An Alternating Direction Method for Frame-Based Image Deblurring with Balanced Regularization

Shoulie Xie and Susanto Rahardja Signal Processing Department, Institute for Infocomm Research, Singapore Email: {slxie, rsusanto}@i2r.a-star.edu.sg

Abstract—In this paper, we propose an efficient algorithm for solving a balanced approach in frame-based image deblurring. The balanced approach is usually formulated as a minimization problem involving an ℓ_2 data-fidelity term, an ℓ_1 regularizer on sparsity of frame coefficients, and a penalty on distance of sparse frame coefficients to the canonical frame coefficients. The balanced approach bridges synthesis-based and analysisbased approaches. Our algorithm is based on a variable splitting strategy and the classical alternating direction method (ADM). This paper shows how the proposed algorithm can be applied to solve the balanced approach efficiently. More precisely, a regularized version of the Hessian matrix of the ℓ_2 data-fidelity term is involved, and by exploiting fast tight frame and circular structure of the observation matrix, the matrix can perform efficiently for image deblurring application. Convergence of the proposed algorithm is guaranteed by the existing ADM theory. Numerical simulations illustrate the efficiency of our proposed algorithm in frame-based image deblurring.

I. INTRODUCTION

A classical and important research topic in image processing is image deblurring. The goal of this task is to recover the unknown true image $u \in \mathbb{R}^n$ from a noisy blurred measurement $y \in \mathbb{R}^m$ that is often modeled by

$$y = Bu + n \tag{1}$$

where B is a convolution operator, n is a white Gaussian noise with variance σ^2 . In frame-based image deblurring, the unknown image u is represented as u = Wx, where $W \in \mathbb{R}^{n \times d}$ denotes a frame, $x \in \mathbb{R}^d$ is the frame coefficients. In general, the frame may be redundant [11], [20]. In this paper, the redundant and normalized tight frame (Parseval frame) is used, i.e., $WW^T = I$. Thus, $u = W(W^Tu)$ for every vector $u \in \mathbb{R}^n$. The components of the vector W^Tu are called the canonical coefficients representing u. So the framebased image deblurring can be described as: the coefficients x are estimated from the noisy image first, then the unknown image u can be constructed as a linear combination of a few columns of frame W.

Since tight wavelet frame systems are redundant, the mapping from the image u to its coefficients is not one-toone, i.e., the representation of u in the frame domain is not unique. Three formulations utilizing sparseness of the frame coefficients are studied, namely analysis based approach, synthesis based approach and balanced approach. The balanced approach [4], [6], [23] can be formulated as:

$$\min_{x} \frac{1}{2} \|BWx - y\|_{2}^{2} + \frac{\gamma}{2} \|(I - W^{T}W)x\|_{2}^{2} + \lambda^{T}|x|_{1}$$
 (2)

where $\gamma > 0$, λ is a given nonnegative weight vector. The first term denotes penalty on the data fidelity, the last term penalizes the sparsity of coefficient vector, the second term penalizes the distance between the frame coefficients x and the range of W^T , i.e., the distance to the canonical frame coefficients of u. The larger γ makes the frame coefficients x is closer to the range of W^T , i.e., the frame coefficients x is closer to the canonical frame coefficients of u.

When $\gamma = 0$, the problem (2) is reduced to

$$\min_{x} \frac{1}{2} \|BWx - y\|_{2}^{2} + \lambda^{T} |x|_{1}.$$
 (3)

This is called the synthesis based approach where only the sparsity of the frame coefficients is penalized and the estimated image is synthesized by the sparsest coefficients..

On the other extreme, when $\gamma = \infty$, the term $||(I - W^T W)x||_2^2$ must be 0 if the problem (2) has solution. This implies that x is in the range of W^T , i.e., $x = W^T u$ for some $u \in \mathbb{R}^n$. Thus the problem (2) can be rewritten as

$$\min_{u \in \mathbb{R}^n} \frac{1}{2} \|Bu - y\|_2^2 + \lambda^T |W^T u|_1.$$
(4)

This is called analysis based approach as the coefficient is in the range of the analysis operator. It is noted that in (4) only the sparsity of canonical wavelet frame coefficients is penalized, which corresponds to the smoothness of the underlying image.

