MISSING REGION RECOVERY BY PROMOTING BLOCKWISE LOW-RANKNESS

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ABSTRACT

In this paper, we propose a novel missing region recovery method by promoting blockwise low-rankness. It is natural to assume that images often have local repetitive structures. Hence, any small block extracted from an image is expected to be a low-rank matrix. Based on this assumption, we formulate missing region recovery as a convex optimization problem via newly introduced *block nuclear norm* which promotes blockwise low-rankness of an image with missing regions. An iterative scheme for approximating a global minimizer of the problem is also presented. The scheme is based on the alternating direction method of multipliers (ADMM) and allows us to restore missing regions efficiently. Experimental results reveal that the proposed method can recover missing regions with detailed local structures.

Index Terms— Interpolation, Convex optimization, Low-rankness, Image restoration, Inpainting

1. INTRODUCTION

Recovery of missing regions in images is required for many applications, e.g., digital effect (object removal), image restoration (scratch or text removal in photograph), image coding and transmission (recovery of the missing blocks).

A major approach of missing region recovery is *example based* (e.g. [1] [2]). This approach utilizes small blocks, from known regions, which is intuitively suitable to recover missing regions. As a result, it needs heuristic and complex procedures such as segmentation, edge detection, and block matching. Therefore, in case of the example based approach, the optimality of the recovered image is not guaranteed.

On the other hand, many image recovery methods based on convex optimization techniques have been proposed in the last decade (e.g. [3] [4] [5]). Because of the convexity, the optimality of the recovered image is guaranteed exactly in this approach. In case of missing region recovery, *total variation (TV) based* approach is studied in [6]. The TV based approach is effective for recovery of nontextured and small missing regions. Moreover, it is fast and free from any heuristic and complex procedures. However, the TV based approach tends to produce over-smoothing effects especially in textured/large missing regions.

In this paper, we propose a novel missing region recovery method based on convex optimization techniques. The key observation of our method is that any local region in an image often have repetitive structures in itself. In other words, if we see any small block extracted from an image as a matrix, it can be expected to be low-rank — it contains similar rows and columns. We call this property as *blockwise low-rankness* of an image. Hence, it is very natural to assume that the lost structures in missing regions can be restored by promoting blockwise low-rankness. Moreover, even if a missing region contains texture, our method can recover it efficiently as long as the missing region originally possesses repetitive structures same in the surrounding region.

First, we formulate missing region recovery as a convex optimization problem by employing newly introduced *block nuclear norm* which promotes blockwise low-rankness of an image. Then, we reformulate the problem into a standard form of an iterative algorithm called alternating direction method of multipliers (ADMM) [7]. This form allows us to approximate a global minimizer of the problem by using an ADMM based algorithm which is free from any heuristic and complex procedure. It also guarantees the optimality of a recovered image. The proposed method is tested on a set of various images with missing regions.

2. PRELIMINARIES

2.1. Proximity Operator

In this paper, we use the proximity operator which was introduced originally by Moreau in 1962 [8] in convex analysis.

Definition 2.1 (Proximity operator) Let $\Gamma_0(\mathbb{R}^N)$ be the class of proper lower semicontinuous convex function on $\mathbb{R}^N$. For any $h \in \Gamma_0(\mathbb{R}^N)$, $\gamma \in (0, \infty)$ and $\mathbf{y} \in \mathbb{R}^N$, the minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^N} h(\mathbf{x}) + \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{x}\|_2^2 \quad (1)$$

admits a unique solution, which is denoted by $\mathrm{prox}_{\gamma h}(\mathbf{y})$ called the proximity operator of index γ of h.

Implementation of the proximity operator depends on the function h. We introduce two examples of h as follows those will be utilized in our method.

