

STATISTICS FOR COMPLEX RANDOM VARIABLES REVISITED

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ABSTRACT

Complex random signals play an increasingly important role in array, communications, and biomedical signal processing and related fields. However, the mathematical foundations of complex-valued signals and tools developed for handling them are scattered in literature. There appears to be a need for a concise, unified, and rigorous treatment of such topics. In this paper such a treatment is provided. Moreover, we establish connections between seemingly unrelated objects such as real differentiability and circularity. In addition, a novel complex-valued extension of Taylor series is presented and a measure for circularity is proposed.

Index Terms— complex random variables, complex moments, \mathbb{R} -differentiability, circularity, Taylor's series

1. INTRODUCTION

As the applications of complex random signals are becoming increasingly more advanced, it is evident that simplistic adaptation of techniques developed for real-valued signals to the complex-valued case may not be adequate, or may lead to suboptimal results or intractable calculations. This issue is now widely recognized in the signal processing research community. Unfortunately, the obtained fundamental results are scattered in open literature. Our intention is to provide a rigorous and unified treatment of properties of complex-valued random signals and related processing tools

We introduce a novel complex-valued extension of the Taylor series, we establish relationship between the moments, characteristic functions and cumulants, and finally we propose a novel measure of circularity. For the recent work in the area of complex valued random signals, see, e.g., [1, 2, 3] and references therein.

The rest of this paper is organized as follows. In Section 2 the preliminaries of the complex field are considered, and the linear maps are revisited. Linearity considerations lead to two types of differentiation in the complex domain. This is treated in Section 3, where it is also derived a generalization of the standard complex Taylor's series along with important specific cases. In Section 4 complex random variables are introduced, and the existence properties of the fundamental expectation operator are derived. It is shown in Section 5 how the results from the previous sections lead to derivation and characterization of complex random variable quantities such as moments, cumulants, and circularity. Finally, Section 6 concludes the paper. Due to the lack of space we have omitted some of the proofs, which are available from the authors by a request.

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2. COMPLEX FIELD AND FUNCTIONS

Let \mathbb{R} denote the set of real numbers. The set of complex numbers, denoted by \mathbb{C} , is the plane $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ equipped with *complex addition* and *complex multiplication* making it the *complex field*. The *complex conjugate* of $z = (x, y) = x + jy \in \mathbb{C}$ is defined as $z^* \triangleq (x, -y) = x - jy$. With this notation we can write the *real part* and the *imaginary part* of a complex number z as $\text{Re}(z) \triangleq x = \frac{1}{2}(z + z^*)$ and $\text{Im}(z) \triangleq y = \frac{j}{2}(z^* - z)$, respectively. The *modulus* of $z = x + jy$ is defined as the nonnegative real number $|z| \triangleq \sqrt{x^2 + y^2} = \sqrt{zz^*}$.

The *complex exponential*, denoted by $\exp(z)$, is defined as the complex number $\exp(z) \triangleq \exp(x)\{\cos(y) + j\sin(y)\}$, where $\exp(x)$ for real-valued x denotes the usual exponential function. Any nonzero complex number has a polar representation, $z = |z|\exp(j\theta)$, where $\theta = \arg(z)$ is called the *argument* of z . The unique argument $\theta = \text{Arg}(z)$ on the interval $-\pi \leq \theta < \pi$ is called the *principal argument*. The *complex logarithm* of $z \neq 0$, denoted by $\log(z)$, is defined as the complex number $\log(z) \triangleq \log|z| + j\text{Arg}(z) + j2n\pi$ where n is an arbitrary integer. The particular value of the logarithm given by $\log|z| + j\text{Arg}(z)$ is called the *principal logarithm* and will be denoted by $\text{Log}(z)$.

