FRAME BOUNDS ESTIMATION OF FREQUENCY WARPING OPERATORS

Salvatore Caporale, Luca De Marchi and Nicolò Speciale

ARCES/DEIS University of Bologna
Viale Risorgimento 2, 40136, Bologna, Italy
scaporale@arces.unibo.it

ABSTRACT

In this work we provide an accurate analytical estimation of the frame bounds for frequency warping operators of arbitrary shaped non-smooth warping maps. We deal with both the Nonuniform Fourier Transform approximation and the aliasing suppressed form of the frequency warping operator. The estimation procedure is obtained by using an analytical model of the aliasing operators which has been previously introduced. The provided estimations can be used as design formulas for the parameters, such as degree of smoothness and redundancy, involved in the definition of the frequency warping operators.

Index Terms— Frequency Warping, Frame Bounds.

1. INTRODUCTION

Warping was originally introduced in signal processing as a new paradigm allowing the generalization of time-frequency transformations techniques suffering of some restrictions in the way the time-frequency plane can be tiles. The basic idea is to adaptively apply a preliminary invertible transformation to the input signal so that it can match the characteristics of the following signal processing [1]. This adapted transformation consists of a deformation of either of the frequency or of the time axis. Here we focus on the first kind.

Frequency warping has been commonly addressed by a Laguerre transform approach [2]. Although giving an interpretation key in the framework of digital filters, this approach has many drawbacks, such as high computational complexity and restrictions in the choice of the warping map.

So, frequency warping has been conveniently placed in the framework of finite length transforms [3], allowing more degrees of freedom in the design of the warping map, that is in the way the transformation can be adapted to the input signal features. Some computational issues have been treated and solved [4, 5] about frequency warping operators of arbitrary non-smooth maps. Thanks to the invertible property and the intrinsic redundancy, frequency warping can also be seen as a *frame* operator. Here we cope with problem of estimating the frame bounds in case of non-smooth maps.

2. FREQUENCY WARPING OPERATORS

Given a discrete-time signal, we want to introduce a deformation of the periodic frequency axis f with a proper warping function $w(f)$. In order to guarantee invertibility, $w(f)$ has to be chosen so that it maps f axis on itself. The warping function $w(f)$ is defined in the interval $[-1/2, 1/2]$ (or equivalently in $[0, 1]$) and extended as $w(f + k) = k + w(f)$, with $k \in \mathbb{Z}$. Moreover, it must be an odd function in order to guarantee that a real signal is transformed into a real signal. The frequency warping operator can be written as the composition of an inverse discrete Fourier transform $\mathbb{F}$ and a modified discrete Fourier transform $\mathbb{F}_w$:

$$\mathbb{W} = \mathbb{F}^{-1}\mathbb{F}_w \tag{1}$$

where $\mathbb{F}_w$ is defined as follows:

$$[\mathbb{F}_w s](f) = \sqrt{\dot{w}(f)} \sum_{n \in \mathbb{Z}} s(n) e^{-j2\pi n w(f)}. \tag{2}$$

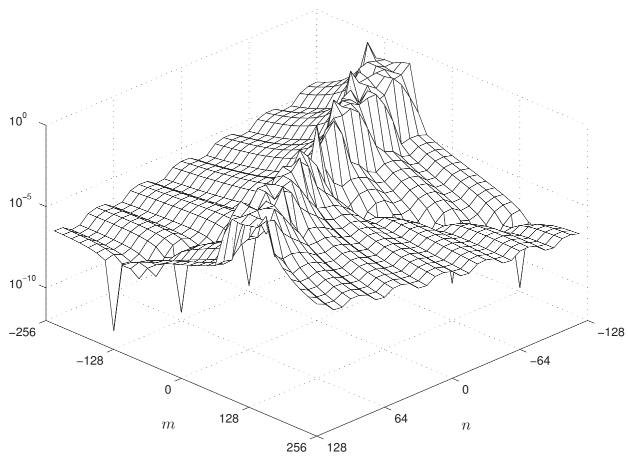
The term $\sqrt{\dot{w}(f)}$, where $\dot{w}$ represents the derivative, has been introduced in order to make the operator be unitary, i.e. preserve orthogonality. By doing so, the operator kernel is a matrix of infinite dimensions whose elements are given by:

$$\mathbb{W}(m, n) = \int_0^1 \sqrt{\dot{w}(f)}\, e^{j2\pi(mf - nw(f))} df \quad m, n \in \mathbb{Z}. \tag{3}$$

