A UNIFIED VIEW OF NON-STATIONARY SINUSOIDAL PARAMETER ESTIMATION METHODS USING SIGNAL DERIVATIVES

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ABSTRACT

In this paper, we present a unified view of three non-stationary sinusoidal parameter estimation methods which are based on taking linear transforms of a signal and its derivatives. These methods, the Distribution Derivative Method (DDM), the Generalized Derivative Method (GDM), and the Generalized Reassignment Method (GRM), are shown to be subcases of a more general method which results in a system of linear equations from which we can solve for the parameter estimators. While the GDM and GRM are known to be theoretically equivalent, we show that they are also equivalent to the DDM in one special case. Matrix formulations are established for the GDM and GRM with a polynomial log-amplitude, polynomial phase sinusoidal signal model, and a bias in previous frequency slope estimators is explicitly demonstrated.

***Index Terms*—** Parameter estimation, non-stationary sinusoidal analysis, reassignment method, derivative method.

1. INTRODUCTION

The sinusoid plus noise model has been useful in many applications, including speech and music analysis and synthesis, and digital audio effects. To model the sinusoidal part of an observed signal we need to estimate the sinusoidal parameters: amplitude, phase, and frequency. Much work has been done for the stationary case, where the parameters are assumed to be constant within the analysis frame, but for signals with strong amplitude or frequency modulations it is necessary to use non-stationary sinusoidal analysis, where the amplitude and frequency parameters are allowed to evolve within the analysis frame. Those amplitude and frequency modulations are modelled with extra sinusoidal parameters.

It has been convenient to model the amplitude variations on a log scale, in which case the "non-stationary sinusoid" is defined as an exponential function with a complex polynomial function of time argument. Estimators have been designed for a first-order amplitude and frequency modulation non-stationary sinusoidal model using reassignment operators [1, 2], signal derivatives [2, 3], Gaussian windows [4], and phase-corrected vocoders [5].

More recently, methods have been proposed where the modulations are generalized to any order of modulation and can be represented not only by a polynomial, but any linear combination of continuous real functions of time, for example, sinusoidal modulations to represent tremolo and vibrato. The first of these methods, the Distribution Derivative Method (DDM), is formulated in terms of linear transforms on a distribution's (the signal) derivative [6] and the second, the Generalized Derivative Method (GDM)[1], in terms of Fourier transforms of windowed signal derivatives [7]. A third formulation was proposed, the Generalized Reassignment Method (GRM), which is equivalent to the GDM and moves the differentiation from the signal to the analysis window. This can be more convenient as signal derivatives are usually unknown and have to be estimated using derivative filters [2].

The purpose of this paper is to unify these three methods under one framework. This will show how these methods are related, how we can use a mix of these three methods, and it will further motivate practical comparisons of the three methods. The paper is laid out as follows. In Section 2, we develop a unified view of the three methods and we show when they are equivalent. In Section 3, we give the matrix formulation for the general case, we develop examples with alternative matrix formulations for the GDM and GRM, and we demonstrate a bias in a previous frequency slope estimator.

2. PARAMETER ESTIMATION

2.1. The Signal Model

Our goal is to estimate the non-stationary sinusoidal parameters of $s(t)$ from the observed signal $\check{s}(t) = s(t) + \nu(t)$, where $\nu(t)$ is an additive noise. The signal model for $s(t)$ is:

$$s(t) = \exp\left(\alpha_0 + \sum_{k=1}^{K} \alpha_k p_k(t)\right), \qquad (1)$$

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[1] Not to be confused with the GDM in [2], which precedes [7].

where α_k are the $K+1$ complex non-stationary sinusoidal parameters for the amplitude (real part) and phase (imaginary part) modulations up to order K. We can use Eq. (1) for a multi-component signal if we can reasonably separate the contributions from each non-stationary sinusoidal component using linear transforms such as the Fourier transform. The types of modulation are represented by the K real functions $p_k(t)$. A reasonable choice is $p_k(t) = t^k$ as it can approximate an arbitrary modulation with a Taylor expansion of order K.