Example 2.1 Define $h : \mathbb{R}^{n\times n(=N)} \to [0,\infty)$ by

$$h : \mathbf{x} \mapsto \|\mathbf{x}\|_* = \sum_{k=1}^{r} \sigma_k(\mathbf{x}) \tag{2}$$

where r is the rank of $\mathbf{x}$ and $\sigma_k(\mathbf{x})$ ($k = 1,\dots,r$) is the k-th singular value of $\mathbf{x}$. $\|\cdot\|_*$ is so called nuclear norm. It is known that the best low-rank approximation of a matrix is obtained by suppressing its nuclear norm. The nuclear norm has been utilized in many applications, for example, in principal component analysis [9] [10] and tensor recovery [11]. Let $\mathbf{x} = \mathbf{U_x}\mathbf{\Sigma_x}\mathbf{V_x}^*$ be singular value decomposition of the matrix $\mathbf{x} \in \mathbb{R}^{n_v\times n_h}$ and $\mathbf{\Sigma_x} = \mathrm{diag}(\sigma_1(\mathbf{x}),\dots,\sigma_r(\mathbf{x}))$ be a diagonal matrix containing the singular values of $\mathbf{x}$ Then, define $\tilde{\mathbf{\Sigma}}_\mathbf{x}$ as a diagonal matrix containing the γ-shrunk singular values of $\mathbf{x}$, i.e.,

$$\tilde{\mathbf{\Sigma}}_\mathbf{x} := \mathrm{diag}(\max\{\sigma_1(\mathbf{x})-\gamma,0\},\dots,\max\{\sigma_r(\mathbf{x})-\gamma,0\}). \tag{3}$$

The proximity operator of (2) is given by

$$\mathrm{prox}_{\gamma h}(\mathbf{x}) = \mathbf{U_x}\tilde{\mathbf{\Sigma}}_\mathbf{x}\mathbf{V_x}^* \tag{4}$$

Example 2.2 For a given non-empty closed convex set $C \subset \mathbb{R}^N$, define the indicator function $\iota_C : \mathbb{R}^N \to [0,\infty]$ by

$$\iota_C(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in C, \\ \infty, & \text{otherwise.} \end{cases} \tag{5}$$

The proximity operator of ι_C is given by the metric projection onto C, i.e.,

$$\mathrm{prox}_{\gamma\iota_C}(\mathbf{x}) = P_C(\mathbf{x}) := \operatorname*{argmin}_{\mathbf{y}\in C} \|\mathbf{x}-\mathbf{y}\|_2. \tag{6}$$

2.2. Alternating Direction Method of Multipliers (ADMM)

Problem 2.1 Suppose $\mathbf{G} \in \mathbb{R}^{N_z\times N_y}$ is full-column rank, consider the optimization problem for $f \in \Gamma_0(\mathbb{R}^{N_y})$ and $g \in \Gamma_0(\mathbb{R}^{N_z})$:

$$\min_{(\mathbf{y},\mathbf{z})\in\mathbb{R}^{N_y}\times\mathbb{R}^{N_z}} f(\mathbf{y}) + g(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{z} = \mathbf{G}\mathbf{y}. \tag{7}$$

The ADMM algorithm [7] shown in Algorithm 2.1 is an iterative algorithm to approximate a solution of problem (7).

Algorithm 2.1 (ADMM)

1: Set $k = 0$, choose $\mathbf{z}^{(0)}$ and $\mathbf{b}^{(0)}$.
2: **while** a stop criterion is not satisfied **do**
3: $\mathbf{y}^{(k+1)} = \operatorname*{argmin}_{\mathbf{y}\in\mathbb{R}^{N_y}} \left\{ f(\mathbf{y}) + \frac{1}{2\gamma}\|(\mathbf{z}^{(k)} - \mathbf{G}\mathbf{y} - \mathbf{b}^{(k)})\|_2^2 \right\}$
4: $\mathbf{z}^{(k+1)} = \operatorname*{argmin}_{\mathbf{z}\in\mathbb{R}^{N_z}} \left\{ g(\mathbf{z}) + \frac{1}{2\gamma}\|(\mathbf{z} - \mathbf{G}\mathbf{y}^{(k+1)} - \mathbf{b}^{(k)})\|_2^2 \right\}$
5: $\mathbf{b}^{(k+1)} = \mathbf{b}^{(k)} + \mathbf{G}\mathbf{y}^{(k+1)} - \mathbf{z}^{(k+1)}$
6: $k \leftarrow k+1$
7: **end while**

3. PROPOSED METHOD

In this section, we newly introduce *block nuclear norm* which is a convex function and approximates blockwise rank information of an image. Then, we formulate missing region recovery as a convex optimization problem via the block nuclear norm and an iterative scheme to solve it.

