COMPRESSED SENSING USING FREBAS TRANSFORM IN MAGNETIC RESONANCE PHASE SCRAMBLING FOURIER TRANSFORM TECHNIQUE

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ABSTRACT

The application of compressed sensing (CS) to MRI has the potential for significant scan time reductions. The signal obtained in the phase-scrambling Fourier transform (PSFT) imaging technique shows better performances compared to standard Fourier transform base imaging when it is applied to CS. Sparsity of signal is an essential condition for compressed sensing. In this work, we adopt a FREBAS transform as a sparsifying transform function instead of wavelet transform to improve the quality of images in PSFT based CS. It was shown that the directionality of FREBAS transform and the usage of successive thresholding in FREBAS domain offers fairly good images, particularly at low sampling rates. In addition, proposed method is robust to the choice of sampling trajectory of NMR signal.

Index Terms— L1-norm, FREBAS, MRI, sparse

1. INTRODUCTION

Recently, a new theory called compressed sensing (CS) has been applied to MR image reconstruction with great success [1]. The CS theory states that a signal with a sparse representation can be reconstructed from much fewer measurements than previously suggested by the conventional Nyquist sampling theory [2]. S. Ito and Y. Wiaux have shown independently that the use of quadratic phase modulation prior to data acquisition can greatly improve the accelerating factor of CS [3,4]. Quadratic phase modulation on the object function can be realized by introducing the phase-scrambling Fourier imaging technique (PSFT) which was proposed by Maudslay and Wedeen independently [5,6]. Sparse representation is the key of the CS theory, and reconstructed images greatly depend on the sparsifying transform function. Most of the prior work in CS MRI used the discrete cosine transforms or wavelet transform for sparsifying transform function.

In this paper, we propose to adopt the FREBAS transform [7] as a sparsifying transform function in the PSFT-CS. FREBAS transform consist of two different Fresnel transform algorithms, which offers multi-resolution image decomposition with highly directional representation. Unlike the wavelet transform, the calculation of FREBAS is rather simple with three FFTs and three quadratic phase modulations, which makes it easy to implement it in the CS procedure. In addition, FREBAS has a distinctive feature that an optional scaling parameter can be taken in the image decomposition. Therefore, the successive thresholding of FREBAS domain with different scaling parameter is expected to encourage incoherency between measurement matrix and sparsifying transform function. In experimental simulations, we find that proposed CS reconstruction based FREBAS transform outperforms equivalent reconstruction using common wavelet transform.

2. METHOD

2.1. Phase-Scrambling Fourier Imaging Technique

Phase-Scrambling Fourier Transform (PSFT) imaging is a technique whereby a quadratic field gradient $\Delta B=b(x^2+y^2)$ is added to the pulse sequence of conventional FT imaging in synchronization with the field gradient for phase encoding [3,4]. The signal obtained in PSFT is given by Eq. (1),

$$v(k_x,k_y)=\iint\left[\rho(x,y)e^{-j\gamma b\tau\left(x^2+y^2\right)}\right]e^{-j\left(k_x x+k_y y\right)}dxdy \quad (1)$$

where $\rho(x,y)$ represents the spin density distribution in the subject, γ is the gyromagnetic ratio, and b and τ are the coefficient and impressing time, respectively, of the quadratic field gradient. Like the standard Fourier Imaging technique, spin density distribution $\rho(x,y)$ can be obtained by taking the inverse Fourier transform of the signal and multiplying the quadratic phase term $\exp[j\gamma b\tau(x^2+y^2)]$.

The maximum value of $\gamma b\tau$ can be determined as $\gamma b\tau_{\max}=\pi/(N\Delta x^2)$ by the sampling theorem, $\Delta x\ \{\partial(\gamma b\tau x^2)/\partial x\}|_{x=N\Delta x/2}=\pi$. The parameter $\gamma b\tau$ can expressed by Eq.(2) using the coefficient h and normalized parameter $\gamma b\tau_{\max}$,

$$\gamma b\tau= h\ \gamma b\tau_{\max} \quad . \quad (2)$$

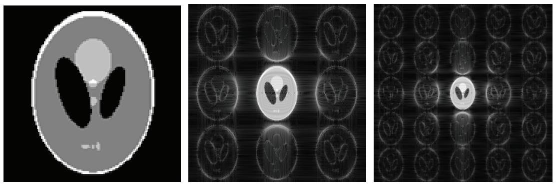
The PSFT signal spread over in k-space in accordance with the parameter h.

