COMPLETE CHARACTERIZATION OF THE PARETO BOUNDARY OF INTERFERENCE-COUPLED WIRELESS SYSTEMS WITH POWER CONSTRAINTS — THE LOG-CONVEX CASE

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ABSTRACT

In this paper we analyze the structure of certain power-constrained utility sets, based on the axiomatic framework of log-convex interference functions. Log-convex interference functions contain convex and linear interference functions as a special case. We analyze the boundary of the set. It is shown how Pareto optimality of boundary points depends on the interference coupling between the users. Finally, we investigate feasible sets of signal-to-interference-plus-noise ratios for individual power constraints and a sum power constraint. We show certain properties that are desirable, e.g. in the context of cooperative game theory.

Index Terms— **resource allocation, interference functions, SIR feasible set, game theory, Nash bargaining**

1 INTRODUCTION

Performance tradeoffs in multiuser systems occur when users share a common resource or if they are coupled by mutual interference. This is typical for wireless systems, and also for certain wireline connections, e.g. twisted-pair copper wires used for DSL transmission. The achievable performances are commonly characterized by the *utility set*, sometimes referred to as utility region or quality-of-service (QoS) region. The utility region $\mathcal{U}$ is defined as the set of all achievable utility vectors $\boldsymbol{u} = [u_1, \ldots, u_K]^T$, where $K \geq 2$ is the number of users.

Many resource allocation strategies crucially depend on the structure of the set $\mathcal{U}$, so a thorough understanding of its boundary is needed. Some often-made assumptions are *comprehensiveness*, *convexity*, and *Pareto optimality*. These properties are often implicitly assumed (see e.g. [1]). Comprehensiveness can be interpreted as free disposability of utility. Convexity allows the application of well-known concepts from optimization and game theory. Pareto optimality of the boundary means that no resources are wasted.

For interference-coupled wireless systems, the set $\mathcal{U}$ can largely depend on the physical layer. Thus, convexity and Pareto optimality need not be fulfilled. Examples are interference mitigation and avoidance strategies, which can have a large impact on the structure of the resulting utility set. Although some particular utility sets are well-understood, there is no general theory for analyzing utility tradeoffs caused by multiuser interference.

This paper builds on previous results [2], [3], where the structure of log-convex interference functions and resulting utility sets was analyzed. The contributions of this paper are:

- In Section 2 we study the boundary of the region of signal-to-interference-plus-noise ratios (SINR). We assume log-convex interference functions and power constraints. A necessary and sufficient condition for Pareto optimality is derived. These results automatically extend to arbitrary utility sets resulting from monotone mappings of the SINR. Examples include capacity, bit error rate, minimum mean square error, etc.
- In Section 3 we show that the SINR sets from Section 2 have certain convexity properties which are desirable in the context of cooperative game theory. One motivation for this analysis is Nash's bargaining theory [1], which can be extended to certain log-convex utility sets [3].

Some notational conventions are: Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the lth component. The notation $\boldsymbol{y} \geq 0$ means that $y_l \geq 0$ for all components l. $\boldsymbol{x} \gneqq \boldsymbol{y}$ means component-wise inequality with strict inequality for at least one component. Similar definitions hold for the reverse directions. Finally, $\boldsymbol{x} \neq \boldsymbol{y}$ means that the vector differ in at least one component. The set of non-negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$.

2 INTERFERENCE-COUPLED WIRELESS SYSTEMS BASED ON LOG-CONVEX INTERFERENCE FUNCTION

In this section we study the SINR region resulting from log-convex interference functions and power constraints. It will be shown how Pareto optimality of boundaries points depends on the interference coupling between the users.