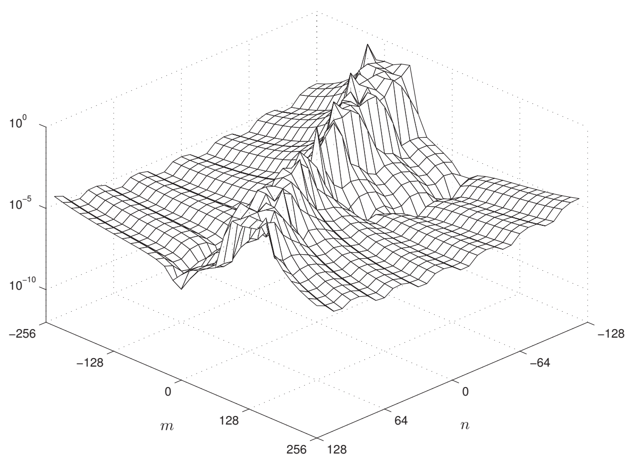
Then the input sequence is limited to N samples, while the output sequence length M has to respect the requirement $M > N \max \dot{w}$ in order to guarantee invertibility. Being represented by an $M \times N$ invertible transformation, frequency warping is a *frame*. As M tends to infinite, the matrix columns tends to be orthogonal and the operator tends to be a *tight* frame, i.e. it can be inverted by applying the transpose operator. So, the first kind of warping operator we consider is given by:

$$\mathbb{W}_{M,N} : s \mapsto \sum_{n \in \mathbb{Z}_N} \mathbb{W}(m, n) s(n) \quad m \in \mathbb{Z}_M. \tag{4}$$

$\mathbb{Z}_N$ and $\mathbb{Z}_M$ are given by $\mathbb{Z}_N = \{-N/2, \ldots, N/2 - 1\}$ and $\mathbb{Z}_M = \{-M/2, \ldots, M/2 - 1\}$. We will refer to this kind of



(a) Absolute value of the entries of a truncated warping matrix



(b) Absolute value of the entries of an aliasing-affected warping matrix

Fig. 1: Addressed warping operators are depicted, aliasing free (a) and aliasing affected (b). Although the two operators are almost equal, the presence of aliasing in (b) can cause a considerable difference in frame bounds

operator as *truncated* warping operator. An example of it is depicted in fig. 1(a).

An approximation of (4) is obtained by sampling the (1):

$$\mathcal{W}_{M,N} = \mathcal{F}_M^{-1}\mathcal{F}_{w,M,N} \tag{5}$$

where $\mathcal{F}_M$ is the $M \times M$ Discrete Fourier Transform and $\mathcal{F}_{w,M,N}$ is an orthogonalized Nonuniform Discrete Fourier Transform on M discrete frequencies $f_k = k/M$, $k \in \mathbb{Z}_M$:

$$[\mathcal{F}_{w,M,N}s](m) = \sqrt{\dot{w}(f_k)} \sum_{n\in\mathbb{Z}_N} s(n)e^{-j2\pi n w(f_k)}. \tag{6}$$

Operator (5) will be addressed as *frequency sampled* warping operator. An example is shown in fig. 1(b). It has been introduced in [3] since operator (4), according to the formulation (1), involves an integral so it is hardly computable. Its frequency sampled version (5) is computable by fast algorithms

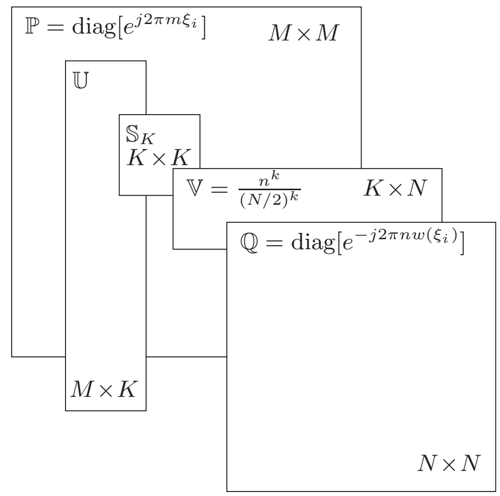


Fig. 2: Schematic structure of the computation of matrix $\mathbb{A}_{M,N}$. The resulting algorithm is fast since the total amount of multiplications is $\propto K(M+N)$ and $K \ll N$.

such as the Fast Fourier Transform (FFT) and the Nonuniform Fast Fourier Transform (NUFFT) [6] instead.

