AN ITERATIVE ALGORITHM FOR PHASE RETRIEVAL WITH SPARSITY CONSTRAINTS: APPLICATION TO FREQUENCY DOMAIN OPTICAL COHERENCE TOMOGRAPHY

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ABSTRACT

We address the problem of phase retrieval, which is frequently encountered in optical imaging. The measured quantity is the magnitude of the Fourier spectrum of a function (in optics, the function is also referred to as an object). The goal is to recover the object based on the magnitude measurements. In doing so, the standard assumptions are that the object is compactly supported and positive. In this paper, we consider objects that admit a sparse representation in some orthonormal basis. We develop a variant of the Fienup algorithm to incorporate the condition of sparsity and to successively estimate and refine the phase starting from the magnitude measurements. We show that the proposed iterative algorithm possesses Cauchy convergence properties. As far as the modality is concerned, we work with measurements obtained using a frequency-domain optical-coherence tomography experimental setup. The experimental results on real measured data show that the proposed technique exhibits good reconstruction performance even with fewer coefficients taken into account for reconstruction. It also suppresses the autocorrelation artifacts to a significant extent since it estimates the phase accurately.

Index Terms— Sparsity, Phase retrieval, Optical Coherence Tomography, Fienup iterations.

1. INTRODUCTION

In imaging modalities such as electron microscopy, crystallography, astronomy, coherence tomography, and wavefront sensing, measurements of a complex-valued signal are made with sensors that can capture only the intensity. These magnitude-only measurements are acceptable for applications such as digital photography. However in imaging applications, the phase of the object carries critical information. In most diffraction-limited imaging applications, the object diffractions, the object diffractions, the object magnitude of the Fourier spectrum. The measurement is actually the intensity or the magnitude of the Fourier spectrum. From the magnitude spectrum, one is required to estimate and reconstruct the phase of the object, based on some assumptions on the object. This, in a nutshell, is the basic problem definition of *phase retrieval*. Therefore, phase retrieval is essentially an inverse

problem. Some of the first solutions for the problem were proposed by Fienup [1] and Gerchberg-Saxton [2, 3]. Quatieri et al. [4] developed iterative algorithms for reconstructing a minimum-phase sequence from the magnitude of its Fourier transform, where the iterations involve repeated application of the causality constraint in the time domain.

Fienup and Gerchberg-Saxton algorithms are iterative techniques. The starting point is the Fourier magnitude spectrum and the initial phase is assumed to be random/all-zero. Within each iteration, the measurement is inverted to obtain an estimate of the object. Then, the object-domain conditions are imposed. The oftenemployed constraint is that the object has compact support and in some cases, that it is also positive-valued. These conditions enable a better estimate of the phase. The new phase estimate is then coupled with the measured magnitude spectrum and the iterations are repeated all over, until convergence is achieved. In this paper, we impose sparsity conditions on the object to be recovered, that is, the object is assumed to admit a sparse representation in some suitably chosen orthonormal basis. This is a reasonably valid constraint for most practical objects. We modify the Fienup algorithm to incorporate the sparsity condition as well. We then address the issue of convergence and show that by retaining the K largest transform coefficients in every iteration, the reconstruction error is non-increasing and therefore certainly reaches a stagnation point. This aspect is verified both theoretically and experimentally.

For the imaging modality, we consider frequency-domain opticalcoherence tomography (FDOCT), which is an efficient interferometric technique that is suitable for noninvasive three-dimensional imaging of biological specimens. Typically, one can achieve millimeter penetration depths with micrometer-range axial resolution. The primary medical applications of FDOCT are tissue imaging, dermatology, and ophthalmology. It is known that in FDOCT, the measurement is the intensity of the Fourier transform of the sum of the object and reference waves. The object wave is a combination of backscattered waves coming from the interfaces between layers (corresponding to changes in refractive index values across layers). The goal is to reconstruct the object wave from the magnitude spectrum. Direct inversion is known to lead to the so-called autocorrelation artifacts [5], which may be particularly disturbing for visualizing some specimens. An iterative phase-retrieval algorithm would be an ideal tool to progressively reconstruct the object from the measurement by successively refining the phase estimate. Such a technique would also potentially suppress the autocorrelation artifacts.