2.1 Interference Functions

Consider K users, with transmit powers $\boldsymbol{p} = [p_1, \ldots, p_K]^T$. The noise power at each receiver is σ_n^2. Hence, the SINR at each receiver depends on the *extended power vector*

$$\underline{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ \sigma_n^2 \end{bmatrix} = [p_1, \ldots, p_K, \sigma_n^2]^T \ . \tag{1}$$

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The resulting SINR of user k is $\mathrm{SINR}_k(\underline{\boldsymbol{p}}) = p_k/\mathcal{I}_k(\underline{\boldsymbol{p}})$, where $\mathcal{I}_k$ is the interference (plus noise) as a function of $\underline{\boldsymbol{p}}$. We use the axiomatic framework proposed in [4].

Definition 1. We say that $\mathcal{I} : \mathbb{R}_+^{K+1} \mapsto \mathbb{R}_+$ is an *interference function* if the following axioms are fulfilled:

A1	(conditional positivity)	$\mathcal{I}(\underline{\boldsymbol{p}}) > 0$ if $\underline{\boldsymbol{p}} > 0$
A2	(scale invariance)	$\mathcal{I}(\alpha\underline{\boldsymbol{p}}) = \alpha\mathcal{I}(\underline{\boldsymbol{p}})$ for all $\alpha > 0$
A3	(monotonicity)	$\mathcal{I}(\underline{\boldsymbol{p}}) \geq \mathcal{I}(\underline{\boldsymbol{p}}')$ if $\underline{\boldsymbol{p}} \geq \underline{\boldsymbol{p}}'$
A4	(strict monotonicity)	$\mathcal{I}(\underline{\boldsymbol{p}}) > \mathcal{I}(\underline{\boldsymbol{p}}')$ if $\underline{\boldsymbol{p}} \geq \underline{\boldsymbol{p}}'$ and $\underline{p}_{K+1} > \underline{p}'_{K+1}$

A simple example is $\mathcal{I}(\underline{\boldsymbol{p}}) = \boldsymbol{v}^T\boldsymbol{p} + \sigma_n^2$, where $\boldsymbol{v} \in \mathbb{R}_+^K$ is a vector of interference coupling coefficients. It was shown in [5] that the axiomatic framework A1–A4 is closely connected with the framework of *standard interference functions* [6]. For any constant noise power $\underline{p}_{K+1} = \sigma_n^2$, the function $Y(\boldsymbol{p}) = \mathcal{I}(\underline{\boldsymbol{p}})$ is standard. Conversely, any standard interference function can be expressed within the framework A1–A4.

2.2 The SINR Region Under Power Constraints

Consider the *SINR feasible region* for users $\mathcal{K} = \{1, 2, \ldots, K\}$, with individual power constraints $\boldsymbol{p} \leq \boldsymbol{p}^{\max} = [p_1^{\max}, \ldots, p_K^{\max}]^T$, which is defined as the sub-level set

$$\mathcal{S}(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max}) = \{\boldsymbol{\gamma} \in \mathbb{R}_{++}^K : C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max}) \leq 1\} \tag{2}$$

where $\boldsymbol{\mathcal{I}} = [\mathcal{I}_1, \ldots, \mathcal{I}_K]^T$ and $\boldsymbol{\gamma}$ is a vector of SINR values, whose feasibility is determined by the min-max optimum

$$C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max}) = \inf_{\boldsymbol{0} < \boldsymbol{p} \leq \boldsymbol{p}^{\max}} \Big(\max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k} \Big) . \tag{3}$$

The structure of the SINR set $\mathcal{S}(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ depends on the properties of the indicator function $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$, which in turn depends on the properties of the underlying interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$, as well as on the chosen power constraints $\boldsymbol{p}^{\max}$.

Sub-level sets of convex functions are convex, thus $\mathcal{S}(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ is a closed convex set from $\mathbb{R}_+^K$ if $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ is *convex* (see e.g. [7]). However, convexity of $C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ does generally not hold, so SINR regions are typically non-convex.