Obviously, the problem (2) balances the sparsity of the frame coefficients and the smoothness of the image, hence is called the balanced approach. It is also noted that when the columns of W form an orthonormal basis, the above three approaches are exactly the same. However, for the redundant tight frame W, these three approaches can not be derived from one another. It is hard to draw definitive conclusions as to which approach is better since each has its own favorable data sets and applications. In the literature, these three approaches are developed independently. For example, the analysis based approach was studied in [14], [15] and reference therein. The synthesis based approach was studied in [12], [16]-[19] and reference therein. The balanced approach started from [7], [8] for high resolution image reconstruction. It was further applied to various applications in [4]–[6], [23]. Since the balanced approach can be viewed as a way to balance the analysis and synthesis based approaches, and it can provide a balance between sharpness of the features and smoothness for the recovered images, we will only consider the balanced approach in this paper for frame-based image restoration.

Recently, the fast iterative shrinkage-thresholding algorithm (FISTA) in [2] was adopted to solve the balanced approach in frame-based image restoration [23] where FISTA is also called accelerated proximal gradient (APG). The FISTA in [2] is based on several variants of APG algorithms which were developed earlier by Nesterov and Nemirovski [21], [22]. These AGP algorithms can speed up the performance of the popular iterative shrinkage-thresholding algorithms (ISTA) [9], [10], [19]. They also have been adapted in various applications [2], [3], [23]. Although the convergence speed of FISTA is faster than ISTA, both of them essentially only use the gradient information and the first-order approximation of smooth function.

A fast algorithm based on variable splitting and classical alternating direction method was proposed for solving the analysis and synthesis based approaches in image restoration [1]. The fast speed of this algorithm comes from the fact that it uses a regularized version of the Hessian of the ℓ_2 datafidelity term, which can be computed efficiently for these standard image restoration, while the previously mentioned algorithms essentially only use the gradient information. This motivates us to adapt the alternating direction method to solve the balanced approach problem in the frame-based image deblurring. In this paper, we show that the proposed ADMbased algorithm involves a regularized version of the Hessian of the data fidelity term and penalty on the distance term of the sparse coefficients to the canonical frame coefficients. And we also show that in the frame-based image deblurring, the regularized Hessian matrices and their inverses can be computed efficiently by exploiting the special convolution structures of the observation matrices and a tight Parseval frame W with fast computational algorithms. Therefore the results of this paper show that the balanced approach problem in image deblurring can be solved efficiently by using ADM algorithm, and numerical experiments also show that the speed of the proposed algorithm is faster than the previous state of the art method such as ISTA and FISTA.

II. STANDARD ALTERNATING DIRECTION METHOD

Consider an unconstrained optimization problem of the form

$$\min_{u \in \mathbb{R}^n} f_1(u) + f_2(Gu),\tag{5}$$

where f_1 and f_2 are closed, proper convex functions, and $G \in \mathbb{R}^{d \times n}$. Variable splitting consists in creating a new variable, say v, to serve as the argument of f_2 , under the constraint that Gu = v. This leads to the constrained problem

$$\min_{u \in \mathbb{R}^n} f_1(u) + f_2(v), \qquad \text{subject to } Gu = v \tag{6}$$

which is clearly equivalent to the unconstrained problem (5). A natural way to address (5) is the so-called alternating direction method of multipliers (ADMM) [1], [13]:

Algorithm ADMM

- 1) Set k = 0, choose $\mu > 0$, v_0 and d_0 .
- 2) repeat
- 3) $u_{k+1} \in \arg\min_u f_1(u) + \frac{\mu}{2} ||Gu v_k d_k||_2^2$.

- 4) $v_{k+1} \in \arg\min_{v} f_2(v) + \frac{\mu}{2} \|Gu_{k+1} v d_k\|_2^2$.
- 5) $d_{k+1} = d_k (Gu_{k+1} v_{k+1}).$
- 6) $k \leftarrow k+1$.

7) until stopping criterion is satisfied.

Here $\mu \ge 0$ is called AL penalty parameter and d_k corresponds to the vector of Lagrange multipliers at the iteration k.

The convergence of ADMM is guaranteed by the theorem in [13] if f_1 and f_2 are closed, proper convex functions, and $G \in \mathbb{R}^{d \times n}$ has full column rank.