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1.1. Review of some recent literature

There has been some recent work on addressing the phase retrieval problem within the framework of sparsity. Moravec et al. [6] made seminal contributions by considering the phase retrieval problem within the framework of Compressive Sensing. They consider a magnitude-only compressive sensing approach and derive sufficient conditions for accurate signal recovery. Essentially, they rely on a signal's compressibility rather than its support to reconstruct it from the Fourier transform magnitude measurements. Another recent related contribution is that of sparse spectral factorization developed by Yu and Vetterli [7]. They address the problem of recovering a signal from its autocorrelation, which plays a critical role in X-ray crystallography. They present a sufficient condition for the recovery technique to give rise to a solution that is unique up to a sign change, time-shift, and time-reversal. In this paper, we do not consider a Compressive Sensing scenario. Instead we rely on standard measurements that are obtained with an existing experimental setup, but enforce the sparsity condition in the reconstruction technique. We do not assume a rational transfer function model for the signal generation. Therefore, our technique is nonparametric.

1.2. Organization of the paper

In Section 2, we give an overview of the FDOCT signal acquisition approach and present the signal model. In Section 3, we formulate the inverse problem in the framework of sparsity. In Section 4, we propose a Fienup-type iterative algorithm for reconstructing the signal from the FDOCT measurements. In Section 5, we show results on synthesized as well as real FDOCT data. We also present comparisons with the standard Fourier inversion method and show that the proposed technique yields tomograms with lower background noise levels. It also suppresses the autocorrelation artifacts since it estimates the phase accurately.

2. FDOCT SIGNAL ACQUISITION AND MODEL

In Figure 1, we show a Michelson interferometer-based FDOCT setup. The output of a broadband light source is split into two beams, one directed towards a mirror (which generates the reference signal) and the other towards an object. The light reflected from the object is primarily due to the refractive index changes at interfaces of layers within the object. The reference and object waves interfere and the resulting interference pattern magnitude is measured by the spectrometer. For three-dimensional imaging, it is necessary to scan the sample laterally.

Let f(z) denote the amplitude of the light field generated due to scattering by the object, as a function of depth z. The spectrometer measurements are a function of the wavelength, but they are subsequently mapped onto the wavenumber $k = 2\pi/\lambda$. In terms of the frequency variable $\omega = -2kn$, we can express the measurements as

$$I(\omega) = S(\omega) \left| 1 + \int_{-\infty}^{\infty} f(z) e^{-j\omega z} dz \right|^2,$$
(1)

where $S(\omega)$ is the source power spectrum. The FDOCT inverse problem can now be formulated explicitly as the task of computing f(z) given $I(\omega)$ and $S(\omega)$, which is equivalent to retrieving the phase of $\delta(z) + f(z)$ (where $\delta(z)$ denotes the unit impulse function)



Fig. 1. Schematic of the FDOCT setup.

given the measurements of the square of its Fourier magnitude spectrum. Note that ω is discrete since the measurements are taken only at a discrete set of wavelengths.

3. PROBLEM FORMULATION WITHIN SPARSITY FRAMEWORK

Let **f** be an *N*-dimensional signal that is sparse in basis Ψ , where Ψ is an $N \times N$ orthonormal matrix. That is, we can write $\mathbf{f} = \Psi \mathbf{a}$, where **a** has only few non-zero entries. Let $\operatorname{supp}(\mathbf{a}) = \{i: a(i) \neq 0\}$ be the support of **a** and $|\operatorname{supp}(\mathbf{a})| = K$, that is, **a** is a *K*-sparse vector. Let Φ be the $N \times N$ DFT matrix. Then, the DFT of the signal **f** is given by $\mathbf{F} = \Phi \mathbf{f}$. Now, given only the magnitudes of the entries of **F**, we would like to estimate the phase and thereby the original signal **f**.

Let f(n) denote the *n*-th entry of the vector **f**. Since the signal with entries f(n), and that with entries $\pm f(n \pm m)$ have same Fourier magnitude for any integer *m*, any phase retrieval algorithm will only be able to recover the sequence up to this indeterminacy.

4. PROPOSED PHASE RETRIEVAL ALGORITHM

Next, we present the new algorithm for phase retrieval. The algorithm starts with an all-zero initialization of the phase and in each iteration, the inverse Fourier transform is computed and we find the best K-sparse representation in the orthonormal basis Ψ . The Fourier transform of the K-sparse approximation is then computed to obtain the refined phase estimate. This procedure is repeated until convergence is achieved. The analysis for the convergence performance of the new algorithm is given in the appendix. The convergence analysis is carried out on the mean-square error (MSE) measure proposed by Fienup for phase retrieval problems:

$$E_k = \frac{1}{N} \sum_{\ell=0}^{N-1} \left(|F_k(\ell)| - |F(\ell)| \right)^2,$$
(2)

where ℓ denotes the frequency index, $F_k(\ell)$ is the Fourier spectrum (discrete-time Fourier transform) of the estimated sparse signal after the k^{th} iteration. We have been able to show that this error measure is non-increasing with iterations, which implies Cauchy convergence and guarantees that the iterations do not result in diverging estimates,

Algorithm 1 To recover a signal f, which is K-sparse in basis Ψ , given only the magnitudes of the entries of F, where $\mathbf{F} = \Phi \mathbf{f}$.