The SINR region under a sum power constraint is defined as

$$\mathcal{S}(\boldsymbol{\mathcal{I}}, P_{\mathrm{tot}}) = \{\boldsymbol{\gamma} \in \mathbb{R}_{++}^K : C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, P_{\mathrm{tot}}) \leq 1\} \tag{4}$$

where $\quad C(\boldsymbol{\gamma}, \boldsymbol{\mathcal{I}}, P_{\mathrm{tot}}) = \inf_{\boldsymbol{p} > 0; \|\boldsymbol{p}\|_1 \leq P_{\mathrm{tot}}} \Big(\max_{k \in \mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k} \Big)$.

2.3 Log-Convex Interference Functions

Having introduced general interference functions in Section 2.1, we will now discuss the important sub-class of *log-convex interference functions*. Throughout this paper, all interference functions are assumed to be log-convex.

For explanation, consider the function $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$, which is said to be *log-convex* on $\mathbb{R}^{K+1}$ if $\log f$ is convex, or equivalently

$$f\big((1-\lambda)\hat{\boldsymbol{s}} + \lambda\check{\boldsymbol{s}}\big) \leq f(\hat{\boldsymbol{s}})^{1-\lambda} f(\check{\boldsymbol{s}})^{\lambda}, \quad \forall \lambda \in (0,1),\ \hat{\boldsymbol{s}}, \check{\boldsymbol{s}} \in \mathbb{R}^{K+1} .$$

Definition 2. We say that the interference function $\mathcal{I}$ is a *log-convex interference function* if $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^{K+1}$.

Note, that the log-convexity in Definition 2 is based on a change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ (component-wise exponential). Such a technique was already used by Sung [8] in the context of linear interference functions, and later in [9], [10].

2.4 Characterization of the Boundary for Individual Power Constraints

Consider log-convex interference functions and individual power limits $\boldsymbol{p}^{\max}$. Let $\boldsymbol{\gamma} > 0$ be an arbitrary boundary point of the resulting region $\mathcal{S}(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$. The set of all power vectors achieving $\boldsymbol{\gamma}$ is

$$\mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max}) = \{\boldsymbol{0} \leq \boldsymbol{p} \leq \boldsymbol{p}^{\max} : p_k \geq \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})\} . \tag{5}$$

For the following analysis, it is important to note that the set $\mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$ can contain multiple elements. This is most easily explained by an example:

Example 1. Consider a 2-user Gaussian multiple access channel (MAC) with successive interference cancellation, normalized noise $\sigma_n^2 = 1$, and a given decoding order $1, 2$. The SINR of the users are

$$\mathrm{SINR}_1(\boldsymbol{p}) = \frac{p_1}{p_2 + 1} , \qquad \mathrm{SINR}_2(\boldsymbol{p}) = p_2 .$$

Assuming power constraints $p_1 \leq p_1^{\max} = 1$ and $p_2 \leq p_2^{\max} = 1$, we obtain an SINR region as depicted in Fig. 1.

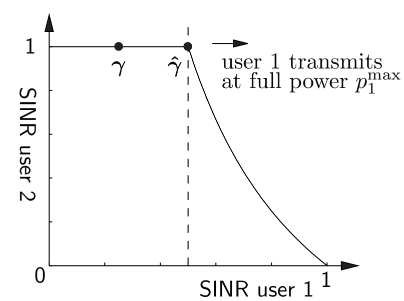

Fig. 1. SINR feasible set for the 2-user MAC channel described in Example 1.

Consider the boundary point $\boldsymbol{\gamma}$ depicted in Fig. 1. This point is achieved by $\boldsymbol{p}^* = [p_1^{\max}/2, p_2^{\max}]^T$, so $\boldsymbol{p}^* \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$. This vector achieves $\boldsymbol{\gamma}$ with component-wise minimum power. However, $\boldsymbol{p}^*$ is not the only element of $\mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$. Because of interference cancellation, we can increase the power (thus the SINR) of user 1, without reducing the SINR at user 2. If both users transmit with maximum power $\boldsymbol{p}^{\max}$ then the corner point $\hat{\boldsymbol{\gamma}}$ is achieved. This power vector is also contained in $\mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$ because $\hat{\boldsymbol{\gamma}} \geq \boldsymbol{\gamma}$, so the SIR targets $\boldsymbol{\gamma}$ are still fulfilled. These effects will be further discussed in the context of Pareto optimality in Section 2.6.