III. PROPOSED METHOD

The balanced approach problem (2) can be rewritten as the constrained optimization problem by variable splitting:

$$\min_{x,v \in R^n} \frac{1}{2} \|BWx - y\|_2^2 + \frac{\gamma}{2} \|(I - W^T W)x\|_2^2 + \lambda^T |v|_1$$

subject to $x = v$ (7)

Then applying ADMM to the problem (7), the steps 3) - 5) in Algorithm ADMM can be replaced with

3a) $x_{k+1} = \arg\min_{x,v} \|BWx - y\|_2^2 + \gamma \|(I - W^T W)x\|_2^2 + \mu \|u - v_k - d_k\|_2^2.$

4a)
$$v_{k+1} = \arg\min_{v} \lambda^{T} |v|_{1} + \frac{\mu}{2} ||x_{k+1} - v - d_{k}||_{2}^{2}$$
.

5a) $d_{k+1} = d_k - (x_{k+1} - v_{k+1}).$

The minimization problem in the step 4a) with respect to v can be solved by the soft thresholding method [9]:

$$v_{k+1} = \operatorname{soft}(v_k, \lambda/\mu) \tag{8}$$

where $v'_{k} = x_{k+1} - d_{k}$ and $\operatorname{soft}(x, \tau) = \operatorname{sign}(x) \odot \max\{|x| - \tau, 0\}$ with \odot denoting the component-wise product, i.e., $(x \odot y)_{i} = x_{i}y_{i}$ and sign being the signum function.

Note that the step 3a) is a strictly convex quadratic minimization problem with respect to x, hence it can be reduced to the following linear system:

$$x_{k+1} = \mathcal{A}^{-1}(W^T B^T y + \mu(v_k + d_k))$$
(9)

where

$$\mathcal{A} = W^T B^T B W + \gamma (I - W^T W) + \mu I.$$
(10)

The matrix \mathcal{A} can be seen as a regularized version of Hessian matrix $W^T B^T B W$ by adding the terms $\gamma(I - W^T W)$ and μI . In general, the computations of this matrix and its inverse are not affordable for general large-size matrix B. However, in standard image deblurring problem, B represents a convolution operator, the matrix-vector products can be performed with the help of the fast Fourier transform (FFT). Thus the matrix-vector products involving $W^T B^T B W$ can be quickly solved by additional fast wavelet frame algorithm [20]. However, since the last two terms is added into \mathcal{A} , it is not straightforward to obtain the inverse of \mathcal{A} where the fast computations can be employed explicitly. In the following part of this section, we will derive a formula that can compute the inverse of \mathcal{A} efficiently.

Using the Sherman-Morrison-Woodbury matrix inversion formula and $WW^T = I$, we can obtain (due to the space limitation, the detailed derivation steps are omitted here):

$$\mathcal{A}^{-1} = \frac{1}{\mu} [\alpha I + (1 - \alpha) W^T W - W^T \mathcal{F} W] \quad (11)$$

where $\alpha = \frac{\mu}{\mu + \gamma}$ and

$$\mathcal{F} = B^T (\mu I + B B^T)^{-1} B. \tag{12}$$

Define

$$r_k = W^T B^T y + \mu(v_k + d_k). \tag{13}$$

In view of (8), (12), (11),(9) and (13), we can obtain the following algorithm to solve the balanced approach optimization in frame-based image deblurring.

Algorithm ADMM for balanced approach (ADMM-B)

Set k = 0, choose μ > 0, v₀ and d₀.
 repeat
 x_{k+1} = ¹/_μ(αr_k + (1 − α)W^TWr_k − W^TFWr_k).
 v_{k+1} = soft(v'_k, ^λ/_μ).
 d_{k+1} = d_k − (x_{k+1} − v_{k+1}).
 k ← k + 1.

7) until stopping criterion is satisfied.

It is noted that $W^T B^T y$ does not change during the algorithm and can be precomputed. Since G = I, it is obvious that the convergence of the proposed ADMM algorithm in this paper can be guaranteed by the existing ADMM theory.

Remark: Obviously, it contains both frame-based synthesis and analysis approaches as special cases. In fact, $\alpha = 1$ ($\gamma = 0$), it is synthesis-based approach [1]. And $\alpha = 0$ ($\gamma = \infty$), it is analysis-based approach [14].