3.1. The block nuclear norm and its proximity operator

Let $\mathbf{x} \in \mathbb{R}^{n\times n(=N)}$ be an image, $\mathbf{d}_\mathbf{x}^{(p,q)} \in \mathbb{R}^{m\times m(=M)}$ be a block of $\mathbf{x}$ whose upper-left pixel coordinate is (p,q). We propose the block nuclear norm defined as

$$\|\mathbf{x}\|_{b*} := \sum_{i=0}^{\frac{m}{\delta}-1}\sum_{j=0}^{\frac{m}{\delta}-1}\sum_{k=0}^{\frac{n}{m}-1}\sum_{l=0}^{\frac{n}{m}-1} \left\| \mathbf{d}^{(mk+1,ml+1)}_{\mathrm{S}_{\delta i,\delta j}(\mathbf{x})} \right\|_*, \tag{8}$$

where $\mathrm{S}_{i,j}(\mathbf{x})$ denotes the procedure of i horizontal and j vertical periodic shift, δ denotes the shift step number which controls an overlap level. (Note that, for simplicity, (i) we only treat a square image and block, (ii) n and m are divisible by m and δ respectively). The block nuclear norm is equal to the sum of the singular values of all target blocks of an image (which are allowed to be overlapped).

By simple algebra, we can confirm that the proximity operator of $\|\cdot\|_{b*}$ is given by

$$\mathrm{prox}_{\gamma\|\cdot\|_{b*}}(\mathbf{x}) := \frac{\delta^2}{M}\sum_{i=0}^{\frac{m}{\delta}-1}\sum_{j=0}^{\frac{m}{\delta}-1} \mathrm{S}_{-\delta i,-\delta j}(\tilde{\mathbf{x}}_{i,j}), \tag{9}$$

where

$$\tilde{\mathbf{x}}_{i,j} := \mathrm{BP}\left(\mathrm{prox}_{\frac{\gamma m}{\delta}\|\cdot\|_*}, \mathrm{S}_{\delta i,\delta j}(\mathbf{x}), m\right), \tag{10}$$

'BP(f, $\mathbf{x}$, m)' denotes the process of applying f to each non-overlapped $m \times m$ size block of $\mathbf{x}$ respectively. In other words, the proximity operator of $\|\cdot\|_{b*}$ is equivalent to apply singular value shrinkage to all target blocks and averaging pixel values in overlapped areas.

3.2. A convex optimization problem via the block nuclear norm for missing region recovery

Let $\mathbf{v} \in \mathbb{R}^N$ be an image with K missing regions (all missing regions are zero padded). The proposed convex optimization problem for missing region recovery is formulated as follows.

Problem 3.1 (proposed convex optimization problem)

$$\min_{\mathbf{x}\in\mathbb{R}^N} \|\mathbf{x}\|_{b*} + \iota_{C_1}(\mathbf{x}) + \iota_{C_2}(\mathbf{x}) + \iota_{C_3}(\mathbf{x}), \tag{11}$$

where C_1, C_2, and C_3 are the nonempty closed convex sets which represent constraints to obtain a desirable recovered image (detailed explanation is given in Remark 3.1).

Remark 3.1 (Roles of functions in (11))

$\|\cdot\|_{b*}$: suppressing this function is expected to recover missing regions because it is expected that an image often has local repetitive structures.