For an arbitrary $\boldsymbol{p} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$, consider the fixed point iteration

$$p_k^{(n+1)} = \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}}^{(n)}), \quad p_k^{(0)} = p_k, \quad \forall k \in \mathcal{K} . \tag{6}$$

The limit $\boldsymbol{p}^* = \lim_{n \to \infty} \boldsymbol{p}^{(n)} > 0$ is special because it achieves $\boldsymbol{\gamma}$ with component-wise minimum power [6].

Lemma 1. *The vector $\boldsymbol{p}^*$ fulfills $p_k \geq \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})$ with minimum component-wise power. That is, for all $\boldsymbol{p} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$ we have $\boldsymbol{p} \geq \boldsymbol{p}^*$.*

The next lemma shows that the inequality constraint in (5) is always fulfilled with equality for at least one component.

Lemma 2. *Consider an arbitrary $\boldsymbol{p} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$. There always exists a $k \in \mathcal{K}$ such that $p_k = \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})$.*

Definition 3. For an arbitrary given boundary point $\boldsymbol{\gamma} > 0$, let $\overline{\mathcal{K}}$ be the set of all $k \in \mathcal{K}$ such that there exists a $\boldsymbol{p}^{(k)} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$ with $p_k^{(k)} > \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}}^{(k)})$. In our example Fig. 1 this is the first user, whose power can be increased without decreasing the performance of user 2.

We are only interested in the case where $\overline{\mathcal{K}}$ is non-empty. Otherwise the fixed point is the unique solution, which is trivial. The next theorem shows that there always exists a vector $\hat{\boldsymbol{p}}$ for which strict inequality holds for all $k \in \overline{\mathcal{K}}$ simultaneously.

Theorem 1. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be log-convex interference functions. Assume that $\boldsymbol{\gamma}$ is an arbitrary boundary point such that $\overline{\mathcal{K}}$ is non-empty. Then there exists a vector $\hat{\boldsymbol{p}} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$ such that*

$$\hat{p}_k > \gamma_k \mathcal{I}_k(\underline{\hat{\boldsymbol{p}}}), \quad \forall k \in \overline{\mathcal{K}}\,, \tag{7}$$

and for all $\boldsymbol{p} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$ we have

$$p_k = \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}}), \quad \forall k \in \mathcal{K} \backslash \overline{\mathcal{K}}\,. \tag{8}$$

The following corollary is an immediate consequence of Theorem 2.

Corollary 1. *Let $\boldsymbol{p}^*$ and $\hat{\boldsymbol{p}}$ be defined as in Lemma 1 and Theorem 2, respectively. We have $\hat{\boldsymbol{p}} \geq \boldsymbol{p}^*$ (Lemma 1) and thus for all $k \in \overline{\mathcal{K}}$*

$$\hat{p}_k > \gamma_k \mathcal{I}_k(\underline{\hat{\boldsymbol{p}}}) \geq \gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}}^*) = p_k^*\,. \tag{9}$$

The following theorem shows that these "oversized powers" from $\overline{\mathcal{K}}$ have no impact on the interference experienced by the other users $\mathcal{K} \backslash \overline{\mathcal{K}}$. That is, the interference is the same as if we would use the minimum-power vector $\boldsymbol{p}^*$. Also, the powers of users $\mathcal{K} \backslash \overline{\mathcal{K}}$ cannot be oversized.

Theorem 2. *Consider an arbitrary $\boldsymbol{p} \in \mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max})$. For all $k \in \mathcal{K} \backslash \overline{\mathcal{K}}$, we have $\mathcal{I}_k(\underline{\boldsymbol{p}}^*) = \mathcal{I}_k(\underline{\boldsymbol{p}})$ and $p_k^* = p_k$.*

2.5 Interference Coupling

The structure of the SINR region depends on the interference coupling in the system. For axiomatic interference functions it is not obvious how to define interference coupling.