The *open disk* with center $c = a + jb \in \mathbb{C}$ and radius $r > 0$ is defined as $B(c, r) \triangleq \{z \in \mathbb{C} : |z - c| < r\}$. Throughout the paper \mathcal{U} will stand for an *open set* in \mathbb{C} , i.e. for each $c \in \mathcal{U}$ there exists $r > 0$ such that $B(c, r) \subset \mathcal{U}$. A *function* f of the complex variable $z = x + jy$ is a rule that assigns to each value z in \mathcal{U} one and only one complex number $w = u + jv \triangleq f(z)$. The real and imaginary part of the function $f(z)$ are real-valued functions of real variables x and y , i.e., $u = u(x, y) \triangleq u(z) : \mathcal{U} \rightarrow \mathbb{R}$ and $v = v(x, y) \triangleq v(z) : \mathcal{U} \rightarrow \mathbb{R}$.

Because \mathbb{C} is an \mathbb{R} -vector space as well as a \mathbb{C} -vector space, there are two kinds of linear functions. A function $L : \mathbb{C} \rightarrow \mathbb{C}$ is called \mathbb{R} -linear, if $L(az_1 + bz_2) = aL(z_1) + bL(z_2)$ for all $z_1, z_2 \in \mathbb{C}$, and scalars $a, b \in \mathbb{R}$. If the defining equation is valid also for all complex scalars $a, b \in \mathbb{C}$, then L is \mathbb{C} -linear. For example, the complex conjugation $z \mapsto z^*$ is \mathbb{R} -linear but not \mathbb{C} -linear.

It is easily seen that a function L is \mathbb{R} -linear if and only if $L(z) = \alpha z + \beta z^*$, $\alpha, \beta \in \mathbb{C}$, and a function T is \mathbb{C} -linear if and only if $T(z) = \alpha z$. This result can be generalized to multivariate mappings $\mathbb{C}^n \rightarrow \mathbb{C}^p$, and the form of \mathbb{R} -linear mappings remains the same: it is the sum of two \mathbb{C} -linear mappings, the first one acting on the argument of the function and the latter on the conjugate of the argument. Therefore, the well-known estimation technique in signal processing known as *widely linear estimation* [4] is in fact estimation with respect to \mathbb{R} -linear estimators instead of the \mathbb{C} -linear ones used in standard linear estimation of complex data. Hence, we prefer the name \mathbb{R} -linear estimation for the technique.

3. $\mathbb{C}\mathbb{R}$ -CALCULUS AND TAYLOR'S THEOREM

The essential idea of differentiation is *linearization* of a function around a point. Since there two types of linear mappings in \mathbb{C} , there are also two types of differentiations. The less restrictive is obtained by linearization with respect to \mathbb{R} -linear functions. A function $f : \mathcal{U} \rightarrow \mathbb{C}$ is said to be \mathbb{R} -differentiable at $c \in \mathcal{U}$, if there exists an \mathbb{R} -linear function $L : \mathbb{C} \rightarrow \mathbb{C}$, called \mathbb{R} -differential, such that

$$f(c+h) - f(c) = L(h) + |h|\varepsilon(h) \text{ and } \lim_{|h| \rightarrow 0} \varepsilon(h) = 0.$$

The function f is \mathbb{R} -differentiable, if it is \mathbb{R} -differentiable at every point $c \in \mathcal{U}$. Now the differential L is \mathbb{C} -linear if and only if f is \mathbb{C} -differentiable, i.e., f has the usual *complex derivative* (\mathbb{C} -derivative) defined as the differential quotient limit.

Functions that are \mathbb{C} -differentiable in a domain are called *holomorphic*, and they are the subject of the standard complex analysis. However, \mathbb{C} -differentiability is too stringent condition for the application domain targeted in this paper. Namely, many functions related to random variables and optimization are real-valued, and only real-valued holomorphic functions are constants. Hence, we are mainly interested in developing a useful framework around \mathbb{R} -differentiation.