2.2. Some Mathematical Properties of Interest

We review some mathematical properties that will be of use in designing our parameter estimators. With $\gamma(t)$, an atom of a linear transform, we denote the linear transform of a signal $x(t)$ by the inner product $\langle x, \gamma \rangle$:

$$\langle x, \gamma \rangle = \int_{-\infty}^{+\infty} x(t)\gamma^*(t)\,dt\,. \tag{2}$$

Using *integration-by-parts* (IBP) on the inner product we have:

$$\Big[x^{(m)}\gamma^{(n)}\Big]_{-\infty}^{+\infty} = \langle x^{(m+1)}, \gamma^{(n)} \rangle + \langle x^{(m)}, \gamma^{(n+1)} \rangle\,, \tag{3}$$

where the superscript (m) denotes differentiation m times.[2] If $\gamma^{(n)}(t)$ or $x^{(m)}(t)$ go to zero at $t = \pm\infty$ we have zero on the left-hand side of Eq. (3), and IBP simplifies to:

$$\langle x^{(m+1)}, \gamma^{(n)} \rangle = -\langle x^{(m)}, \gamma^{(n+1)} \rangle\,. \tag{4}$$

2.3. Solving for Model Parameters

Taking the derivative of our signal model Eq. (1) we get a linear equation with respect to the model parameters α_k, $k > 0$:

$$s'(t) = \sum_{k=1}^{K} \alpha_k s(t) p'_k(t)\,, \tag{5}$$

and taking the mth derivative, $m > 0$, assuming $p_k(t)$ are m-times differentiable, we get:

$$s^{(m)}(t) = \sum_{k=1}^{K} \alpha_k (s(t)p'_k(t))^{(m-1)}\,. \tag{6}$$

While we could use time-domain linear equations to design sinusoidal parameter estimators, it is often more useful to transform the signal to another domain, such as the frequency domain, to estimate the non-stationary sinusoidal parameters. Using the atom $\gamma(t)$, we apply a linear transform to both sides of Eq. (6):

$$\langle s^{(m)}, \gamma \rangle = \sum_{k=1}^{K} \alpha_k \langle (s \cdot p'_k)^{(m-1)}, \gamma \rangle\,. \tag{7}$$

[2] We also use the convention: $x'(t) = x^{(1)}(t), x''(t) = x^{(2)}(t), \ldots$

Since the derivatives of the observed signal are usually unknown and have to be estimated, we can move the differentiation from the signal to the atom by using IBP b times on the left-hand side of Eq. (7), with $b \le m$, and d times on the right-hand side, with $d \le m-1$, assuming $\gamma(t)$ is n-times differentiable, where $n = \max(b, d)$, and its derivatives up to order $n-1$ go to zero at $t = \pm\infty$. We end up with:

$$(-1)^b \langle s^{(m-b)}, \gamma^{(b)} \rangle = (-1)^d \sum_{k=1}^{K} \alpha_k \langle (s \cdot p'_k)^{(m-d-1)}, \gamma^{(d)} \rangle\,. \tag{8}$$

To solve for the non-stationary parameters α_k, $k > 0$, we need K independent linear equations. Once we have solved for the model parameters, the non-stationary *parameter estimators* $\hat{\alpha}_k$, $k > 0$, are given by replacing the signal model $s(t)$ by the observed signal $\check{s}(t)$. Since the observed signal will usually not exactly match the signal model, we can find a linear least squares fit using L equations with $L \ge K$. The parameter α_0, which represents the initial log-amplitude and phase, can be estimated using the other estimations $\hat{\alpha}_k$ for $k > 0$ [6].

2.4. A Unified View

To present a unified view of the three methods, we need to look at the possible ways of generating L independent linear equations from Eq. (8). To the ith equation, we associate one atom: $\gamma_i(t)$, one derivative order: m_i, and one pair: (b_i, d_i) such that $b_i \le m_i$ and $d_i \le m_i - 1$. The pair (b_i, d_i) allows us to allocate orders of differentiation to the signal or to the atom via IBP. Choosing R unique atoms and M unique derivative orders provides us with $M + R - 1$ equations, so we need $M + R - 1 = L$. A specific method is then determined by the M values of m_i, b_i, d_i, and the set of R atoms $\gamma_i(t)$. We call these choices the "configuration parameters".

Now we will show which configuration parameters lead to the DDM [6], the GDM [7], and the GRM [7]. These are also summarized in Table 1.

