the proof of lemma 2 is similar.

3.2 DOA estimation

Let $\mathbf{r}_s = [\sigma_1^2, \sigma_2^2, \cdots, \sigma_K^2]^T$, $\mathbf{S}_x = \mathbf{R}_x - \sigma_n^2\mathbf{I}_M$, from (7-8) we get

$$\mathbf{S}_x = \mathbf{CA}(diag(\mathbf{r}_s))(\mathbf{CA})^H , \quad (19)$$

In light of property 1, we have

$$vec(\mathbf{S}_x) = ((\mathbf{CA})^* \odot (\mathbf{CA}))\mathbf{r}_s = (\mathbf{C}^* \otimes \mathbf{C})(\mathbf{A}^* \odot \mathbf{A})\mathbf{r}_s , (20)$$

From lemma 2, we can conclude that $\mathbf{C}^* \otimes \mathbf{C}$ can be determined by $\tilde{\mathbf{c}} = vec(\mathbf{cc}^H)$. On the other hand, $(\mathbf{A}^* \odot \mathbf{A})$ is a matrix function of $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_K]^T$. Once the number of signals K is known, reasonable estimates of DOAs can be obtained by minimizing the following cost function with respect to $\tilde{\mathbf{c}}$, $\boldsymbol{\theta}$ and $\mathbf{r}_s$

$$J = \| vec(\mathbf{S}_x) - (\mathbf{C}^* \otimes \mathbf{C})(\mathbf{A}^* \odot \mathbf{A})\mathbf{r}_s \|^2 , \quad (21)$$

where $\|\bullet\|$ indicates the Euclidean norm of a vector. A necessary condition for uniqueness of the solution is $K(2M-K) \geq 2K + 2(P^2-1)$.

Since a closed-form solution of such a problem is extremely difficult to be found, we propose an alternating minimization iterative technique as follows.

1) minimize J with respect to $\tilde{\mathbf{c}}$

$$\min_{\tilde{\mathbf{c}}} \| vec(\mathbf{S}_x) - (\mathbf{C}^* \otimes \mathbf{C})(\mathbf{A}^* \odot \mathbf{A})\mathbf{r}_s \|^2 , \quad (22)$$

A priori is that the first element of $\tilde{\mathbf{c}}$ is 1, *i.e.* $\tilde{c}_1 = 1$. Let $\mathbf{b} = (\mathbf{A}^* \odot \mathbf{A})\mathbf{r}_s$, and $\mathbf{b} = [\mathbf{b}_1^T, \mathbf{b}_2^T, \cdots, \mathbf{b}_M^T]^T$, $\mathbf{b}_m \in \mathbb{C}^M$. In the light of lemma 2, we can write (22) as

$$\min_{\tilde{\mathbf{c}}} \| vec(\mathbf{S}_x) - T_2[\mathbf{b}]\tilde{\mathbf{c}} \|^2 \quad s.t. \quad \tilde{c}_1 = 1 , \quad (23)$$

If we let $T_2[\mathbf{b}]_{:,1}$ and $T_2[\mathbf{b}]_{:,2:K}$ to be the first column and the sub-matrix contains all columns except the first one of $T_2[\mathbf{b}]$, respectively, the solution of (23) can be given exactly

$$\tilde{\mathbf{c}} = \begin{bmatrix} 1 \\ (T_2[\mathbf{b}]_{:,2:K})^{\#}(vec(\mathbf{S}_x) - T_2[\mathbf{b}]_{:,1}) \end{bmatrix} , \quad (24)$$

where $^{\#}$ denotes Moore-Penrose inverse.