0 for ℓ Step 1: Initialize $k \leftarrow 0, \phi_k(\ell) \leftarrow$ = $0, 1, 2, \cdots, (N-1)$, set the error threshold ϵ . **Step 2**: Compute $F_k(\ell) = |F_k(\ell)| \exp(j\phi_k(\ell))$ = $\mathbb{F}{f_k(n)}$, where \mathbb{F} is the DFT operator. **Step 3**: Find $F'_k(\ell) = |F(\ell)| \exp(j\phi_k(\ell))$. Step 4: Find $f'_k(n) = \mathbb{F}^{-1}{F'_k(\ell)}$ for n = $0, 1, 2, \cdots, (N-1)$, where \mathbb{F}^{-1} is the inverse DFT operator. Step 5: Calculate $\mathbf{a}'_k = \Psi^{-1}\mathbf{f}'_k$. Obtain $\mathbf{a}_{k+1} = \arg\min_{\mathbf{a}\in S} ||\mathbf{a} - \mathbf{a}'_k||^2$ by keeping only the first K largest entries in \mathbf{a}'_k and setting other entries equal to zero, where $\mathcal{S} = \{\mathbf{f} : |\mathrm{supp}(\mathbf{f})| = K\}.$ **Step 6**: Find $\mathbf{f}_{k+1} = \mathbf{\Psi} \mathbf{a}_{k+1}$. Compute $\mathbb{F}\{f_{k+1}(n)\} =$ $|F_{k+1}(\ell)|\exp(\mathrm{j}\phi_{k+1}(\ell)).$ **Step 7**: Set $k \leftarrow k+1$. If $E_k = \frac{1}{N} \sum_{\ell=0}^{N-1} [|F_k(\ell)| - |F(\ell)|]^2 \le \epsilon$, terminate, else go to step 2.

irrespective of the initialization. Similar convergence behavior was found to hold for the original Fienup and Gerchberg-Saxton algorithms.

5. SIMULATION RESULTS

We present results on simulated as well as real FDOCT data. Since the FDOCT signal essentially contains a combination of backscattered signals, which arise only when there is a change in refractive index across layers, the FDOCT signal is already sparse in the spatial domain. Therefore, for the FDOCT case, it is valid to consider $\Psi = \mathbf{I}$, which is the $N \times N$ identity matrix. It must be noted that the proposed algorithm (cf. Section 4) is valid for any orthonormal Ψ in general.

5.1. Synthesized data

Based on the above motivation, we have generated the sparse signal by randomly choosing K locations for the non-zero entries from the N possible locations and the magnitudes of the non-zero entries are taken to be random samples of a zero-mean Gaussian distributed random variable with unity variance. We conduct four different experiments with different values of N and K. The sparse signals are reconstructed from their Fourier spectrum magnitude by using the proposed algorithm. The mean-square error is analyzed in each case as a function of the number of iterations. These results are shown in Figure 2. We observe that the error decreases with increase in number of iterations. In some instances of the simulation, we found that the MSE stagnates after some iterations, whereas in some other instances, the MSE continued to decrease although marginally after sufficient iterations.

5.2. Experimental FDOCT data

The experimental configuration is the same as that employed in [5]. We briefly recall the critical parameters. A Ti-Sapphire laser with wavelength $\lambda_0 = 800$ nm and bandwidth $\Delta \lambda = 135$ nm has been



Fig. 2. Convergence performance of the proposed algorithm: MSE versus number of iterations

used as the light source. The beam is collimated using a lens with focal length 8.2 mm and is split in reference and sample arms. The power at the sample is approximately 3 mW, the axial resolution in air is 3 μ m and the lateral resolution is 1.3 μ m. The integration time is 43 μ s and the line rate is 5 kHz. The number of samples in a measurement is 2048.

We have demonstrated the performance of our method on two specimens: (i) a glass sample with three interfaces and (ii) an onionpeel sample. We also compare the reconstructions with the standard Fourier technique after applying the so-called *background subtraction*. Our technique does not need the background subtraction step. The results are shown in Figure 3. The reconstruction with the proposed technique has been made assuming a small sparsity level K = 320, which is much smaller compared with the number of measurements (2048 in this case). We observe that the proposed phase retrieval technique yields tomograms with relatively higher signalto-noise ratio (SNR). We have shown an extended dynamic range to illustrate this point. In addition, we observe from Figures 3(c) and (d) that the autocorrelation artifacts are also suppressed. If the sparsity level is increased, going by the error metric in (4), one can expect an improvement in reconstruction quality.

