A PDE CHARACTERIZATION OF THE INTRINSIC MODE FUNCTIONS

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ABSTRACT

For the first time, a proof of the sifting process (SP) and so the empirical mode decomposition (EMD), is given. For doing this, lower and upper envelopes are modeled in a more convenient way that helps us prove the convergence of the SP towards a solution of a partial differential equation (PDE). We also prove that such a PDE has a unique solution, which ensures the uniqueness of the EMD decomposition. The new formulation of envelopes has another benefit. In fact, it removes interpolation problems and related issues. Not only helps the modelization of envelopes to give a mathematical framework on the EMD, but also, as confirmed by the numerical simulations, the PDE-based EMD improves a lot the classical EMD.

***Index Terms*—Eigenfunctions, eigenvalues, partial differential equations, spectral analysis.**

I. INTRODUCTION

The Empirical mode decomposition (EMD) was introduced by Huang et al. [1] for analyzing linear and non-stationary time series. Any given signal is then decomposed by the EMD into a sum of intrinsic mode functions (IMFs), which are generated at each scale going from fine to coarse, by an iterative procedure, the sifting process (SP). Many numerical simulations have been done for a better understanding of the EMD behaviors [2], [3]. The crucial part in the EMD is the SP. It is during this step that the oscillatory functions of the signal, the so-called IMFs, are extracted.

A lack of a theoretical background is one of the weaknesses of the EMD algorithm. What makes the SP hard to study is the definition of the so-called *local mean*. Indeed, it involves the upper and lower envelopes of the signal, which are not easy to handle for calculus whatever the interpolations one could use. One important thing to keep in mind is that interpolation always creates additional information that has nothing to deal with the original data.

The main scope of this work is to provide a theoretical framework on the EMD for a better understanding of the method, and then, to definitely get rid of its main criticism. This will be essentially done by slightly changing the definition of the *local mean* by modeling lower and upper envelopes in more suitable and explicit forms.

In the next section, we make a brief survey on mathematical works on EMD. EMD principles are recalled in section 3. The convergence of the SP and the PDE characterization of IMFs are done in section 4. Numerical results are shown in section 5.

II. EMD RELATED WORKS

Many studies have recently been done on the comprehension of the EMD. A quadratic programming is proposed in [4] as another way in computing the *local mean*. In [5], the SP is replaced by a fourth order PDE without any mathematical proof. This approach is validated by some numerical simulations. A consistent study on the IMFs is proposed in [6], [7]. However, a mathematical characterization of relaxed IMFs is given independently to the SP. We point out that in neither the works cited above, the information given by SP is taken into account.

III. EMD PRINCIPLES

Following Huang et al. [1], for a given signal denoted by $S(x)$, the EMD algorithm could be summarized as follows:

1) Find all the extrema of $S(x)$.
2) Interpolate the maxima of $S(x)$ (resp. the minima of $S(x)$), denoted by $E_{max}(x)$ $(resp.\ E_{min}(x))$.
3) Compute the *local mean*:
$$m(x) = \frac{1}{2}(E_{max}(x) + E_{min}(x)). \quad (1)$$
4) Extract the detail $d(x) = S(x) - m(x)$.
5) Iterate on $m(x)$.

Thus, any signal $S(x)$ will be decomposed by the EMD in the following way: $S(x) = \sum_{k=1}^{N} f_k(x) + r(x)$, where f_k denotes the k^{th} IMF, and $r(x)$ is the residual. A good review of the EMD process and some implementations can be found in [1], [8]. Two conditions should be fulfilled to get an IMF: **(C1)**: its local mean must be equal to zero, and **(C2)**: its number of extrema and zero crossings must either be equal, or differ at most by one. **(C1)** modifies the global requirement to a local one, and is necessary to ensure that the instantaneous frequency (IF) will not include unwanted fluctuations induced by asymmetric waveforms. **(C2)** is similar to the traditional narrow-band requirement. The above conditions satisfy the physically necessary conditions to define a meaningful IF. The problem with **(C1)** is that it makes the SP dependent on the types of the interpolants (e.g.

splines) used to build the envelopes. The new formulation of the *local mean* will resolve the problem.