Now the existence of the (unique) differential $L(h)$ implies that the *complex partial derivatives* (cpd) $\frac{\partial f}{\partial x}(c) \triangleq \frac{\partial u}{\partial x}(c) + j\frac{\partial v}{\partial x}(c)$ and $\frac{\partial f}{\partial y}(c) \triangleq \frac{\partial u}{\partial y}(c) + j\frac{\partial v}{\partial y}(c)$ exist at c . Furthermore,

$$\begin{aligned} L(h) &= \frac{\partial f}{\partial x}(c)\text{Re}(h) + \frac{\partial f}{\partial y}(c)\text{Im}(h) \\ &= \frac{\partial f}{\partial z}(c)h + \frac{\partial f}{\partial z^*}(c)h^*, \end{aligned} \quad (1)$$

where the mixed cpd's

$$\frac{\partial f}{\partial z}(c) \triangleq \frac{1}{2} \left(\frac{\partial f}{\partial x}(c) - j\frac{\partial f}{\partial y}(c) \right)$$

and

$$\frac{\partial f}{\partial z^*}(c) \triangleq \frac{1}{2} \left(\frac{\partial f}{\partial x}(c) + j\frac{\partial f}{\partial y}(c) \right)$$

are called [5] the \mathbb{R} -derivative and the *conjugate \mathbb{R} -derivative*, respectively. The differential calculus based on these operators is known as *Wirtinger calculus* [6, 1], or, as we prefer, the $\mathbb{C}\mathbb{R}$ -calculus [5]. The variables z and z^* are treated as though they were independent variables, and the usefulness of these operators stem from the fact that they follow formally the same sum, product, and quotient rules as the ordinary partial differentiation. The *chain rules* read as follows:

$$\begin{aligned} \frac{\partial f \circ g}{\partial z}(c) &= \frac{\partial f}{\partial z}(g(c)) \cdot \frac{\partial g}{\partial z}(c) + \frac{\partial f}{\partial z^*}(g(c)) \cdot \frac{\partial g^*}{\partial z}(c), \\ \frac{\partial f \circ g}{\partial z^*}(c) &= \frac{\partial f}{\partial z}(g(c)) \cdot \frac{\partial g}{\partial z^*}(c) + \frac{\partial f}{\partial z^*}(g(c)) \cdot \frac{\partial g^*}{\partial z^*}(c). \end{aligned}$$

The $\mathbb{C}\mathbb{R}$ -calculus extends easily to functions with several variables by the multivariate extension of the differential property (1). The literature about this multivariate extension is very scarce, however, see [5, 7, 8].

As with the usual partial derivatives, it is possible to have higher order \mathbb{R} -derivatives. Functions with continuous \mathbb{R} -derivatives of order m in \mathcal{U} are denoted by $C^m(\mathcal{U})$. We are ready to state the main result in this section.

Theorem 1 (Taylor's \mathbb{R} -theorem) *Assume that $f \in C^{m+1}(\mathcal{U})$. Then*

$$\begin{aligned} f(c+h) - f(c) &= \sum_{p=1}^m \sum_{n=0}^p \frac{h^{*n} h^{p-n}}{n!(p-n)!} \frac{\partial^p f}{\partial z^{p-n} \partial z^{*n}}(c) + |h|^m \varepsilon(h) \end{aligned} \quad (2)$$

and $\lim_{|h| \rightarrow 0} \varepsilon(h) = 0$.

An infinitely continuously differentiable function f is *real analytic*, if the power series in (2) converges.

An import special case occurs when $\frac{\partial f}{\partial z^*} \equiv 0$. This condition is easily seen to be equivalent to Cauchy-Riemann equations. Therefore, f is holomorphic, infinitely \mathbb{C} -differentiable, and the power series (2) converges, i.e., f is *complex analytic*. Taylor's series takes now the usual form from the complex analysis: $f(c+h) - f(c) = \sum_{p=1}^{\infty} \frac{h^p}{p!} \frac{\partial^p f}{\partial z^p}(c)$.