6. CONCLUSION

We have proposed a Fienup-type algorithm to incorporate the constraint of sparsity by iteratively estimating the Fourier phase from the Fourier magnitude for signals that admit a sparse representation in a suitably chosen orthonormal basis. The key aspect of our algorithm is that the sparsity condition is applied in each iteration by retaining only the K coefficients with largest magnitudes and setting others equal to zero and thus refining the estimate of the phase. This allows us to have variable support in each iteration unlike the standard phase retrieval methods in the literature. We do not constrain the signal model to be of the rational transfer function type. We have also analyzed the convergence properties of the algorithm. On the application front, we considered FDOCT and showed that the



Fig. 3. Comparison of reconstruction performance for onion-peel and glass specimen data, respectively; (a) and (c) correspond to the proposed phase retrieval technique with a sparsity level K = 320; (b) and (d) correspond to the standard Fourier reconstruction technique.

proposed method yields better quality tomograms than the standard Fourier technique even with a low sparsity level.

Appendix: Proof of Convergence

Consider E_k , the error in k-th iteration:

$$E_k = \frac{1}{N} \sum_{\ell=0}^{N-1} [|F_k(\ell)| - |F(\ell)|]^2 = \frac{1}{N} \sum_{\ell=0}^{N-1} [|F_k(\ell)| - |F'_k(\ell)|]^2$$

since $|F'_k(\ell)| = |F(\ell)|, \forall k$. Note that $|F(\ell)|$ is the observed Fourier magnitude. Both $F_k(\ell)$ and $F'_k(\ell)$ have same phase $\phi_k(\ell)$ (follows from Step 3 of the algorithm) and hence we write that

$$E_k = \frac{1}{N} \sum_{\ell=0}^{N-1} [|F_k(\ell)| - |F'_k(\ell)|]^2$$

= $\frac{1}{N} \sum_{\ell=0}^{N-1} |F_k(\ell) - F'_k(\ell)|^2 = \sum_{n=0}^{N-1} |f_k(n) - f'_k(n)|^2$

where the last equality follows from Parseval's theorem. We then use the fact that $\mathbf{f} = \Psi \mathbf{a}$ and that Ψ is orthonormal. This implies

$$\sum_{n=0}^{N-1} |f_k(n) - f'_k(n)|^2 = \sum_{n=0}^{N-1} |a_k(n) - a'_k(n)|^2$$

which leads to the inequality

$$E_k = \sum_{n=0}^{N-1} |a_k(n) - a'_k(n)|^2 \ge \sum_{n=0}^{N-1} |a_{k+1}(n) - a'_k(n)|^2.$$
(3)

The above inequality is a direct consequence of the following observation: \mathbf{a}_{k+1} is the best possible K-sparse approximation to \mathbf{a}'_k (because it is formed by retaining the K largest entries in \mathbf{a}'_k and setting others to zero) and \mathbf{a}_k is also a K-sparse vector. In other words, if $S = \{\mathbf{f} : |\text{supp}(\mathbf{f})| = K\}$, we have that

$$\mathbf{a}_{k+1} = \arg\min_{\mathbf{a}\in\mathcal{S}} ||\mathbf{a} - \mathbf{a}_k'||^2 = \arg\min_{\mathbf{a}\in\mathcal{S}} \sum_{n=0}^{N-1} |a(n) - a_k'(n)|^2.$$

Since $\mathbf{a}_k \in \mathcal{S}$, we have the inequality

$$\sum_{n=0}^{N-1} |a_k(n) - a'_k(n)|^2 \ge \sum_{n=0}^{N-1} |a_{k+1}(n) - a'_k(n)|^2.$$
(4)

since Ψ is orthonormal, by Parseval property we have

$$\sum_{n=0}^{N-1} |a_{k+1}(n) - a'_k(n)|^2 = \sum_{n=0}^{N-1} |f_{k+1}(n) - f'_k(n)|^2.$$
(5)

Therefore, we get the following:

$$\sum_{n=0}^{N-1} |f_{k+1}(n) - f'_k(n)|^2 = \frac{1}{N} \sum_{\ell=0}^{N-1} |F_{k+1}(\ell) - F'_k(\ell)|^2$$
$$\geq \frac{1}{N} \sum_{\ell=0}^{N-1} [|F_{k+1}(\ell)| - |F'_k(\ell)|]^2$$
$$= \frac{1}{N} \sum_{\ell=0}^{N-1} [|F_{k+1}(\ell)| - |F(\ell)|]^2$$
$$= E_{k+1},$$

where the above inequality follows as a direct consequence of the triangle inequality. Hence, we have $E_k \ge E_{k+1}$, $\forall k$. Therefore, the error in $(k + 1)^{st}$ iteration is less than or equal to that in the k^{th} iteration.

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