2.2. Signal restoration using compressed sensing

According to the CS theory, a signal $\mathbf{s}$ with a sparse representation in the basis $\mathbf{\Psi}$, can be recovered from the compressed measurements $\mathbf{p}=\mathbf{\Phi}\ \mathbf{s}$, where $\mathbf{\Phi}$ is an M xN measurement matrix (M<<N), if the $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are incoherent.

$$\mathbf{p}=\mathbf{\Phi s}=\mathbf{\Phi\Psi}^{-1}\boldsymbol{\mu}\,, \qquad \|\boldsymbol{\mu}\|_0 = M << N \quad (3)$$

The image is reconstructed from the undersampled k-space data by solving the nonlinear optimization problem: minimize $\|\mathbf{\Psi s}\|_1$ subject to $\|\mathbf{\Phi s} - \mathbf{p}\|_2 < \varepsilon$, where ε is a small



(a) (b) (c)

Fig.1 Examples of FREBAS transform (a) original image, (b), (c) scaling parameter D=3 and 5, respectively.

constant.

Minimizing $\|\Psi\mathbf{s}\|_1$, we use technique based on projection [8]. Algorithm of this class form by **s** successively projecting and thresholding;

$$\breve{\boldsymbol{\mu}}^{(i)} = \breve{\boldsymbol{\mu}}^{(i)} + \frac{1}{\beta}\boldsymbol{\psi}\boldsymbol{\Phi}^{T}\left(\mathbf{p} - \boldsymbol{\Phi}\boldsymbol{\psi}^{-1}\breve{\boldsymbol{\mu}}^{(i)}\right), \tag{4}$$

$$\breve{\boldsymbol{\mu}}^{(\mathbf{i+1})} = \begin{cases} \breve{\boldsymbol{\mu}}^{(\mathbf{i})}, & \left|\breve{\boldsymbol{\mu}}^{(i)}\right| \geq \tau^{(i)}, \\ 0 & \text{else} \end{cases} \tag{5}$$

where, $\boldsymbol{\Phi}^{\mathbf{T}}$ is orthonormal such that $\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathbf{T}}$=I, β is scaling factor, $\tau^{(i)}$ is a threshold set appropriately at each iteration, $\boldsymbol{\Psi}$ is sparsifying transform function.

In our work, we adopt FREBAS transform [7] as a sparsifying transform function which is described in **2.3.** The starting condition of Eq.(4) we used zero-filled reconstructed image for $\mathbf{s}^{(0)}$. Equation (4) is a specific instance of a projected Landweber (PL) algorithm [9]. PL-based CS reconstruction provides reduced computational complexity. Figure 2 shows the schematic of proposed CS method using the FFT instead of calculating $\boldsymbol{\Phi}$ or $\boldsymbol{\Phi}^{\mathbf{T}}$.

2.3. FREBAS TRANSFORM

The FREBAS transform consists of two different algorithms for the Fresnel transform, which is a diffraction equation for use with sound or light waves [7]. When down-scaling is performed, the alias signal that is produced when computing the Fresnel transform signal is separated in the reconstructed image domain, and equivalent band splitting of the Fresnel transformed signal is performed in the image domain. This band splitting of the Fresnel transformed signal domain can be interpreted as the image data being analyzed by convoluting imaged data with sinc functions having different modulation indices.

Considering the one-dimensional signal for reasons of simplicity, a decomposed image $\rho_m(x)$ in the FREBSAS domain can be described equivalently as a convolution integral with the kernel of a band-pass filter function:

$$\begin{aligned}\rho_m(x) &= \rho(x-mX)e^{-jcx^2} * \mathrm{sinc}(2cXx) \\ &= A\left\{\rho(x-mX)e^{-jc(x-mX)^2} * \mathrm{sinc}(2cXx)e^{j2cmXx}\right\}\end{aligned} \tag{6}$$

$$A = e^{jc(x-mX)^2}e^{-jcx^2} \tag{7}$$

where $\rho(x)$ is an image data, $X(=N\Delta x)$ is the field of view of the input image, N is the size of image, Δx is the pixel width, c is a coefficient directly related to the scaling parameter of the FREBAS transform, namely, $D=\pi/(cN\Delta x^2)$,and m ($|m| \leqq D$) is an index of the frequency band in filter-banks. Figure 1 shows the examples of the FREBAS decomposed images for the cases of D = 3 and 5.