Another special case occurs when f is \mathbb{R} -differentiable at $c = 0$, and further independent of $\text{Arg}(z)$. Now, since f can be written as a function of $|z|^2$ alone, it follows from the chain rules that $\frac{\partial^{p+q} f}{\partial z^p \partial z^{*q}}(0) = 0$ whenever $p \neq q$. Hence, if $f \in C^{2m+1}(\mathcal{U})$, then Eq. (2) reduces to

$$f(h) - f(0) = \sum_{p=1}^m \frac{|h|^{2p}}{(p!)^2 4^p} \Delta^p f(0) + |h|^{2m} \varepsilon(h), \quad (3)$$

where the differential polynomial $\Delta \triangleq 4 \frac{\partial}{\partial z^*} \frac{\partial f}{\partial z} = 4 \frac{\partial}{\partial z} \frac{\partial f}{\partial z^*} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is known [6] as the *Laplace operator* playing a crucial role in analysis of harmonic (or potential) functions.

4. COMPLEX RANDOM VARIABLES AND THE EXPECTATION

A *complex random variable* (r.v.) is defined by $Z \triangleq X + jY$, where X and Y are real r.v.'s. The distribution of a complex r.v. Z is identified with the joint (real bivariate) distribution of X and Y ,

$$F_Z(z) = P(Z \leq z) \triangleq P(X \leq x, Y \leq y) = F_{(X,Y)}(x, y),$$

where the functions $F_Z(z)$ and $F_{(X,Y)}(x, y)$ are the *cumulative distribution function* (cdf) of the complex r.v. Z and the joint distribution function of the bivariate random vector (X, Y) , respectively. The *probability density function* (pdf), if it exists, is the nonnegative function $f_Z(z) \triangleq f_{(X,Y)}(x, y)$ such that $F_{(X,Y)}(x, y) = \int_{-\infty}^y \int_{-\infty}^x f_{(X,Y)}(x, y) dx dy$ for all $(x, y) \in \mathbb{R}^2$. Then $2j \left(\frac{\partial^2 F_Z}{\partial z^2}(z) - \frac{\partial^2 F_Z}{\partial z^* 2}(z) \right) = \frac{\partial^2 F_{(X,Y)}}{\partial x \partial y}(x, y)$ exists and equals $f_Z(z)$ almost everywhere.

The *mean* (or the *expectation*) of Z , defined as $E_Z[Z] \triangleq E_X[X] + jE_Y[Y]$ is said to exist if both real expectations $E_X[X]$ and $E_Y[Y]$ exist, i.e., if the corresponding integrals are defined and finite. It follows from the definition of integration that integrability and absolute integrability are equivalent for a real r.v. X , i.e., $E_X[X]$ exists if and only if $E_X[|X|]$ exist. Thus, the existence of $E_Z[Z]$ implies the existence of $E_X[|X|]$ and $E_Y[|Y|]$. For a real r.v. X it is well-known that $|E_X[X]| \leq E_X[|X|]$. The same holds for complex r.v.'s. Explicitly, $E_Z[Z]$ exists if and only if $E_Z[|Z|]$ exists, and if $E_Z[f(Z)]$ exists for a Borel measurable function $f : \mathbb{C} \rightarrow \mathbb{C}$, then $|E_Z[f(Z)]| \leq E_Z[|f(Z)|]$.

For the *characteristic function* (cf) we may write

$$\begin{aligned}\Phi_Z(z) &\triangleq \Phi_{(X,Y)}(x,y) = E_{(X,Y)}[\exp\{j(xX + yY)\}] \\ &= E_Z[\exp\{j\text{Re}(z^*Z)\}] = E_Z[\exp\{\frac{j}{2}(z^*Z + zZ^*)\}],\end{aligned}$$

and the *second characteristic function* (scf) is defined in the neighborhood of zero by $\Psi_Z(z) \triangleq \text{Log}[\Phi_Z(z)]$.