1) Computing \mathcal{F} : In image deblurring, B represents a periodic convolution, so \mathcal{F} can be computed in Fourier domain that has fast algorithm. In fact, B can be factorized as

$$B = U^T D U \tag{14}$$

where U represents the 2-D discrete Fourier transform (DFT) with $U^T = U^{-1}$, D is a diagonal matrix containing the DFT coefficients of B. Thus we have

$$\mathcal{F} = B^T (\mu I + BB^T)^{-1} B = U^T D^* (|D|^2 + \mu I)^{-1} D U$$
(15)

where D^* denotes complex conjugate and $|D|^2$ the squared absolute values of the entries of D. Since all the matrices in $D^*(|D|^2 + \mu I)^{-1}D$ are diagonal, it can be computed with O(n) cost, while the products by U and U^T can be computed with $O(n \log n)$ cost using FFT. Thus products by matrix \mathcal{F} have $O(n \log n)$ cost. Hence, x_{k+1} can be computed with $O(n \log n)$ cost using the fast tight frame W and the above fast algorithm of \mathcal{F} .

IV. SIMULATION RESULTS

In this section, numerical simulations are used to illustrate the performance of our proposed algorithm in the framebased image deblurring with balanced regularization. In [23], the FISTA algorithm was shown to be more efficient than the existing algorithms such as the split Bregman iteration, proximal forward-backward splitting or ISTA. Hence, in this section, we only need to compare our proposed ADMM-B algorithm with the FISTA algorithm. Our simulations are written in MATLAB and are performed on a Dell computer with Intel Xeon CPU 2.66GHz and 4GB of RAM under Windows XP.

We consider deblurring problems on the well-known Cameraman image with sized 256×256 pixels. The blur operator B is applied via FFT, the image is blurred by a 9×9 uniform blur and followed by additive normal noise with zero mean and standard variance $\sigma = 0.56$. W is a redundant Haar wavelet frame with four levels. To compare the speed of the algorithms, we run them until they reach the same value of the objective function. In our simulations, we choose $\gamma = 1$, $\lambda = 0.0075$ and $\mu = 0.1\lambda$. The number of iterations, computation times, and improvement in SNR (ISNR) are the average values over 10 instances and are presented in Table I. The average ISNR was computed as $10 \log_{10}(\sum_{k} ||u - y_k||^2 / \sum_{k} ||u - \hat{u}||^2)$, where u is the original image, y_k is the observed image at the kth iteration, and \hat{u}_k is the corresponding estimated image. To visually illustrate the relative speed of the algorithms, Fig. 1 plots the evolution of the objective function versus time. The deblurred image produced by our proposed algorithm is shown in Fig. 2 The simulations in this section show that our proposed algorithm is clearly faster than the FISTA algorithm.

TABLE I Image Deblurring: Comparison

| Algorithm | Iterations | CPU time (seconds) | ISNR (dB) |
|-----------|------------|--------------------|-----------|
| FISTA | 102 | 58.23 | 7.63 |
| ADMM-B | 4 | 5.34 | 8.54 |



Fig. 1. Objective function evolution

V. CONCLUSIONS

An efficient ADM-based algorithm for solving the balanced approach optimization problem in frame-based image deblurring is presented. The balanced approach equalizes the analysis-based and synthesis-based approaches in image restoration. The proposed ADM-based algorithm can be used to solve this optimization problem efficiently by exploiting the fast tight frame transform algorithm and the special convolutional structure of observation matrix in the standard image deblurring problem. Theoretical and experimental results have shown that the proposed ADMM-B algorithm in this paper is faster than previous state-of-the- art methods.