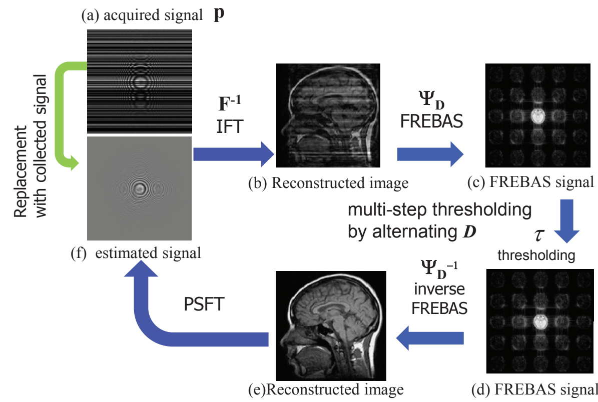


Fig.2 Schematic of compressed sensing using FREBAS transform. Ψ_{D} is FREBAS transform using scaling parameter D.

The FREBAS transform is similar to the wavelet transform in that it decomposes input images in the image domain; however, the FREBAS transform differs from the standard wavelet transform in the sense that 1) the FREBAS transform with scaling parameter D decomposes the input image into D^2 smaller images having the same scale size $1/D$, 2) the calculation consists of three FFTs and three quadratic phase modulations, 3) scaling parameter D can take on not only integer values but also real values, 4) complex-value decomposition which ease to transform images with phase variations, and 5) redundant transform. FREBAS domain signal has a real and imaginary part in general, therefore, the size of FREBAS transform signal would be double of input image sizes, if input image is a real-value data. Since the size of FREBAS signal is increased, FREBAS is a redundant transform. The basis function of FREBAS transform is considered as sinc($2cXx$)exp$\{j2cmXx\}$ by Eq.(7). Since the inner product of sinc($2cXx$)exp$\{j2cmXx\}$ having different m number would be zero, which means that the FREBAS is a orthogonal transform. The calculation steps of FREBAS transform are as follows;

1) Fourier transform (FT),
2) quadratic phase modulation $\exp\{j(D\pi/N)(i_x^{\,2}+i_y^{\,2})\}$
3) inverse Fourier transform (IFT),
4) quadratic phase modulation $\exp\{j(\pi/(DN))(i_x^{\,2}+i_y^{\,2})\}$
5) Fourier transform (FT)
6) quadratic phase modulation $\exp\{j(D\pi/N)(i_x^{\,2}+i_y^{\,2})\}$,

where i_x, i_y are the index of signal data.

Discrete wavelet transform in the standard dyadic decomposition form is widely used for image compression; however, it is known to be somewhat deficient in several aspects, specifically, it lacks shift invariance and significant directional selectivity. On the other hand, FREBAS has many directional representations; therefore, a much higher degree of directional features can be tracked.

3. SIMULATION EXPERIMENTS

In the simulation experiments, PSFT signal is calculated numerically according to the Eq.(1) using the MR volunteer image data. MR image data were collected on Toshiba 1.5T MRI scanner with 3D gradient echo sequence. The normalized parameter $\gamma b\tau_{\max}$ is 1.23 rad/cm for the conditions of Δx=Δy=0.1 cm and N=256x256. We introduce the successive 2-step thresholding in the FREBAS domain (PSFT-CS-FREBAS 2-step) using the different scaling parameter D to improve the incoherency between measurement matrix and sparsifying transform function (FREBAS). The best combination of scaling parameters D is 6 and 9, which is determined by preliminary simulations. We used Cartesian sampling because it is by far the most widely used in practice. Fully sampled PSFT signal data was calculated and then randomly picked for the phase encoding direction to be a given reduction factor. Since the energy of PSFT signal spreads widely over the k-space dependent on the parameter $\gamma b\tau$ (or h), and do not concentrated strongly on the center of k-space, it is not necessary to concentrate the sampling trajectory on the central region of the k-space.

As a comparison, iterative thresholding method based on wavelet is examined. (PSFT-CS-Wavelet). Wavelet used in the simulation experiments is 4-level dyadic decomposition with daubechies 4. Representative results using PSFT-CS-FREBAS (proposed method) and PSFT-CS-Wavelet for 20% and 25% of full data with h=0.6 are shown in Fig. 3. Figure 4 shows the images for different trajectory using the same amount of signal 25% for the condition of h=0.4.