IV. A MATHEMATICAL CHARACTERIZATION OF IMFS

Let S be a continuous signal defined in Ω. We assume that Ω is an open bounded set of $\mathbb{R}$. The whole SP is fully determined by the sequence $(h_n)_{n\in\mathbb{N}}$ defined by:

$$\begin{cases} h_{n+1} = h_n - \frac{1}{2}(\hat{h_n} + \check{h_n}) \\ h_0 = S. \end{cases} \tag{2}$$

where $\hat{h_n}$ (resp. $\check{h_n}$) denotes the continuous interpolate of the maxima (resp. minima) of h_n. Let Φ be the operator such that: $\forall n \in \mathbb{N}$, $\Phi(h_n) = \frac{1}{2}(\hat{h_n} + \check{h_n})$. Let f^n be the application such that: $f^0 = I_d$, I_d is the identity operator; and $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n\ times}$. Then, $\forall n \in \mathbb{N}$, we have $h_n = (I_d - \Phi)^n h_0$. Let $\Psi = I_d - \Phi$, then $h_n = \Psi^n h_0$, $\forall n \in \mathbb{N}$. What we aim is to study the convergence of the sequence $(h_n)_{n\in\mathbb{N}}$. To do so, we introduce two operators that will act similarly like the upper and lower envelopes. Let $\delta > 0$. Let h be a real valued function defined $\forall x \in \Omega$ as:

$$S_\delta h(x) = \sup_{|y|<\delta} h(x+y),$$

δ chosen such that $x + y \in \Omega$.

Theorem 1: Let $h \in C^2(\Omega)$. For $\delta > 0$ small, we have the asymptotic development of S_δ:

$$S_\delta h(x) = h(x) + \sup_{|z|\leq 1} F_\delta(z) + o(\delta^2) \tag{3}$$

$\forall x \in \Omega$, where $F_\delta(z) = \delta z h'(x) + \dfrac{\delta^2}{2} z^2 h''(x)$. The values of $\sup_{|z|\leq 1} F_\delta(z)$ are given in case 1, case 2 and case 3 below.

Proof: Let $x \in \Omega$ and let $z \in \Omega$ such that $|z| < 1$ and $x + \delta z \in \Omega$. Because $h \in C^2(\Omega)$, we write:

$$h(x+\delta z) = h(x) + \delta z h'(x) + \frac{\delta^2}{2} z^2 h''(x) + o(\delta^2),$$

for a small δ. Let $\epsilon > 0$. Thus:

$$-\epsilon\delta^2 + h(x) + \sup_{|z|\leq 1} F_\delta(z) \leq S_\delta h(x) \leq \sup_{|z|\leq 1} F_\delta(z) + h(x) + \epsilon\delta^2.$$

x and ϵ are chosen arbitrary; thus, $\forall x \in \Omega$, we have:

$$S_\delta h(x) = h(x) + \sup_{|z|\leq 1} F_\delta(z) + o(\delta^2).$$

An extremum of F_δ is obtained $\Leftrightarrow z = z_0 = -\frac{1}{\delta}\dfrac{h'(x)}{h''}$.

$|z_0 \leq 1| \Leftrightarrow |h'(x)| \leq \delta|h''(x)|$. We have $F_\delta(z_0) = -\dfrac{1}{2}\dfrac{h'(x)}{h''(x)}$, $F_\delta(1) = \delta h'(x) + \dfrac{\delta^2}{2}h''(x)$ and $F_\delta(-1) = -\delta h'(x) + \dfrac{\delta^2}{2}h''(x)$. So:

- $F_\delta(1) > F_\delta(-1)$, if h is strictly increasing,
- $F_\delta(-1) > F_\delta(1)$, if h is strictly decreasing.

Taking into account these facts, we have three cases:

- **case 1:** h is strictly increasing such that:
  - $h' > \delta|h''|$, or
  - $h'' \neq 0,\ h' \leq \delta|h''|\ and\ 0 < \delta h' - \dfrac{1}{2}\dfrac{h'}{h''} - \dfrac{\delta^2}{2}h'' < 2\delta h'$; or
  - $h'' \neq 0,\ h' \leq \delta|h''|\ and\ -\delta h' + \dfrac{1}{2}\dfrac{h'}{h''} + \dfrac{\delta^2}{2}h'' > 0$.