The following dependency matrix is independent of the choice of $\boldsymbol{p}$.

$$[\boldsymbol{D}_{\mathcal{I}}]_{kl} = \begin{cases} 1 & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that } \mathcal{I}_k(\underline{\boldsymbol{p}} + \delta \underline{\boldsymbol{e}}_l) \\ & \text{is not constant for some values } \delta > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The non-zero entries in $\boldsymbol{D}_{\mathcal{I}}$ mark the transmitter/receiver pairs which are coupled by interference. A zero entry means that no interference is received, no matter how large the transmission power is. As an example, think of users that are assigned to different orthogonal resources, or separated by adaptive interference rejection techniques.

2.6 Analysis of the Pareto Optimal Boundary

Thus far we have focused on the interference coupling aspects. Now, we will analyze the resulting utility set. In this paper, "utility" can stand for some arbitrary performance measure, which depends on the SINR by a strictly monotone and continuous function ϕ defined on $\mathbb{R}_+$. The utility of user k is

$$u_k(\boldsymbol{p}) = \phi_k\big(\mathrm{SINR}_k(\boldsymbol{p})\big)\,, \quad k \in \mathcal{K}\,. \tag{11}$$

Examples are MMSE: $\phi(x) = 1/(1+x)$, BER: $\phi(x) = Q(\sqrt{x})$, high-SNR approximation of BER: $\phi(x) = x^{-\alpha}$, with diversity order α, or capacity: $\phi(x) = \log(1+x)$.

Let γ_k be the inverse function of ϕ_k, then $\gamma_k(u_k)$ is the minimum SINR level needed by the kth user to satisfy the QoS target u_k. Let $\boldsymbol{u} \in \mathcal{U}$ be a vector of QoS values, then the associated SINR vector is

$$\boldsymbol{\gamma}(\boldsymbol{u}) = [\gamma_1(u_1), \ldots, \gamma_K(u_K)]^T\,. \tag{12}$$

QoS values $\boldsymbol{u} \in \mathcal{U}$ are feasible if and only if $C\big(\boldsymbol{\gamma}(\boldsymbol{u}), \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max}\big) \leq 1$. The QoS feasible set is the sub-level set

$$\mathcal{U} = \{\boldsymbol{u} : C(\boldsymbol{\gamma}(\boldsymbol{u}), \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max}) \leq 1\}\,. \tag{13}$$

We are now interested in the boundary of $\mathcal{U}$, which is characterized by $C(\boldsymbol{\gamma}(\boldsymbol{u}), \boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max}) = 1$. The boundary is denoted by $\partial\mathcal{U}$.

Definition 4. A boundary point $\boldsymbol{u} \in \partial\mathcal{U}$ is said to be *Pareto optimal* if there is no $\hat{\boldsymbol{u}} \in \partial\mathcal{U}$ with $\hat{\boldsymbol{u}} \gneqq \hat{\boldsymbol{u}}$.

From a practical point of view, this means that it is not possible to improve the performance of one user without decreasing the performance of another user.

Lemma 3. *A boundary point $\boldsymbol{u} \in \partial\mathcal{U}$ is Pareto optimal if and only if $\boldsymbol{\gamma}(\boldsymbol{u}) \in \partial S(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ is Pareto optimal.*

With Lemma 3 we know that, for any utility set according to the above definition we can analyze Pareto optimality by focusing on the underlying SINR set. The results transfer automatically to the corresponding utility sets.

With Theorem 1 we show the following result.

Theorem 3. *Let $\boldsymbol{p}^*$ be defined as in Lemma 1. A boundary point $\gamma \in \partial\mathcal{S}(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ is Pareto optimal if and only if for $\boldsymbol{p}^* = \boldsymbol{p}^*(\gamma)$ we have*

$$\mathcal{P}(\boldsymbol{\gamma}, \boldsymbol{p}^{\max}) = \{\boldsymbol{p}^*\}\,. \tag{14}$$

That is, the set (5) consists of a single vector $\boldsymbol{p}^*$.