ι_{C_1}: a constraint on the average of the pixel values in missing regions represented by the following non-empty closed convex set

$$C_1 = \{\mathbf{x} \in \mathbb{R}^N \,|\, \text{mean}(\mathbf{x}, \mathcal{M}_{\mathbf{x}}^{(k)}) = \text{mean}(\mathbf{x}, \mathcal{P}_{\mathbf{x}}^{(k)}) \quad \text{for } k = 1, \ldots, K\} \quad (12)$$

where $\mathcal{M}_{\mathbf{x}}^{(k)}$ is the index set of k-th missing region pixels of $\mathbf{x}$, and $\mathcal{P}_{\mathbf{x}}^{(k)}$ is the index set of pixels which surround $\mathcal{M}_{\mathbf{x}}^{(k)}$. 'mean($\mathbf{x}, \mathcal{I}$)' denotes the process of calculating an average of the pixel values of $\mathbf{x}$ of which index is in $\mathcal{I}$. This constraint is required for matching brightness between a missing region and its surrounding regions.

ι_{C_2}: a constraint on the numerical range of the pixel values represented by the following non-empty closed convex set

$$C_2 = \{\mathbf{x} \in \mathbb{R}^N \,|\, x_i \in [0, 255] \quad \text{for } i = 1, \ldots, N\}, \quad (13)$$

where x_i denotes the i-th element of $\mathbf{x}$.

ι_{C_3}: a fidelity constraint with respect to $\mathbf{v}$ represented by the following non-empty closed convex set

$$C_3 = \{\mathbf{x} \in \mathbb{R}^N \,|\, x_i = v_i \text{ for } i \notin \mathcal{M}_{\mathbf{v}}^{(k)}, k = 1, \ldots, K\}, \quad (14)$$

where v_i denotes the i-th element of $\mathbf{v}$, and $\mathcal{M}_{\mathbf{v}}^{(k)}$ denotes the index set of k-th missing region pixels of $\mathbf{v}$. By this constraint, known pixel values are precisely maintained.

ι_{C_1}, ι_{C_2}, ι_{C_3} are the indicator function and its composition $\iota_{C_1} + \iota_{C_2} + \iota_{C_3}$ takes 0 as long as $\mathbf{x} \in C_1 \cap C_2 \cap C_3$, otherwise it takes ∞.

We demonstrate that Problem 3.1 can be translated into a special example of Problem 2.1, hence the ADMM (Algorithm 2.1) can be applicable to Problem 3.1.

Problem 3.2 (ADMM-applicable form of Problem 3.1) Let $\mathbf{z}_i = \mathbf{x} \in \mathbb{R}^N$ $(i = 1, \ldots, 4)$, $\mathbf{y} = \mathbf{x}$, $\mathbf{z} = [\mathbf{z}_1^T \cdots \mathbf{z}_4^T]^T \in \mathbb{R}^{4N}$, $\mathbf{G} = [\mathbf{I}_N \cdots \mathbf{I}_N]^T \in \mathbb{R}^{4N \times N}$ ($\mathbf{I}_N$ is $N \times N$ identity matrix), $f : \mathbf{y} \mapsto 0$, and $g : \mathbf{z} \mapsto \|\mathbf{z}_1\|_{b*} + \iota_{C_1}(\mathbf{z}_2) + \iota_{C_2}(\mathbf{z}_3) + \iota_{C_3}(\mathbf{z}_4)$. Then, Problem 3.1 can be rewritten as

$$\min_{\mathbf{y}, \mathbf{z}} f(\mathbf{y}) + g(\mathbf{z}) \quad \text{s.t.} \quad \mathbf{z} = \mathbf{G}\mathbf{y}. \quad (15)$$

Problem 3.2 is the same form of (7) and $\mathbf{G}$ is obviously full column rank, therefore, we can solve Problem 3.1 by applying the ADMM (Algorithm 2.1) to Problem 3.2.