The relationship between $\mathbb{W}_{M,N}$ and $\mathcal{W}_{M,N}$ has been investigated in [4, 5]. The sampling operation in the Fourier domain introduces aliasing in the warped signal domain represented by the transformation $\mathbb{A}_{M,N}$:

$$\mathbb{W}_{M,N} = \mathcal{W}_{M,N} - \mathbb{A}_{M,N} \tag{7}$$

In case the warping map w is not a *smooth* function, that is $w \in \mathcal{C}^\sigma$, an analytical computational model for $\mathbb{A}_{\mathbb{M},\mathbb{N}}$ has been introduced. By supposing that w has singularities on $\xi_i \in [0,1]$ satisfying $M\xi_i \in \mathbb{N}$, we have:

$$\mathbb{A}_{M,N} = \sum_i [\mathbb{PUSVQ}](\xi_i) \tag{8}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are diagonal matrixes obtained from the vectors $e^{j2\pi m\xi_i}$ and $e^{-j2\pi n w(\xi_i)}$ respectively. $\mathbb{V}$ and $\mathbb{U}$ are a $K \times N$ and a $M \times K$ matrix respectively:

$$\mathbb{V}(k,n) = \frac{n^k}{(N/2)^k} \quad n \in \mathbb{Z}_N, k \in \mathbb{N} \tag{9}$$

$$\mathbb{U}(m,k) = \frac{(-1)^{k-1}}{2^k(k-1)!}\, D^k\zeta\left(m/M\right)\ m \in \mathbb{Z}_M, k \in \mathbb{N} \tag{10}$$

where D^k represents the k-th derivative and ζ is an analytically known function. Finally, $\mathbb{S}$ is a $K \times K$ lower triangular matrix, whose entries depend on the warping map w and on N and M only. K is an arbitrary value which is shown to be small in comparison to N, as it appears from fig. 3. A schematic representation of the algorithm is depicted in fig. 2.

The decomposition (8) derives from a representation for the *tails* of matrix $\mathbb{W}$ for $m \notin \mathbb{Z}_M$. We refer to this matrix as

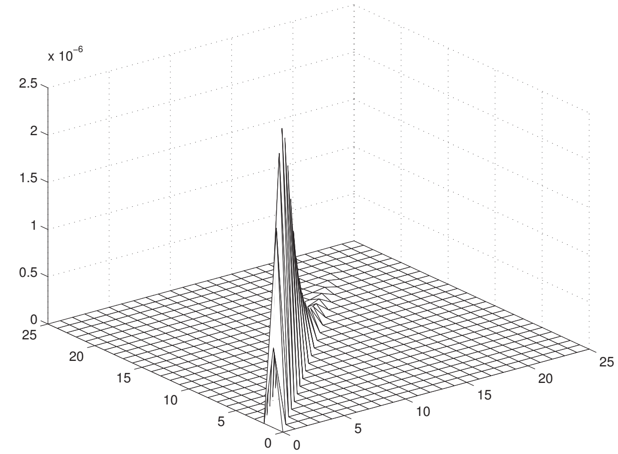


Fig. 3: Absolute value of matrix $\mathbb{S}$ entries for $\sigma = 1$, $N = 2^8$ and $M = 2N$. Most of the energy is concentrated along the σ-th lower diagonal.

error matrix, since it is the error introduced by truncation:

$$\mathbb{E}_{M,N} = \sum_i [\mathbb{PYSVQ}](\xi_i) \tag{11}$$

where $\mathbb{P}$ is now obtained from vector $e^{j2\pi m \xi_i}$ with $m \notin \mathbb{Z}_M$ and $\mathbb{Y}$ is given by:

$$\mathbb{Y}(m,k) = \frac{m^{-(i+1)}}{(M/2)^{-(i+1)}} \quad m \notin \mathbb{Z}_M, i \in \mathbb{N}. \tag{12}$$

3. FRAME BOUNDS ESTIMATION

For discrete operators, frame bounds are actually given by the minimum and the maximum eigenvalue of the product between the transposed operator and the operator itself. Frame bounds will be addresses, as usual in literature, by A and B, $B \geq A$. Since they have to be positive, the resulting matrix has to be positive definite. Moreover, they define many properties of considered transformation, in particular the quantity $|B/A - 1|$ is the maximum euclidian norm of the error vector which can be obtained when the output is scaled by a factor in $[B^{-1}, A^{-1}]$. In case of frequency warping, as M tends to infinite, A and B tends to 1, so the scaling factor is 1 and the maximum error norm results to be the maximum between $B - 1$ and $1 - A$. So, by estimating the error matrix norm, one gets an estimation of the frame bounds. Hence here we try to estimate:

$$\varepsilon = \|\mathbb{W}'_{M,N}\mathbb{W}_{M,N} - \mathbb{I}\| \tag{13}$$

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For estimating ε we point out that (13) is equivalent to $\|\mathbb{E}'_{M,N}\mathbb{E}_{M,N}\|$ and we take advantage of decomposition (11). With proper hypothesis when multiple singularities are considered, we reduce to the following:

$$\varepsilon = \|\mathbb{V}'\mathbb{S}^\dagger\mathbb{Y}'\mathbb{YSV}\|$$

then $\mathbb{S}$ is substituted by the σ-th lower diagonal, $\mathbb{Y}'\mathbb{Y}$, which is analytically computable and proportional to M, is substituted by the σ-th entry of the main diagonal and the effect of $\mathbb{V}$ is represented by a factor N. Finally we get:

$$\varepsilon \simeq \frac{\varrho(M/N)}{\pi^{2\sigma+2}(2\sigma+1)} \cdot \frac{N}{M^{2\sigma+1}} \cdot \Delta^2 \tag{15}$$

Δ represents the differential value between ξ^+ and ξ^- of:

$$Dw(\xi)^{-1/2} D^{\sigma+1} w(\xi)$$

while ϱ is:

$$\varrho = \max_x \left[\left(\frac{M}{NDw(\xi)} \right)^{-x+\sigma} \frac{2x - \sigma + 1}{2(\sigma+1)!} \prod_{l=0}^{\sigma-1} (x - l) \right]^2 \tag{16}$$

which corresponds to the square maximum of the σ-th lower diagonal of $\mathbb{S}$ (see fig. 3). A quasi–exact analytical solution has been provided for ϱ. The expression (15) has to be slightly modified in the case $\sigma = 0$.

In order to estimate ϵ we take advantage of equation (7):

$$\epsilon \simeq 2\|\mathbb{W}^\dagger_{M,N}\mathbb{A}_{M,N}\| = 2\|\mathbb{A}^\dagger_{M,N}\mathbb{W}_{M,N}\mathbb{W}^\dagger_{M,N}\mathbb{A}_{M,N}\|^{1/2}.$$

For $\sigma > 0$ the product $\mathbb{W}_{M,N}\mathbb{W}^\dagger_{M,N}$ can be safely substituted by its main diagonal, whose shape can be analytically derived starting from the warping map. Then the model (8) is substituted and it turns out:

$$\epsilon \simeq 2\|\mathbb{V}'\mathbb{S}^\dagger\mathbb{U}'\mathbb{DUSV}\|^{1/2} \tag{17}$$

where $\mathrm{diag}(\mathbb{D}) = \mathrm{diag}(\mathbb{W}_{M,N}\mathbb{W}^\dagger_{M,N})$. Then only an estimation for $\mathrm{diag}(\mathbb{U}'\mathbb{DU})$ is needed to trace the estimation of ϵ back to the form (15). Matrix $\mathbb{D}$ has only $N \max Dw$ significant values, so that it actually selects the central values of U columns, that is $\mathbb{U}(m,k)$ with $m \in \mathbb{Z}_{\lceil N \max Dw \rceil}$. The σ-th entry of $\mathrm{diag}(\mathbb{U}'\mathbb{DU})$ results to be the energy of $\mathbb{U}(m,\sigma)$ with $m \in \mathbb{Z}_{\lceil N \max Dw \rceil}$, which behaves like a constant in case σ is odd and like m in case σ is even. So energies are proportional to $M^{2 \mathrm{mod}_2(\sigma+1)}$:

$$\epsilon \simeq \frac{1}{2} \left(\frac{\varrho(M/N)}{\pi^{2\sigma+2}} \cdot \frac{\kappa N}{M^{2(\sigma+1+\mathrm{mod}_2(\sigma+1))}} \cdot \Delta^2 \right)^{1/2}. \tag{18}$$

In case σ is odd κ is exactly proportional to N according to values of $D^k\zeta(0)$ (see eq. (10)). In case σ is even κ is roughly proportional to N^3 but has to be numerically computed.

The estimations (15) and (18) have been obtained by imposing the convergence to the exact values for M tending to ∞. So, apart from a possible lack in accuracy for M close to $N \max Dw$, these estimations describe the analytical dependency of the frame bounds on the design variables N, M and σ. Moreover, they allow to evaluate the advantage obtained by using the aliasing free frequency warping operator rather than the frequency sampled one. Finally, the solution of (16) allows to estimate the required K to make the computation of the aliasing matrix $\mathbb{A}_{M,N}$ converge.