2.7 Strict Monotonicity

In this section we introduce the additional property of *strict monotonicity*. We begin by a definition. Based on the dependency matrix $\boldsymbol{D}_{\mathcal{I}}$, as defined by (10), we introduce the *dependency set*

$$\mathsf{L}_k = \{l \in \mathcal{K} : [\boldsymbol{D}_{\mathcal{I}}]_{kl} = 1\}\,. \tag{15}$$

This is the set of transmitters which have impact on user k.

Definition 5 (strict monotonicity)**.** $\mathcal{I}_k(\underline{\boldsymbol{p}})$ is said to be strictly monotonic if $\boldsymbol{p}^{(1)} \geq \boldsymbol{p}^{(2)}$, with $p_l^{(1)} > p_l^{(2)}$ for some $l \in \mathsf{L}_k$, implies $\mathcal{I}_k(\underline{\boldsymbol{p}}^{(1)}) > \mathcal{I}_k(\underline{\boldsymbol{p}}^{(2)})$.

In other words, $\mathcal{I}_k(\underline{\boldsymbol{p}})$ is strictly increasing in at least one power component.

The assumption of strict monotonicity enables us to derive a link between the dependency matrix $\boldsymbol{D}_{\mathcal{I}}$ and Pareto optimality. This is summarized by the next theorem.

Theorem 4. *Let $\mathcal{I}_1, \ldots \mathcal{I}_K$ be log-convex interference functions which are strictly monotonic on their respective dependency set. The following statements are equivalent.*

- *The dependency matrix $\boldsymbol{D}_{\mathcal{I}}$ is irreducible.*
- *Every boundary point is Pareto optimal.*

3 Logarithmic Convexity and Nash Bargaining

It was shown in [3] that the game-theoretic framework of *Nash bargaining* can be extended to a certain family of non-convex sets $\mathcal{ST}_c$, as defined in the following Definition 7. The properties of such sets are illustrated in Fig. 2. In this section we will extend the results [3].

To this end, consider the bijective continuous mapping $\log(\boldsymbol{u}) = [\log u_1, \ldots, \log u_K]^T$, where $\boldsymbol{u} \in \mathcal{U} \cap \mathbb{R}^K_{++}$. The image set of $\mathcal{U}$ is

$$\mathcal{L}og(\mathcal{U}) = \{\boldsymbol{q} = \log(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{U} \cap \mathbb{R}^K_{++}\} . \quad (16)$$

Definition 6. We say that a set $\mathcal{U} \subseteq \mathbb{R}^K_+$ is a *log-convex* set if $\mathcal{L}og(\mathcal{U})$ is convex.

Definition 7. By $\mathcal{ST}$ we denote the set of all closed downward-comprehensive utility sets $\mathcal{U} \subset \mathbb{R}^K_+$ such that the image set $\mathcal{Q} = \mathcal{L}og(\mathcal{U})$ is convex and the following additional property is fulfilled: For any $\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}} \in PO(\mathcal{Q})$ (the Pareto optimal boundary), the connecting line $\boldsymbol{q}(\lambda) = (1-\lambda)\hat{\boldsymbol{q}} + \lambda\hat{\boldsymbol{q}}$, with $\lambda \in (0,1)$, is contained in the interior of $\mathcal{Q}$. By $\mathcal{ST}_c$ we denote the set of all $\mathcal{U} \in \mathcal{ST}$, which are additionally bounded, thus compact.