Let us explain how to calculate each step of the ADMM (described in Algorithm 2.1) for Problem 3.2. The fact that $f(\mathbf{y}) = 0$ turns Step 3 of the ADMM for Problem 3.2 into

$$\begin{aligned} \mathbf{y}^{(k+1)} &= \operatorname*{argmin}_{y} \left\{ \frac{1}{2\gamma} ||(\mathbf{z}^{(k)} - \mathbf{G}\mathbf{y} - \mathbf{b}^{(k)})||_2^2 \right\} \\ &= \frac{1}{4} \sum_{i=1}^{4} (\mathbf{z}_i^{(k)} - \mathbf{b}_i^{(k)}) \end{aligned} \quad (16)$$

(Note that $\mathbf{b} = [\mathbf{b}_1^T \cdots \mathbf{b}_4^T]^T \in \mathbb{R}^{4N}$). Step 4 of the ADMM for Problem 3.2 can be separated with respect to $\mathbf{z}_1, \ldots, \mathbf{z}_4$, i.e.,

$$\begin{aligned} \begin{bmatrix} \mathbf{z}_1^{(k+1)} \\ \vdots \\ \mathbf{z}_4^{(k+1)} \end{bmatrix} &= \operatorname*{argmin}_{\mathbf{z}_1, \ldots, \mathbf{z}_4} \left\{ \sum_{j=1}^{4} \left(g_j(\mathbf{z}_j) + \frac{1}{2\gamma} \|\mathbf{z}_j - \mathbf{y}^{(k+1)} - \mathbf{b}_i^{(k)}\|_2^2 \right) \right\} \\ &= \text{prox}_{\gamma g_j}(\mathbf{y}^{(k+1)} + \mathbf{b}_i^{(k)}) \quad \text{for } j = 1, \ldots, 4 \end{aligned} \quad (17)$$

where $g_1(\mathbf{z}_1) = ||\mathbf{z}_1||_{b*}$, $g_2(\mathbf{z}_2) = \iota_{C_1}(\mathbf{z}_2)$, $g_3(\mathbf{z}_3) = \iota_{C_2}(\mathbf{z}_3)$, and $g_4(\mathbf{z}_4) = \iota_{C_3}(\mathbf{z}_4)$. The proximity operator of g_1 can be calculated by (9). Similarly by (6), the proximity operators of g_2, g_3, and g_4 are simply the metric projection onto each set respectively, i.e., $\mathbf{z}_2^{k+1} = P_{C_1}(\mathbf{y}^{(k+1)} + \mathbf{b}_2^{(k)})$, $\mathbf{z}_3^{k+1} = P_{C_2}(\mathbf{y}^{(k+1)} + \mathbf{b}_3^{(k)})$, and $\mathbf{z}_4^{k+1} = P_{C_3}(\mathbf{y}^{(k+1)} + \mathbf{b}_4^{(k)})$, where, for $i = 1, \ldots, N$ and $k = 1, \ldots, K$,

$$P_{C_1}(\mathbf{x}) = \begin{cases} x_i + \text{mean}(\mathcal{P}_{\mathbf{x}}^{(k)}) - \text{mean}(\mathcal{M}_{\mathbf{x}}^{(k)}), & \text{if } i \in \mathcal{M}_{\mathbf{x}}^{(k)}, \\ x_i, & \text{otherwise}, \end{cases} \quad (18)$$

$$P_{C_2}(\mathbf{x}) = \begin{cases} 0, & \text{if } x_i < 0, \\ x_i, & \text{if } 0 \leq x_i \leq 255, \\ 255, & \text{if } x_i > 255, \end{cases} \quad (19)$$

$$P_{C_3}(\mathbf{x}) = \begin{cases} v_i, & \text{if } i \notin \mathcal{M}_{\mathbf{x}}^{(k)}, \\ x_i, & \text{otherwise}. \end{cases} \quad (20)$$

4. NUMERICAL EXPERIMENTS

The eleven standard test image (256×256[pixel]) are randomly corrupted with eight 16×16 blocks. To verify the inherent performance of the block nuclear norm itself, we design the TV based missing recovery method which utilizes the TV norm instead of the block nuclear norm in (11). Moreover, to compare substantial recovery performance of the proposed method, we compare it to Fadili *et al.*'s method [12] which is one of the state-of-the-art missing region recovery method. The proximity operator of the TV norm is approximated by the fast gradient projection method [4]. We choose $m = 32$ and $s = 4$ for the block size and the shift step number of $\|\cdot\|_{b*}$, respectively. These parameters produce good performance on average. γ is set as 1 and the iteration number is fixed as 50.