Distributed Derivative Method (DDM): For the DDM, we choose $R = L$ (R different atoms) and $M = 1$, with $m_i = 1$ and $(b_i, d_i) = (1, 0)$.

Generalized Derivative Method (GDM): For the GDM, we choose $R = 1$ and $M = L$, with $m_i = 1, \ldots, L$ and $(b_i, d_i) = (0, 0)$, assuming the signal is L-times differentiable. In [7] they used the atom: $\psi_\omega(t) = h^*(t)e^{j\omega t}$, where $h(t)$ is a tapered analysis window, but any linear transform atom could be used such that it goes to zero at $t = \pm\infty$.

Generalized Reassignment Method (GRM): For the GRM, we choose $R = 1$ and $M = L$, with $m_i = 1, \ldots, L$ and $(b_i, d_i) = (m_i, m_i - 1)$. This allocates all the differentiation to the atom. The atom we select must be L-times differentiable, and must go to zero at $t = \pm\infty$ for all derivative orders up to $L - 1$. A method of building sufficiently differentiable atoms from $\psi_\omega(t)$ is given in [7]. The authors of [8] recently used a polynomial phase Fourier kernel with the GRM.

Table 1 Examples of configuration parameters for generating L equations from Eq. (8)

# of atoms (R)	# of derivative orders (M)	m_i	(b_i, d_i)	Method used in literature
L	1	$m_i = 1$	$(1, 0)$	DDM [6]
1	L	$m_i = 1, \ldots, L$	$(0, 0)$	GDM [7]
1	L	$m_i = 1, \ldots, L$	$(m_i, m_i - 1)$	GRM [7, 8]

The configuration parameters which apply to the three methods given are the extreme cases: the equations of the DDM are given by using different atoms, the GDM uses signal derivatives, and the GRM moves those derivatives completely to the atom. Beyond these three configurations exist many possibilities which could be tailored to the constraints on the differentiability of the atoms used and on the availability of signal derivatives.

2.5. Theoretical Equivalence: Special Case

While the GRM and GDM are theoretically equivalent via IBP, the DDM is only equivalent to the GRM and GDM in one special case. If we choose increasing orders of derivatives of one atom for the L atoms of the DDM such that $\gamma_i(t) = \gamma^{(i)}(t)$, then the DDM is equivalent to the GRM, and thus the GDM.

3. MATRIX FORMULATION

3.1. General Case

To solve our system of equations we can write it in matrix form and use typical matrix decompositions. We want to solve for the vector $\boldsymbol{\alpha}$, which contains the non-stationary sinusoidal parameters for $k > 0$, from the matrix equation: $\boldsymbol{A}_s\boldsymbol{\alpha} = \boldsymbol{b}_s$, where the elements of the matrices are:

$$\boldsymbol{A}_{s\{i,k\}} = (-1)^{d_i} \langle (s \cdot p_k')^{(m_i - d_i - 1)}, \gamma_i^{(d_i)} \rangle \tag{9}$$

$$\boldsymbol{\alpha}_{\{i,1\}} = \alpha_i \tag{10}$$

$$\boldsymbol{b}_{s\{i,1\}} = (-1)^{b_i} \langle s^{(m_i - b_i)}, \gamma_i^{(b_i)} \rangle\,. \tag{11}$$

$\boldsymbol{A}_s$ is $L \times K$, $\boldsymbol{\alpha}$ is $K \times 1$, and $\boldsymbol{b}_s$ is $L \times 1$. We solve for $\boldsymbol{\alpha}$ by taking the pseudoinverse of $\boldsymbol{A}_s$:

$$\boldsymbol{\alpha} = (\boldsymbol{A}_s^H \boldsymbol{A}_s)^{-1} \boldsymbol{A}_s^H \boldsymbol{b}_s\,. \tag{12}$$

Replacing $s(t)$ with the observed signal $\check{s}(t)$ we get the estimation of the parameter vector $\boldsymbol{\alpha}$:

$$\hat{\boldsymbol{\alpha}} = (\boldsymbol{A}_{\check{s}}^H \boldsymbol{A}_{\check{s}})^{-1} \boldsymbol{A}_{\check{s}}^H \boldsymbol{b}_{\check{s}}\,. \tag{13}$$