5. MOMENTS, CUMULANTS, AND CIRCULARITY

A complex r.v. Z has $p+1$ p :th-order moments defined by

$$\alpha_{n;m} \triangleq E_Z[Z^n Z^{*m}],$$

where n and m are natural numbers such that $p = n+m$. Symmetric moments are redundant in the sense that $\alpha_{m;n} = \alpha_{n;m}^*$. If Z is real all p :th-order moments coincide. The q :th-order *absolute moment* is defined by $\beta_q \triangleq E_Z[|Z|^q]$ for all rational numbers q . Clearly $\beta_{2n} = \alpha_{n;n}$. In an analogous fashion, the quantity defined by

$$\underline{\alpha}_{n;m} \triangleq E_Z[\{Z - E_Z[Z]\}^n \{Z - E_Z[Z]\}^{*m}]$$

is known as the p :th-order *central moment* and $\underline{\beta}_q \triangleq E_Z[|Z - E_Z[Z]|^q]$ as the q :th-order *absolute central moment*. Again there are $p+1$ different central p :th-order moments. The 2nd-order central moment $\underline{\beta}_2$ is called the variance of Z and denoted by $\text{var}(Z)$ whereas the 2nd-order central moment $\underline{\alpha}_{2;0}$ is the *pseudo-variance* [9] of Z and it is denoted by $\text{pvar}(Z)$. It can be shown that if a p :th-order moment exists, then all moments of order $p' \leq p$ exist. Moreover, we have $|\alpha_{n;m}| \leq \beta_{n+m}$. Now there is an important relationship between moments and cf's [10] given in the following theorem.

Theorem 2 *If a p :th order moment of a complex r.v. Z exists, then the cf Φ_Z is p times continuously differentiable in \mathbb{C} , and, moreover,*

$$\frac{\partial^{m+n}\Phi_Z}{\partial z^m \partial z^{*n}}(z) = \left(\frac{j}{2}\right)^{m+n} E_Z[Z^n Z^{*m} \exp(j\text{Re}(z^*Z))]$$

for all natural numbers m, n such that $m+n \leq p$. Especially,

$$\alpha_{n;m} = \left(\frac{2}{j}\right)^{m+n} \frac{\partial^{m+n}\Phi_Z}{\partial z^m \partial z^{*n}}(0). \quad (4)$$

Conversely, if the partial derivative $\frac{\partial^{m+n}\Phi_Z}{\partial z^m \partial z^{*n}}(0)$ of the order $p = m+n$ exists, then all moments of order $\leq p$, if p is even, and all moments of order $< p$, if p is odd, exist.

This gives together with the Taylor's \mathbb{R} -series (2) at zero an expansion for the characteristic function.

Corollary 1 *If a complex r.v. Z has a finite p :th-order moment, then, as $z \rightarrow 0$,*

$$\Phi_Z(z) = 1 + \sum_{m=1}^p \left(\frac{j}{2}\right)^m \sum_{n=0}^m \frac{z^{*n} z^{m-n}}{n!(m-n)!} \alpha_{n;m-n} + o(|z|^p).$$

As in the real r.v. case, besides moments it is useful to define a closely related concept, namely *cumulants*. The complex cumulant of order $p = n+m$, denoted by $\kappa_{n;m}$, is obtained [10] as the moments in Eq. (4), but using the scf instead of the cf. Namely,

$$\kappa_{n;m} \triangleq \left(\frac{2}{j}\right)^{m+n} \frac{\partial^{m+n}\Psi_Z}{\partial z^m \partial z^{*n}}(0).$$

Again there are $p+1$ cumulants $\kappa_{n;m}$ of order p , and symmetric cumulants are redundant in the sense that $\kappa_{n;m} = \kappa_{m;n}^*$. Since the existence relations follow from the differentiation, they are the same as in the case of moments. The symmetric cumulant $\kappa_{n;n}$ is called the *absolute* (or *central*) *cumulant of order $2n$* .