Table 1. Comparison of PSNR [dB].

method \ image	Barbara	Bridge	Building	Cameraman	Girl	Home	Lena	Lighthouse	Mandrill	Woman	Window
TV based	35.04	31.79	35.92	36.51	38.61	37.54	37.95	34.08	35.29	32.81	36.59
Fadili *et al.*'s method [12]	35.40	30.80	32.69	34.80	35.54	38.77	36.52	31.83	34.86	34.20	34.97
Proposed	38.66	33.44	38.46	38.01	40.66	43.37	39.04	36.69	38.58	37.00	37.94

Table 1 presents the comparison of PSNR of recovered images. For all test images, the resulting images of our method achieved higher PSNR than those of the comparative methods. For subjective evaluation, the portions of some resulting images are depicted in Fig. 1. In all TV based recovery results, there are many over-smoothing effects. Fadili *et al.*'s method generated some artifacts in Fig. 1(b) and (c). On the other hand, our results of Fig. 1(b) and (c) look natural and detailed structures are reconstructed efficiently. Conversely, Fig. 1(e) is the typical failure case of our method, because this region has random (less repetitive) structure.

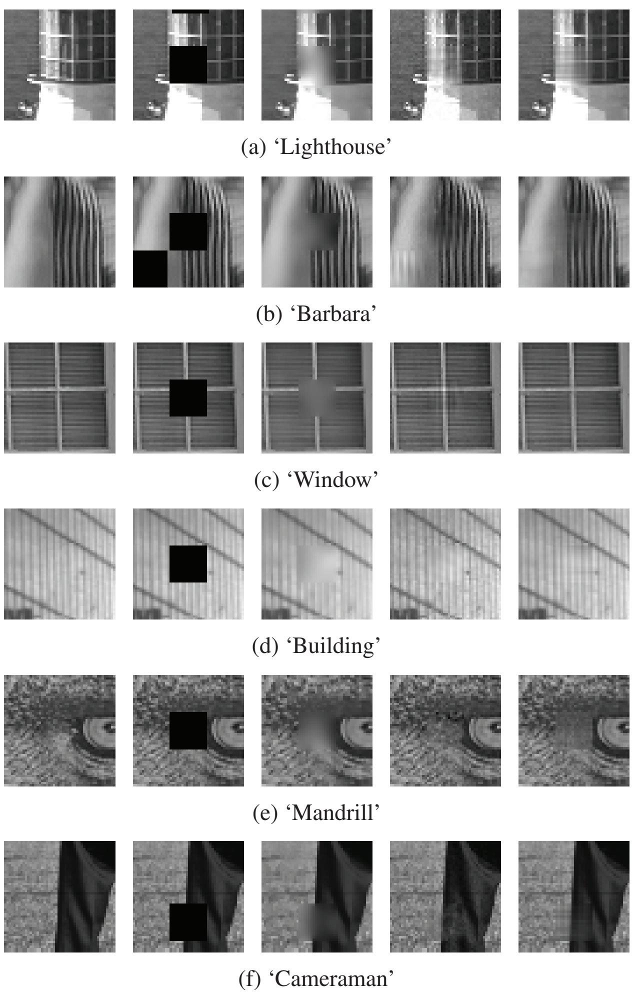

Fig. 1. The portions of the resulting images. From left: Original; Missing; TV based; Fadili *et al.*'s method [12]; Proposed.

5. CONCLUSION

We have proposed the missing region recovery method via the blockwise low-rankness promotion. For recovery of missing regions, we introduced the block nuclear norm as the criterion of local repetitivity in natural image. We formulated missing region recovery as the convex optimization problem via the block nuclear norm. The ADMM based algorithm which can solve the optimization problem efficiently is also presented. Numerical experiments showed that the proposed method works well for missing region recovery.

6. REFERENCES

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