Several important features should be noted from Fig. 3 and Fig.4. When the amount of data is small such as 20%, the reconstructed images by PSFT-CS-Wavelet show stripe-shape artifacts perpendicular to phase encoding direction. Figure 4 indicates that quality of image obtained by PSFT-CS-Wavelet depends on the trajectory of signal strongly even though the amount of signal is the same. It was revealed from Fig.4 (b),(c) that CS reconstruction by PSFT-CS-Wavelet tends to fail when the dense of signal is not sufficient in k-space. Therefore, the reconstruction would fail in the case when the amount of signal is rather small.

On the other hand, FREBAS based CS technique shows very small dependencies on the trajectory of signal and offers fairly good images for all the case shown in this article. We confirmed that proposed PSFT-CS-FREBAS 2-step succeeded in CS reconstruction using 12% of full signal, even though some blurring effects appear on the image.

The consideration of point spread function in helpful to understand the behavior of CS algorithm and measure the

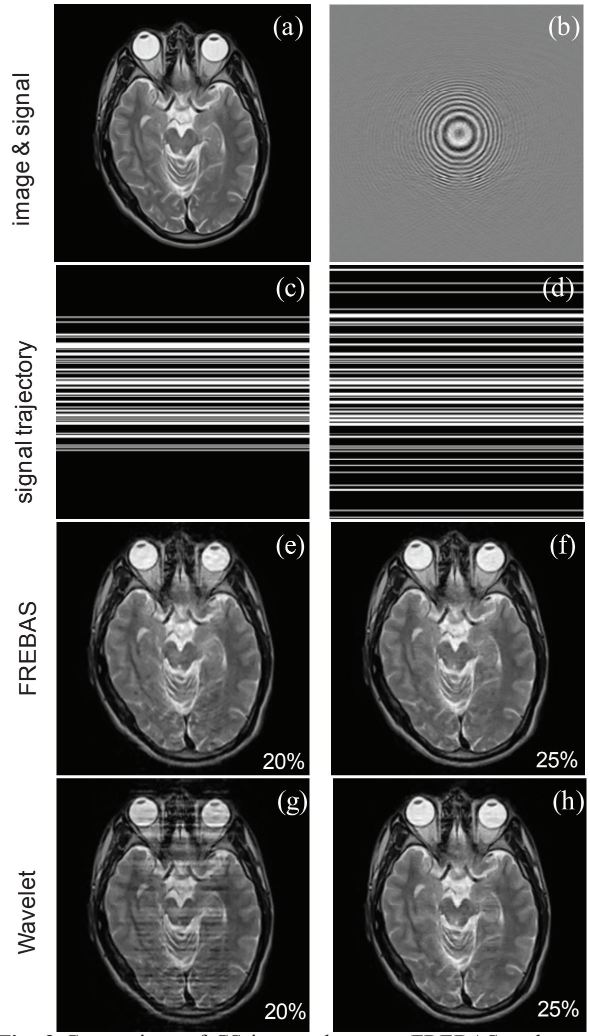


Fig. 3 Comparison of CS images between FREBAS and wavelet. (a) fully scanned image data (gold standard). (b) PSFT signal (h=0.6) (c) and (d) show the signal trajectory for acquiring 20% and 25% of signal, respectively. (e), (f) are images obtained by FREBAS 2-step and (g), (h) are by wavelet for 20%, 25% of data. When the amount of data is small, the wavelet-base CS shows artifact on the image.

incoherence. Let e be a point image (having 1 at a pixel and zeros elsewhere). Then PSF can be obtained by applying e instead of $\psi^{-1}\breve{\mu}^{(0)}$ in Eq.(4). Figure 5 shows the results of PSF examination with h=0.6. Fig.5(a) shows the PSF after random sampling in k-space. In this case, it looks likes random noise added on the point image e. Figs. (b), (c) shows the PSF after PSFT-CS-FREBAS single-step (D=6) and proposed PSFT-CS-FREBAS 2-step (D=6 and 9), respectively. Figure 5(d) is the PSF after PSFT-CS-Wavelet. It was shown that random noises are much more reduced in proposed method compared to PSFT-CS-Wavelet. These results indicate that mutual incoherence between basis of FREBAS transform and measurement matrix (Fourier operator) is smaller than that of between wavelet basis and