3.2. Example 1: GDM

If we choose the configuration parameters for the GDM with the atom $\psi_\omega(t)$ and $p_k(t) = t^k$, we can get an alternative matrix formulation by expanding the differentiation on the product $(s \cdot p_k')$ using the generalized product rule of differentiation, which results in a binomial expansion of the derivatives. We can represent this system of equations by the matrix equation $\mathbf{U}_s\,\mathbf{a} = \boldsymbol{b}_s$, where $\mathbf{U}_s$ is $L \times LK$, $\mathbf{a}$ is $LK \times 1$, and $\boldsymbol{b}_s$ is given by Eq. (11). We use the *partition indices* $(\mathbf{i}, \mathbf{k})$, representing vector partitions of $\mathbf{U}_s$ and $\mathbf{a}_s$, to get:

$$\mathbf{U}_{s\{\mathbf{i},\mathbf{k}\}} = \begin{cases} \dbinom{\mathbf{i}-1}{\mathbf{i}-\mathbf{k}} \boldsymbol{C}_s(\mathbf{i},\mathbf{k}) & \mathbf{i} \geq \mathbf{k} \\ \mathbf{0}_{\{1\times K\}} & \text{otherwise} \end{cases} \tag{14}$$

$$\mathbf{a}_{\{\mathbf{i},1\}} = \boldsymbol{\alpha}\,, \tag{15}$$

and $\boldsymbol{C}_s(\mathbf{i}, \mathbf{k})$ is a $1 \times K$ row vector with elements:

$$\boldsymbol{C}_{s\{1,k\}}(\mathbf{i},\mathbf{k}) = \begin{cases} \dfrac{k!}{(k-\mathbf{i}+\mathbf{k}-1)!} \cdot \\ \mathcal{F}_\omega \left\{ s^{(\mathbf{k}-1)} \mathcal{T}^{k-\mathbf{i}+\mathbf{k}-1} h \right\} & k \geq \mathbf{i}-\mathbf{k}+1 \\ 0 & \text{otherwise}\,, \end{cases} \tag{16}$$

where $\mathcal{F}_\omega\{x\}$ denotes the Fourier transform (FT) of $x(t)$ evaluated at the frequency ω, and where $(\mathcal{T}x)(t) = tx(t)$. We see that with the atom we have selected, the GDM results in a system of equations built on FTs of signal derivatives combined with time-ramped windows. Using a shorthand notation such that $S^{(m)}_{\omega;\mathcal{T}h^{(n)}} = \mathcal{F}_\omega\{s^{(m)}\mathcal{T}h^{(n)}\}$, we give as an example the matrices for $K = L = 2$:

$$\left[\begin{array}{cc|cc} S_{\omega;h} & 2S_{\omega;\mathcal{T}h} & 0 & 0 \\ \hline 0 & 2S_{\omega;h} & S'_{\omega;h} & 2S'_{\omega;\mathcal{T}h} \end{array}\right] \left[\begin{array}{c} \left[\begin{array}{cc} \alpha_1 & \alpha_2 \end{array}\right]^T \\ \left[\begin{array}{cc} \alpha_1 & \alpha_2 \end{array}\right]^T \end{array}\right] = \left[\begin{array}{c} S'_{\omega;h} \\ S''_{\omega;h} \end{array}\right]. \tag{17}$$

From these matrices we can solve for the parameters α_1, α_2:

$$\alpha_2 = \frac{S''_{\omega;h} S_{\omega;h} - (S'_{\omega;h})^2}{2\left((S_{\omega;h})^2 + S'_{\omega;\mathcal{T}h} S_{\omega;h} - S_{\omega;\mathcal{T}h} S'_{\omega;h}\right)} \tag{18}$$

$$\alpha_1 = \frac{S'_{\omega;h}}{S_{\omega;h}} - 2\alpha_2 \frac{S_{\omega;\mathcal{T}h}}{S_{\omega;h}}\,. \tag{19}$$

By replacing $s(t)$ with the observed signal $\check{s}(t)$ these expressions become the parameter estimators $\hat{\alpha}_1, \hat{\alpha}_2$.