  Then:
  $$\sup_{|z|\leq 1} F_\delta(z) = \delta h' + \frac{\delta^2}{2}h''. \tag{4}$$
- **case 2:** h is strictly decreasing such that:
  - $|h'| > \delta|h''|$, or
  - $h'' \neq 0,\ |h'| \leq \delta|h''|\ and\ 2\delta h' < \delta h' - \dfrac{1}{2}\dfrac{h'}{h''} - \dfrac{\delta^2}{2}h'' < 0$; or
  - $h'' \neq 0,\ |h'| \leq \delta|h''|\ and\ \delta h' + \dfrac{1}{2}\dfrac{h'}{h''} + \dfrac{\delta^2}{2}h'' > 0$.

  Then:
  $$\sup_{|z|\leq 1} F_\delta(z) = -\delta h' + \frac{\delta^2}{2}h''. \tag{5}$$
- **case 3:** $h'' \neq 0,\ \delta h' + \dfrac{1}{2}\dfrac{h'}{h''} + \dfrac{\delta^2}{2}h'' < 0\ and\ -\delta h' + \dfrac{1}{2}\dfrac{h'}{h''} + \dfrac{\delta^2}{2}h'' < 0$. Then:
  $$\sup_{|z|\leq 1} F_\delta(z) = -\frac{1}{2}\frac{h'(x)}{h''(x)}. \tag{6}$$

■

In the same manner, we define now:

$$I_\delta h(x) = \inf_{|y|<\delta} h(x+y),$$

Theorem 2: Let $h \in C^2(\Omega)$. For $\delta > 0$ small, we have the asymptotic development of I_δ:

$$I_\delta h(x) = h(x) + \inf_{|z|\leq 1} F_\delta(z) + o(\delta^2) \tag{7}$$

$\forall x \in \Omega$, where $F_\delta(z) = \delta z h'(x) + \dfrac{\delta^2}{2} z^2 h''(x)$. The values of $\inf_{|z|\leq 1} F_\delta(z)$ are given in case $1'$, case $2'$ and case $3'$ below.

Proof: Based on Proof 2, we can write:

$$-\epsilon\delta^2 + h(x) + \inf_{|z|\leq 1} F_\delta(z) \leq I_\delta h(x) \leq \inf_{|z|1} F_\delta(z) + h(x) + \epsilon\delta^2$$

It follows that: $\forall x \in \Omega$

$$I_\delta h(x) = h(x) + \inf_{|z|\leq 1} F_\delta(z) + o(\delta^2).$$

- **case $1'$:**
  - h is strictly increasing such that: $h' > \delta|h''|$, or
  - h is strictly increasing such that: $h'' \neq 0,\ h' \leq \delta|h''|\ and\ 0 < \delta h' - \dfrac{1}{2}\dfrac{h'}{h''} - \dfrac{\delta^2}{2}h'' < 2\delta h'$; or

- h is strictly decreasing such that: $h'' \neq 0,\ h' \leq \delta|h''|\ and\ -\delta h' + \frac{1}{2}\frac{h'}{h''} + \frac{\delta^2}{2}h'' > 0$.

Then:

$$\inf_{|z|\leq 1} F_\delta(z) = -\delta h' + \frac{\delta^2}{2}h''.$$

- **case $2'$:**
  - h is strictly decreasing such that: $|h'| > \delta|h''|$, or
  - h is strictly decreasing such that: $h'' \neq 0,\ |h'| \leq \delta|h''|\ and\ 2\delta h' < \delta h' - \frac{1}{2}\frac{h'}{h''} - \frac{\delta^2}{2}h'' < 0$; or
  - h is strictly increasing such that: $h'' \neq 0,\ |h'| \leq \delta|h''|\ and\ \delta h' + \frac{1}{2}\frac{h'}{h''} + \frac{\delta^2}{2}h'' < 0$.