Fourier operator. This feature contributes to reconstruct superior images in proposed CS reconstruction. The relative reconstruction error calculated by averaging 10 kind of images, defined as $\|\rho - \rho_{cs}\|_2 / \|\rho\|_2$ is shown in Fig.6. RMS of reconstruction error is plotted with respect to the reduction factor of signal. As described earlier, the proposed method shows the better performances for the data smaller than 25%. These results suggest that the proposed CS method is robust to the *k*-space trajectory and the resultant quality of images outperforms the wavelet-based method. The proposed algorithm inherits the fast execution speed of the projection-based CS reconstruction. The execution time for 100 time iterations is 12.1sec using 3.06-GHz Intel Corei7 950 processor.

5. CONCLUSION

In this paper, we examined the application of FREBAS transform as a kind of directional transforms for the compressed sensing in the modified Fourier transform imaging. The directionality of FREBAS transform and the usage of successive application of thresholding encourage superior image quality. The proposed method is robust to sampling trajectory of signal and offers fairly good images, particularly at low sampling rates. The future work is the application to actual MRI signal with phase variations.

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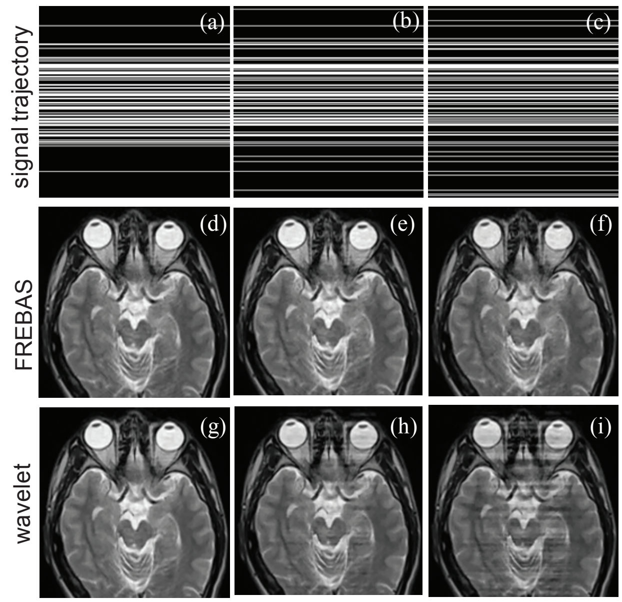

Fig. 4 Comparisons of CS images for the different signal trajectories on condition that the signal has the same amount of data, 25% of full signal (h=0.4); (a),(b),(c) different signal trajectory that cover 25% of *k*-space, (d)-(f) images by PSFT-CS-FREBAS corresponding (a) to (c), (g)-(i) images by PSFT-CS-Wavelet.

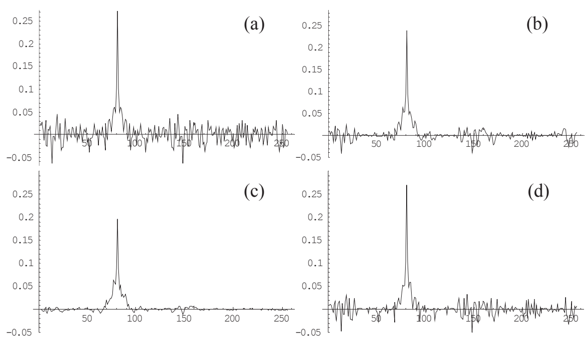

Fig. 5 Comparison of PSF, (a) PSF of undersampled signal, (b) after FREBAS- single-step (D=6), (c) after FREBAS 2-step (proposed; D=6 and 9), (d) CS-wavelet.

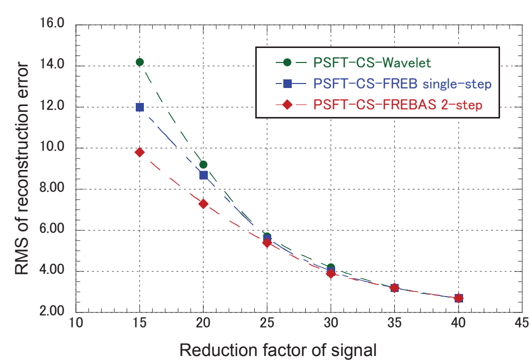

Fig. 6 Reconstruction error with respect to the reduction factor.

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