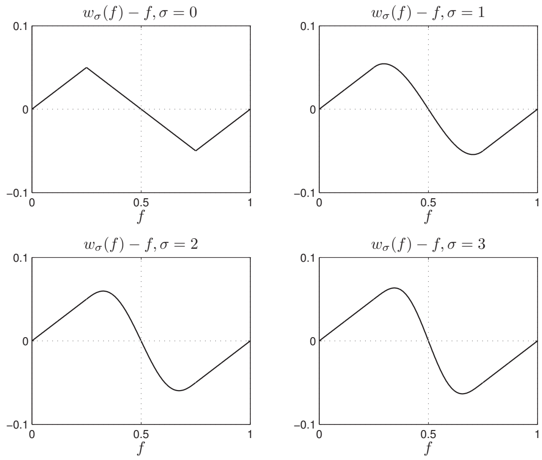


Fig. 4: Frequency deviation, i.e. $w(f) - f$, of the warping maps used in the for evaluating the performances of the frame bounds estimation.

4. EXPERIMENTAL RESULTS

To evaluate performances, we consider frequency maps having a single singularities in $\xi = 1/4$ (a singularity in $\xi = -1/4$ is also present) with different degrees of smoothness:

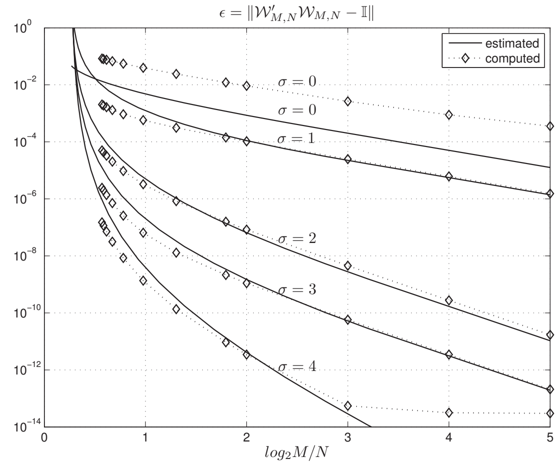
$$w_\sigma(f) = \begin{cases} \frac{6}{5}f & f \in [0, 1/4] \\ \frac{1}{2} + \sum_{i=0}^{\sigma} a_i (f - \frac{1}{2})^{2i+1} & f \in [1/4, 1/2] \end{cases}.$$

Coefficients a_i are obtained by imposing the first σ derivatives to be null on ξ. The frequency deviations of these maps are depicted in fig. 4. N has been fixed to 2^8 and the redundancy M/N varies from its lower allowed value $\max Dw = 6/5$, which is the same for all the considered maps, to the very large value 2^5, to be able to check the asymptotic behavior.

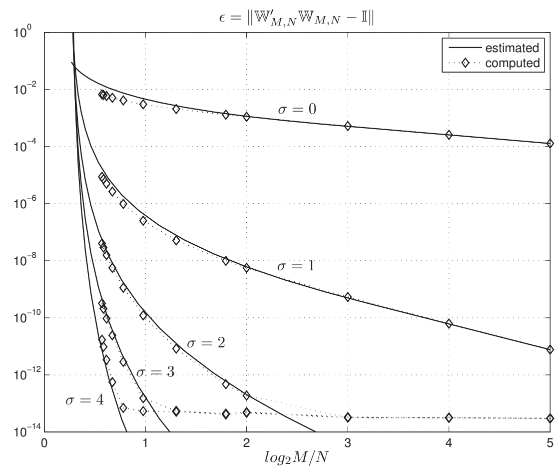
Estimation results are shown in fig. 5 for $\mathcal{W}_{M,N}$ and $\mathbb{W}_{M,N}$ respectively. The second case is shown to be very accurate while the first case is a bit inaccurate for small M but still converges for large M. As we predicted when the approximation (17) has been done, the model completely fails for $\mathcal{W}_{M,N}$ when $\sigma = 0$. The considered maps represent a bad case in the sense that the slope of w on the singularity is equal to the maximum slope. Having a smaller slope on the singularity improves the estimation accuracy.

5. CONCLUSIONS

We dealt with the problem of frame bounds estimation for frequency warping operators of non-smooth warping maps. We gave estimations formulas for both the aliasing affected and the aliasing free form of the frequency warping operator. The estimations are proven to be effective and can be used for designing warping operators satisfying specific requirements in reconstruction accuracy.



(a)



(b)

Fig. 5: Error norm estimations (solid line) and computed error norms (diamonds) of the frequency sampled (a) and truncated warping operator (b) for $\sigma = 0, \ldots, 4$. Computed norms saturates to a lower computational limit.

6. REFERENCES

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