3.3. Example 2: GRM

If instead we want to use the GRM with the atom $\psi_\omega(t)$ and $p_k(t) = t^k$, we still solve for $\mathbf{U}_s\,\mathbf{a} = \boldsymbol{b}_s$, but we can expand the differentiation on the atom so $\boldsymbol{C}_s(\mathbf{i},\mathbf{k})$ and $\boldsymbol{b}_s$ become:

$$\boldsymbol{C}_{s\{1,k\}}(\mathbf{i},\mathbf{k}) = (-j\omega)^{\mathbf{i}-\mathbf{k}} k\,\mathcal{F}_\omega\left\{s\mathcal{T}^{k-1}h^{(\mathbf{k}-1)}\right\} \tag{20}$$

$$\boldsymbol{b}_{s\{i,1\}} = -\sum_{q=0}^{i}\binom{i}{q}(-j\omega)^{i-q}\mathcal{F}_\omega\left\{sh^{(q)}\right\}. \tag{21}$$

The *pyramid-like* scheme in [7, 9] comes from the binomial expansion of the atom derivatives in $\boldsymbol{b}_s$. For $K = L = 2$ we get the following matrices:

$$\left[\begin{array}{cc|cc} S_{\omega;h} & 2S_{\omega;\mathcal{T}h} & 0 & 0 \\ \hline -j\omega S_{\omega;h} & -2j\omega S_{\omega;\mathcal{T}h} & S_{\omega;h'} & 2S_{\omega;\mathcal{T}h'} \end{array}\right]\left[\begin{array}{c} \left[\begin{array}{cc}\alpha_1 & \alpha_2\end{array}\right]^T \\ \left[\begin{array}{cc}\alpha_1 & \alpha_2\end{array}\right]^T \end{array}\right]$$

$$= \begin{bmatrix} j\omega S_{\omega;h} - S_{\omega;h'} \\ \omega^2 S_{\omega;h} + 2j\omega S_{\omega;h'} - S_{\omega;h''} \end{bmatrix}, \tag{22}$$

from which we can solve for α_1 and α_2:

$$\alpha_2 = \frac{S_{\omega;h''}S_{\omega;h} - (S_{\omega;h'})^2}{2\left(S_{\omega;\mathcal{T}h}S_{\omega;h'} - S_{\omega;\mathcal{T}h'}S_{\omega;h}\right)} \tag{23}$$

$$\alpha_1 = j\omega - \frac{S_{\omega;h'}}{S_{\omega;h}} - 2\alpha_2\frac{S_{\omega;\mathcal{T}h}}{S_{\omega;h}}. \tag{24}$$

We note that the frequency slope estimators used in the past [1, 2, 3, 9] are different from the unbiased estimators given by taking the imaginary part of Eq. (18) or (23). Unlike these estimators, which come directly from the signal model, the reassignment method frequency slope estimator in [1] comes directly from the reassignment operators. That estimator, which appears on the left-hand side of the following inequality, is slightly biased because in general $\Im\{z_1\}/\Re\{z_2\} \neq \Im\{z_1/z_2\}$, where:

$$z_1 = \frac{S_{\omega;h''}S_{\omega;h} - (S_{\omega;h'})^2}{(S_{\omega;h})^2} \tag{25}$$

$$z_2 = 2\left(\frac{S_{\omega;\mathcal{T}h}S_{\omega;h'} - S_{\omega;\mathcal{T}h'}S_{\omega;h}}{(S_{\omega;h})^2}\right). \tag{26}$$

The same goes for the frequency slope estimator in [3]; it is a slightly biased version of Eq. (18).

Similar matrix formulations for the DDM are found in [6].

4. CONCLUSIONS

We have presented a unified framework from which we can get the DDM, GDM, and GRM non-stationary sinusoidal parameter estimators. This shows that the three methods are just different configurations of a more general method, and in one case the three methods are equivalent. Matrix formulations of the GDM and GRM were developed for polynomial log-amplitude and phase modulations, and we showed there was a bias in previous frequency slope parameter estimators using reassignment operators and signal derivatives. This work further motivates the practical comparisons of the three methods, and motivates studies on the use of hybrid methods that combine different atoms with derivatives of the signal and derivatives of the atoms. Experiments comparing these three methods with the Cramér-Rao bounds (also comparing with [4]) have been presented in [10].

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