  Then:

$$\inf_{|z|\leq 1} F_\delta(z) = \delta h' + \frac{\delta^2}{2}h''.$$

- **case $3'$:** $h'' \neq 0,\ \delta h' + \frac{1}{2}\frac{h'}{h''} + \frac{\delta^2}{2}h'' > 0\ and\ -\delta h' + \frac{1}{2}\frac{h'}{h''} + \frac{\delta^2}{2}h'' > 0$. Then:

$$\inf_{|z|\leq 1} F_\delta(z) = -\frac{1}{2}\frac{h'(x)}{h''(x)}.$$

■

We finally define the operator m_δ as follow:

$$m_\delta h(x) = \frac{1}{2}\left(S_\delta h(x) + I_\delta h(x)\right),\ \forall x \in \Omega. \qquad (8)$$

m_δ will replace the *local mean* formulated in (1). The first main result of the paper is stated as follow:

Theorem 3: Let $h \in C^2(\Omega)$. Under conditions given in case $1''$ and case $2''$, and for $\delta > 0$ small, $\forall x \in \Omega$ the asymptotic development of m_δ is:

$$m_\delta h(x) = h(x) + \frac{\delta^2}{2}h''(x). \qquad (9)$$

Proof: Let $x \in \Omega$. Thanks to Theorems 1 and 2, we have:

- **case $1''$:** h is strictly increasing such that:
  - $h' > \delta|h''|$, or
  - $h'' \neq 0,\ h' \leq \delta|h''|\ and\ 0 < \delta h' - \frac{1}{2}\frac{h'}{h''} - \frac{\delta^2}{2}h'' < 2\delta h'$.

  Then: $m_\delta h(x) = h(x) + \frac{\delta^2}{2}h''(x) + o(\delta^2)$.
- **case $2''$:** h is strictly decreasing such that:
  - $|h'| > \delta|h''|$, or
  - $h'' \neq 0,\ |h'| \leq \delta|h''|\ and\ 2\delta h' < \delta h' - \frac{1}{2}\frac{h'}{h''} - \frac{\delta^2}{2}h'' < 0$.

  Then: $m_\delta h(x) = h(x) + \frac{\delta^2}{2}h''(x) + o(\delta^2)$.

In either case, we have the same result. Thus, for a small $\delta > 0$ and $\forall x \in \Omega$, we have: $m_\delta h(x) = h(x) + \frac{\delta^2}{2}h''(x) + o(\delta^2)$. ■

Next is the main result of the paper:

Theorem 4: For a small $\delta > 0$, the SP is performed by a parabolic PDE. Then, IMFs are the solutions of the PDE:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = o(1) \\ h(x,0) = S(x),\ \forall x \in \Omega. \end{cases}$$

Proof: Let $x \in \Omega$. Rewriting (2) by considering (8) as the mean, gives:

$$\begin{cases} h_{n+1} = (I_d - m_\delta)h_n \\ h_0 = S,\ \forall x \in \Omega. \end{cases} \qquad (10)$$

Let $\tau > 0$, τ small. Let's now consider the function:

$$\begin{cases} h: & \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R} \\ & (x,t) \mapsto h(x,t) \end{cases}$$

We define: $h(x, n\tau) = h_n(x),\ \forall x \in \Omega$. Using Taylor expansion for t and thanks to Theorem 3, (10) yields:

$$\begin{aligned} h_{n+1}(x) &= -\frac{\delta^2}{2}h''(x) + o(\delta^2) \\ &= h(x, n\tau + \tau) = h(x, n\tau) + \tau\frac{\partial h}{\partial t}(x, n\tau) + o(\tau^2) \end{aligned}$$

So, for $\tau = \delta^2$, we have: $\frac{\partial h}{\partial t} = -\frac{1}{\delta^2}h - \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + o(1)$. Thus, an IMF is the solution of the parabolic PDE:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = o(1) \\ h(x,0) = S(x),\ \forall x \in \Omega. \end{cases}$$

■

Remark 1: In practice, we resolve:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = 0 \\ h(x,0) = S(x),\ \forall x \in \Omega. \end{cases} \qquad (11)$$

Definition 1: h is a δ-IMF if h is a solution of (11) and, for an adequate T, $h(.,T)$ is a null mean function.

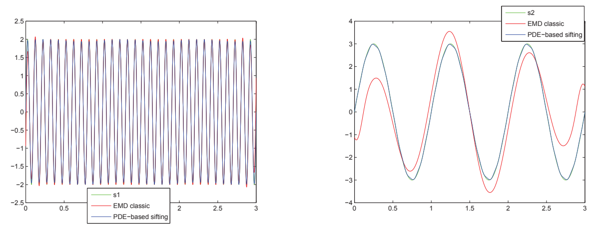
Remark 2: δ-IMFs are extracted with errors of order $o(1)$, which is relatively small.