REFERENCES

- M. Afonso, J. Bioucas-Dias, and M. Figueiredo, "Fast image recovery using variable splitting and constrained optimization," *IEEE Trans. Image Process.*, vol. 19, no. 9, pp. 2345–2356, 2010.
- [2] A. Beck and M. Teboulle, "Fast iterative shrinkage-thresholding algorithm for linear inverse problems," *SIAM J. Imaging Sci.*, vol. 2, no. 1, pp. 183–202, 2009.
- [3] S. Becker, J. Bobin, and E. Candès, "NESTA: A fast and accurate firstorder method for sparse recovery," *SIAM J. Imaging Sci.*, vol. 4, no. 1, pp. 1–39, 2011.
- [4] J.-F. Cai, R. H. Chan, L. Shen, and Z. Shen, "Restoration of chopped and nodded images by framelets," *SIAM J. Sci. Comput.*, vol. 30, pp. 1205–1227, 2008.
- [5] J.-F. Cai, R. H. Chan, and Z. Shen, "A framelet-based image inpainting algorithm," *Appl. Comput. Harmon. Anal.*, vol. 24, pp. 131–149, 2008.
- [6] A. Chai and Z. Shen, "Framelet based deconvolution," J. Computational Mathematics, vol. 28, pp. 289–308, 2010.
- [7] R. H. Chan, T. F. Chan, L. Shen, and Z. Shen, "Wavelet algorithms for high-resolution image reconstruction," *SIAM J. Sci. Comput.*, vol. 24, pp. 1408–1432, 2003.
- [8] R. H. Chan, S. D. Riemenschneider, L. Shen, and Z. Shen, "Tight frame: an efficient way for high-resolution image reconstruction," *Appl. Comput. Harmon. Anal.*, vol. 17, pp. 91–115, 2004.
- Comput. Harmon. Anal., vol. 17, pp. 91–115, 2004.
 [9] P. Combettes and V. Wajs, "Signal recovery by proximal forward-backward splitting," SIAM J. Multiscale Model. Sim., vol. 4, pp. 1168–1200, 2005.
- [10] I. Daubechies, M. De Friese, and C. De Mol, "An iterative thresholding algorithm for linear inverse problems with a sparsity constraint," *Comm. on Pure and Applied Math.*, vol. 57, pp. 1413–1457, 2004.
- [11] I. Daubechies, B. Han, A. Ron, and Z. Shen, "Framelets: MRA-based constructions of wavelet frames," *Appl. Comput. Harmon. Anal.*, vol. 14, pp. 1–46, 2003.
- [12] I. Daubechies, G. Teschke, and L. Vese, "Iteratively solving linear inverse problems under general convex constraints," *Inverse Probl. Imaging*, vol. 1, pp. 29–46, 2007.
- [13] J. Eckstein and D. Bertsekas, "On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators," *Math. Program.*, vol. 5, pp. 293–318, 1992.
- [14] M. Elad, P. Milanfar, and R. Rubinstein, "Analysis versus synthesis in signal priors," *Inverse Problems*, vol. 23, pp. 947–968, 2007.
- [15] M. Elad, J.-L. Starck, P. Querre, and D. L. Donoho, "Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA)," *Appl. Comput. Harmon. Anal*, vol. 19, pp. 340–358, 2005.
- [16] M. Fadili and J.-L. Starck, "Sparse representations and bayesian image inpainting," in *Proc. SPARS'05*, vol. 1, Rennes, France, 2005.
- [17] M. Fadili, J.-L. Starck, and F. Murtagh, "Inpainting and zooming using sparse representations," *The Computer Journal*, vol. 52, no. 1, pp. 64–79, 2009.
- [18] M. Figueiredo and R. Nowak, "An EM algorithm for wavelet-based image restoration," *IEEE Trans. Image Process.*, vol. 12, no. 8, pp. 906– 916, 2003.
- [19] —, "A bound optimization approach to wavelet-based image deconvolution," in *IEEE Int. Conf. on Image Processing (ICIP'05)*, Genoa, Italy, 2005, pp. 782–785.
- [20] S. Mallat, A Wavelet Tour of Signal Processing. New York: Academic, 2009.
- [21] Y. Nesterov, "A method of solving a convex programming problem with convergence rate $o(1/k^2)$," *Soviet Mathematics Doklady*, vol. 269, pp. 543–547, 1983.
- [22] —, "Smooth minimization of non-smooth functions," *Math. Program.*, vol. 103, pp. 127–152, 2005.
- [23] Z. Shen, K.-C. Toh, and S. Yun, "An accelerated proximal gradient algorithm for frame-based image restoration via the balanced approach," *SIAM J. Imaging Sci.*, vol. 4, no. 2, pp. 573–596, 2011.





Blurred and noisy



Estimated using FISTA



Estimated using ADMM-B



Fig. 2. Image deblurring: (a) Original cameraman image with 256×256 pixels. (b) Image blurred with noise. (c) Estimated image using FISTA. (d) Estimated image using our proposed ADMM-B.