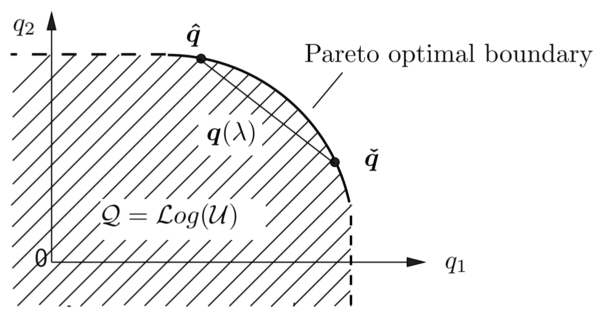

Fig. 2. Illustration of an image set $\mathcal{Q} = \mathcal{L}og(\mathcal{U})$ with $\mathcal{U} \in \mathcal{ST}_c$. The set is strictly convex with the exception of possible boundary segments parallel to the axes (dashed lines). Sets from $\mathcal{ST}_c$ play an important role in cooperative game theory [1]. For any $\mathcal{U} \in \mathcal{ST}_c$ there exists a single-valued Nash bargaining solution.

3.1 Total Power Constraint

Assume that the sum of all transmission powers is limited by P_{tot}. The next theorem shows that the resulting SINR set is strictly convex after the transformation.

Theorem 5. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be arbitrary log-convex interference functions. For all $0 < P_{tot} < +\infty$ the logarithmic transformation of the SINR region $\mathcal{L}og(\mathcal{S}(\boldsymbol{\mathcal{I}}, P_{\text{tot}}))$, as defined by (4), is strictly convex. Thus, the entire boundary is Pareto optimal and $\mathcal{S}(\boldsymbol{\mathcal{I}}, P_{\text{tot}}) \in \mathcal{ST}_c$.*

3.2 Individual Power Constraints

The possible occurrence of decoupled users did not matter under a sum-power constraint, because the users are always coupled by sharing a common power budget. However, in order to analyze the behavior under individual power constraints, we need to take into account the interference coupling. Pareto optimality was already studied in Section 2. In this section we will show under which conditions the Nash bargaining framework can be applied.

We begin by defining strict log-convexity on the dependency set.

Definition 8 (strict log-convexity)**.** A log-convex interference function $\mathcal{I}_k$ is said to be *strictly log-convex* if for all $\hat{\boldsymbol{p}}$, $\check{\boldsymbol{p}}$ for which there is some $l \in \mathsf{L}_k$ with $\hat{p}_l \neq \check{p}_l$, we have

$$\mathcal{I}_k\big(\underline{\boldsymbol{p}}(\lambda)\big) < \big(\mathcal{I}_k(\underline{\hat{\boldsymbol{p}}})\big)^{1-\lambda} \cdot \big(\mathcal{I}_k(\underline{\check{\boldsymbol{p}}})\big)^{\lambda} , \quad \lambda \in (0,1) . \quad (17)$$

where $\boldsymbol{p}(\lambda) = \hat{\boldsymbol{p}}^{1-\lambda} \cdot \check{\boldsymbol{p}}^{\lambda}$.

The following lemma shows that log-convexity implies strict monotonicity (introduced in Section 2.7).

Lemma 4. *Every strictly log-convex interference function is* strictly monotonic *on its dependency set (see Definition 5).*

Note, that the converse of Lemma 4 is not true. The next theorem shows a sufficient condition for the SINR region to be contained in $\mathcal{ST}_c$. It thereby extends the results [3].

Theorem 6. *Let $\mathcal{I}_1, \ldots \mathcal{I}_K$ be strictly log-convex interference functions, which depend on $p_1, \ldots, p_K$. Then the SINR region $\mathcal{S}(\boldsymbol{\mathcal{I}}, \boldsymbol{p}^{\max})$ is contained in $\mathcal{ST}_c$.*

4 Conclusions

In this paper we have analyzed log-convex utility regions, resulting from different assumptions and power constraints and interference coupling. This "hidden convexity" is useful for developing resource allocation strategies that operate on the boundary of the region.

The objective of this paper is to provide a theoretical basis for exploiting log-convexity in interference-coupled multiuser networks. The paper extends previous results [11] and [3].

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