Then the estimate of $\mathbf{C}^* \otimes \mathbf{C}$ can be given as

$$\mathbf{C}^* \otimes \mathbf{C} = \begin{bmatrix} \overline{\mathbf{C}}_{1,1} & \overline{\mathbf{C}}_{1,2} & \cdots & \overline{\mathbf{C}}_{1,M} \\ \overline{\mathbf{C}}_{2,1} & \overline{\mathbf{C}}_{2,2} & \cdots & \overline{\mathbf{C}}_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mathbf{C}}_{M,1} & \overline{\mathbf{C}}_{M,2} & \cdots & \overline{\mathbf{C}}_{M,M} \end{bmatrix} , \quad (25)$$

$$\overline{\mathbf{C}}_{i,j} = \begin{cases} Stoep(\mathbf{c}_{abs(i-j)+1}), & abs(i-j) \leq P-1 \\ \mathbf{0}, & otherwise \end{cases} , \quad (26)$$

where $\mathbf{c}_p \in \mathbb{C}^P$ is a part of $\tilde{\mathbf{c}}$, $\tilde{\mathbf{c}} = [\mathbf{c}_1^T, \mathbf{c}_2^T, \cdots, \mathbf{c}_P^T]^T$.

2) Minimize J with respect to $\boldsymbol{\theta}$

$$\min_{\boldsymbol{\theta}} \| vec(\mathbf{S}_x) - (\mathbf{C}^* \otimes \mathbf{C})(\mathbf{A}^* \odot \mathbf{A})\mathbf{r}_s \|^2 , \quad (27)$$

Since $\mathbf{A}^* \odot \mathbf{A}$ is a known continuously differentiable function of $\boldsymbol{\theta}$, we make use of gradient descent algorithm to get the solution.

3) Minimize J with respect to $\mathbf{r}_s$

$$\min_{\mathbf{r}_s} \| vec(\mathbf{S}_x) - (\mathbf{C}^* \otimes \mathbf{C})(\mathbf{A}^* \odot \mathbf{A})\mathbf{r}_s \|^2 , \quad (28)$$

The closed-form solution is given by

$$\mathbf{r}_s = ((\mathbf{C}^* \otimes \mathbf{C})(\mathbf{A}^* \odot \mathbf{A}))^{\#} vec(\mathbf{S}_x) , \quad (29)$$

3.3 Numerical implementation

Let us suppose that N snapshots of the array output vector $\mathbf{x}(t), t = 1, 2, \cdots, N$ are available for processing. Here we assume the number of signals K and mutual coefficients P are known. The proposed method can be described as follows.

a) Preprocess

The first step is to estimate the array covariance matrix $\mathbf{R}_x$. It can be done by adopting the sample covariance matrix

$$\hat{\mathbf{R}}_x = \frac{1}{N} \sum_{t=1}^{N} \mathbf{x}(t)\mathbf{x}^H(t) , \quad (30)$$

Subsequently, the noise power σ_n^2 is estimated as the mean of the $M-K$ smallest eigenvalues of $\hat{\mathbf{R}}_x$, denoted as $\hat{\sigma}_n^2$.

Then the estimate of $\mathbf{S}_x$ is given by $\hat{\mathbf{S}}_x = \hat{\mathbf{R}}_x - \hat{\sigma}_n^2 \mathbf{I}_M$.

b) Initialization

The initial values of $\mathbf{C}$ can be selected from previous knowledge, or just initialized as an identity matrix when no previous knowledge is available. The initial values of $\boldsymbol{\theta}$ can be obtained using other DOA estimation algorithms, where mutual coupling is taken into account or not. For simplicity, we just incorporate the basic MUSIC algorithm ignoring the mutual coupling. With the initial $\mathbf{C}$ and $\boldsymbol{\theta}$, $\mathbf{r}_s$ is initialized as (29).

The remaining steps of the algorithm are alternating minimization steps described in the last sub-section, and it is implemented iteratively for a preset times.

4. COMPUTER SIMULATIONS

In this section, some computer simulations are reported to demonstrate the behavior of the proposed method. Herein we compare the performance of the proposed method with the one proposed in [8] and MUSIC without taking mutual coupling into account.