The usefulness of cumulants stems from the fact that the scf is additive for independent r.v.'s. That is, if r.v.'s Z_1 and Z_2 are independent, then $\Psi_{Z_1+Z_2}(z) = \Psi_{Z_1}(z) + \Psi_{Z_2}(z)$. This property is preserved for the cumulants as the differentiation is a linear operation.

For the estimation purposes, it is important to know the algebraic relation between cumulants and moments. This relationship was introduced in [11] (see also [10]) in an ad-hoc manner by directly substituting Z and Z^* into a cumulant operator derived for the cumulants of real random vectors. However, the formula holds also for complex r.v.'s, and it can be rigorously proved by extending the underlying Faà di Bruno's formula [12] to complex \mathbb{R} -derivatives. This also gives the explicit formulas for the relationship between multivariate complex cumulants and moments. However, we do not pursue this issue here.

A complex r.v. Z is *circular* [13] if for any real α , the r.v.'s $Z = X + jY$ and $\exp(j\alpha)Z$ have the same distribution. In other words, Z is circular if the joint distribution of X and Y is *spherically symmetric*. It follows directly from the results in [14] that if a r.v. $Z = |Z| \exp(j\Theta)$ is circular, then a uniformly on $[0, 2\pi]$ distributed r.v. Θ and a positive r.v. $|Z|$ are independent. Furthermore, the cf of a circular r.v. depends only on $|z|^2$, and therefore, assuming the moments exist, we can apply the specific case (3) of Taylor's \mathbb{R} -theorem to the (s)cf of the r.v. Z . It follows that all offcentric (i.e., nonabsolute) moments (and cumulants) are identically zero.

Since much of the classical signal processing and communications work in the complex domain has assumed circular r.v.'s, absolute moments (and cumulants) have dominated the literature until recently. Recent literature, e.g. [3, 2, 1, 15], has focused on a weaker assumption than circularity. Namely, instead of presuming that r.v.'s are fully circular, one only assumes that the second order offcentric moments do not vanish, i.e., r.v.'s are assumed to *improper* [9]. However, since there is no a priori reason to assume that the higher order offcentric moments vanish if the signal is known to be noncircular, also higher order offcentric moments need to be considered for the optimal performance. For testing circularity, a generalized likelihood ratio was considered in [16]. Another circularity measure is obtained by calculating the difference between the cf and its circular approximation (3):

$$\delta_Z(z) \triangleq \left| \Phi_Z(z) - 1 - \sum_{n=1}^p \left(-\frac{1}{4}\right)^n \frac{1}{n!2} |z|^{2n} \beta_n \right|,$$

where p is the approximation order and $\delta_Z(z)$ is evaluated on appropriate points $z \in \mathbb{C}$. We will consider this in a future work.

6. CONCLUSIONS

We provided a rigorous and unified treatment of properties of complex-valued random signals and related processing tools. A novel complex-valued extension of the Taylor series was introduced. We establish relationship between the moments, characteristic functions and cumulants, and finally proposed a novel measure of circularity.

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A. PROOFS

Proof of Theorem 1. Denote $h_x = \text{Re}(h)$ and $h_y = \text{Im}(h)$. Since $f = u + jv \in C^{m+1}(\mathcal{U})$, partial derivatives of u and v up to order

m are continuous in \mathcal{U} by definition, which also implies that u and v possess m -th-order Taylor series expansion

$$\begin{aligned} u(c+h) - u(c) &= \sum_{p=1}^m \frac{1}{p!} \sum_{n=0}^p \binom{p}{n} h_x^{p-n} h_y^n \frac{\partial^p u}{\partial x^{p-n} \partial y^n}(c) \\ &\quad + |h|^m \varepsilon_u(h), \\ v(c+h) - v(c) &= \sum_{p=1}^m \frac{1}{p!} \sum_{n=0}^p \binom{p}{n} h_x^{p-n} h_y^n \frac{\partial^p v}{\partial x^{p-n} \partial y^n}(c) \\ &\quad + |h|^m \varepsilon_v(h), \end{aligned}$$