We consider a ULA consisting of $M = 8$ sensors with inter-sensor spacing $d = \lambda / 2$. The simulation model consists of two signal arriving at angles -11° and 7°. We use simulated signal vectors and noise vectors drawn from a complex Gaussian distribution with zero mean and covariance matrices $diag\{[\sigma_1^2, \sigma_2^2, \cdots, \sigma_K^2]^T\}$ and $\sigma_n^2 \mathbf{I}_M$, respectively. The mutual coupling is assumed to be negligible at a distance larger than 1.5λ and hence $P = 3$. The mutual coupling vector is assumed to be $\mathbf{c} = [1, 0.43301 - 0.25j, 0.14142 - 0.14142j]^T$, for which

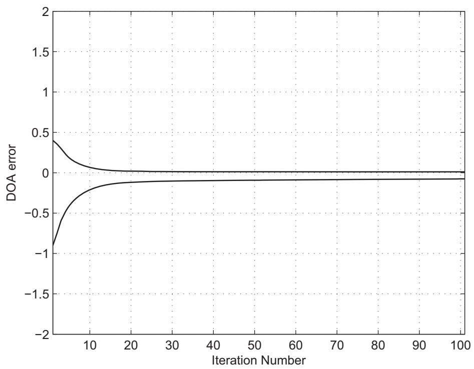

Fig.1 DOA estimation errors versus iteration number

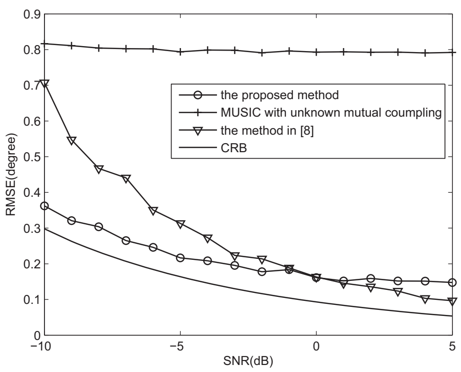

Fig. 2 RMSE of the DOA estimates versus SNR. The number of snapshots is 500.

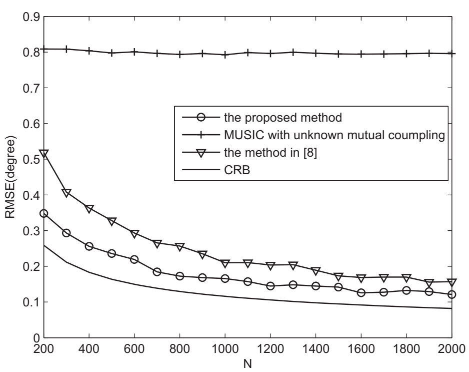

Fig. 3 RMSE of the DOA estimates versus the number of snapshots. SNR is -5dB.

there are no blind angles [8]. Experimental results are shown in Figs. 1-3.

Fig.1 shows the reduction of DOA estimation errors for a trial during the iterative procedure, when the number of snapshots is 500 and the SNR of both signals are 0dB. It demonstrates that convergence is reached in approximately 25 iterations. So it is reasonable to set the number of iteration to 60 as we done in the following experiments.

To compare the statistical performance of the proposed method with the method in [8] and MUSIC, the following Monte Carlo simulations have been realized. 500 Monte Carlo trials for each condition are performed. The root mean squared errors (RMSE) of DOAs of the three methods and the Cramer-Rao Bound (CRB) are compared in the same scenario.

Fig.2 illustrates RMSE of the estimated DOAs versus SNR (from -10 dB to 5dB). Here the number of snapshots is fixed to 500. The results demonstrate that, for low SNR, the DOAs estimations are more accurate by using the proposed method. The reason is that, in the method [8], just the middle subarray is used to estimate the DOAs.

Fig.3 illustrates RMSE of the estimated DOAs versus the number of snapshots (from 200 to 2000). Here the SNR is fixed to -5dB. The results demonstrate that the DOA estimates are more accurate by using the proposed method.

5. CONCLUSIONS

A new method of DOA estimation tailored for ULA in the presence of mutual coupling has been proposed. We deduced a cost function based on the KR product. Taking the symmetric Toeplitz structure into account, we developed an alternating minimization procedure to solve the DOA estimation problem. Computer simulation results have showed the effectiveness of the proposed method.

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