where $\varepsilon_u(h) \rightarrow 0$ and $\varepsilon_v(h) \rightarrow 0$ as $|h| = \sqrt{h_x^2 + h_y^2} \rightarrow 0$. Therefore we now have that

$$\begin{aligned} f(c+h) - f(c) &= [u(c+h) - u(c)] + j[v(c+h) - v(c)] \\ &= \sum_{p=1}^m \frac{1}{p!} \sum_{n=0}^p \binom{p}{n} h_x^{p-n} h_y^n \frac{\partial^p (u + jv)}{\partial x^{p-n} \partial y^n}(c) \\ &\quad + |h|^m [\varepsilon_u(h) + j\varepsilon_v(h)] \\ &= \sum_{p=1}^m \frac{1}{p!} \sum_{n=0}^p \binom{p}{n} h^{p-n} h^{*n} \frac{\partial^p f}{\partial z^{p-n} \partial z^{*n}}(c) + |h|^m \varepsilon(h), \end{aligned}$$

where $\varepsilon \triangleq \varepsilon_u + j\varepsilon_v \rightarrow 0$ as $|h| \rightarrow 0$. The last identity follows by a straightforward application of the binomial theorem to the operators $(h_x \frac{\partial}{\partial x} + h_y \frac{\partial}{\partial y})^p$ and $(h \frac{\partial}{\partial z} + h^* \frac{\partial}{\partial z^*})^p$. Also observe that $\binom{p}{n}/p! = 1/[n!(p-n)!]$.

Proof of Theorem 2. For the first part, notice that all absolute moments of orders $p' \leq p$ exist. Let $h = h_x + jh_y$. Now

$$\frac{\Phi_Z(c + h_x) - \Phi_Z(c)}{h_x} = \mathbb{E}_Z \left[\frac{\exp(jh_x X) - 1}{h_x} \exp(j\text{Re}(c^* Z)) \right]$$

and

$$\frac{\Phi_Z(c + jh_y) - \Phi_Z(c)}{h_y} = \mathbb{E}_Z \left[\frac{\exp(jh_y Y) - 1}{h_y} \exp(j\text{Re}(c^* Z)) \right].$$

Since for all $c \in \mathbb{C}$, $h_x \in \mathbb{R}$

$$\left| \frac{\exp(jh_x x) - 1}{h_x} \exp(j\text{Re}(c^* Z)) \right| \leq |x| \leq |z|,$$

following from the result that $|\exp(jt) - 1| \leq |t|$ for all $t \in \mathbb{R}$, and similarly for $h_y \in \mathbb{R}$, it follows from Lebesgue dominated convergence that

$$\frac{\partial \Phi_Z}{\partial x}(c) = j \mathbb{E}_Z [X \exp(j\text{Re}(c^* Z))]$$

and

$$\frac{\partial \Phi_Z}{\partial y}(c) = j \mathbb{E}_Z [Y \exp(j\text{Re}(c^* Z))].$$

Hence, by the definition of the complex partial derivatives

$$\frac{\partial \Phi_Z}{\partial z}(c) = \left(\frac{j}{2} \right) \mathbb{E}_Z [Z^* \exp(j\text{Re}(c^* Z))]$$

and

$$\frac{\partial \Phi_Z}{\partial z^*}(c) = \left(\frac{j}{2} \right) \mathbb{E}_Z [Z \exp(j\text{Re}(c^* Z))].$$

The general case follows by induction, and the claim about the moments with the substitution $c = 0$.

For the second part of the theorem, the existence of the p -th-order cpd implies the existence of the real partial derivatives of the order p . Hence, the real moments $\alpha_k(X)$ and $\alpha_k(Y)$ exist for $k \leq p$ if p is even and for $k < p$ if p is odd. This implies the existence of β_p for p even and β_{p-1} for p odd, and therefore also all other moments as claimed.