Remark 3: Once a δ-IMF is extracted through (11), we resolve again (11) with the residual between the signal and that extracted δ-IMF as for initial condition, and so on. Finally, the original signal is decomposed into sum of IMFs.

The following last theorem proves the existence and uniqueness of the solution of (11):

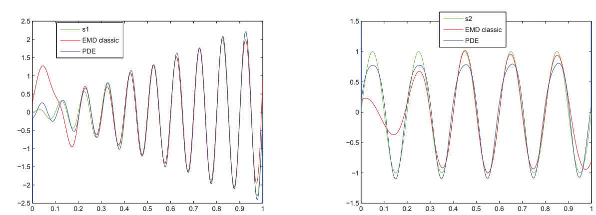
Theorem 5: Let $h \in C^2(\Omega)$. If h is a bandlimited function, then (11) has a unique solution.

Proof: Let $(u_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ be respectively sequences of eigen functions and eigen values of the Laplacian operator, associated with Dirichlet conditions. So, $\forall j \in \mathbb{N}$: $\begin{cases} u_j'' + \lambda_j u_j = 0\ \ in\ \Omega \\ u_j = 0\ \ in\ \partial\Omega. \end{cases}$. Multiplying (11) by u_j and taking the integral, yields: $\int_\Omega (u_j\frac{\partial h}{\partial t} + u_j\frac{1}{\delta^2}h + u_j\frac{1}{2}\frac{\partial^2 h}{\partial x^2})dx = 0$. Let $a_j = \int_\Omega u_j h\, dx$. Thus: $a_j =$



(a) (b)

Fig. 1. IMFs vs δ-IMFs: first modes (a). second modes (b).



(a) (b)

Fig. 2. IMFs vs δ-IMFs: first modes (a). second modes (b).

$\exp\left(-(\frac{1}{\delta^2}-\frac{\lambda_j}{2})t\right)\int_\Omega h_0 u_j dx$. Since $\{(u_j)_{j\in\mathbb{N}}\}$ is an orthonormal basis in $L^2(\Omega)$ [9], then h is uniquely written: $h=\sum_{j\in\mathbb{N}} \prec h, u_j \succ u_j$, where $\prec\cdot,\cdot\succ$ is the inner product in $L^2(\Omega)$. Suppose that there exists $j_0\in\mathbb{N}$ such that $\forall j > j_0$ $\frac{1}{\delta^2}-\frac{\lambda_j}{2}>0$. So, $\lambda_j<\frac{2}{\delta^2}$. This is impossible, because $(\lambda_j)_{j\in\mathbb{N}}$ is positive, increasing and converges towards $+\infty$ [9]. Because h is bandlimited i.e $\exists j_0\in\mathbb{N}$ such that $\forall j>j_0$ $a_j=0$, then h is well defined such as: $h=\sum_{j\le j_0}\exp\left(-(\frac{1}{\delta^2}-\frac{\lambda_j}{2})t\right)u_j\int_\Omega h_0u_jdx$. ∎

V. NUMERICAL RESULTS

Presented results are obtained by implementating equation (11) with an explicit scheme. To better show the relevance and performances of the proposed PDE-based EMD, we compare δ-IMFs to classical IMFs, which are extracted with codes given in [10]. Let's first consider: $s(x)=s_1(x)+s_2(x)$; $s_1(x)=2\sin(20\pi x)$, $s_2(x)=3\sin(2\pi x)$. The signal's components are very well separated by our approach (Fig. 1). δ-IMFs 1 and 2 fit exactly s_1 and s_2 respectively (Figs. 1-(a) and 1-(b)). On the other hand, IMF 1 is almost the same as s_1, expect at the boundaries where we see some little differences (Fig. 1-(a)). IMF 2 totally differs from s_2 (Fig. 1-(b)). Probable reasons for that are the well known boundary problems during the SP. Boundary problems are due to the mean envelopes' estimations by interpolations (splines for examples). δ-IMFs are obtained with Neumann boundary conditions.

The second example is the same as the one considered in [11], and for which authors used a bandwidth criterion for a correct EMD decomposition: $s=s_1+s_2$; $s_1(x)=4\sin(20\pi x)\sin(0.2\pi x)$, $s_2(x)=\sin(10\pi x)$. Despite some attenuation on δ-IMF 2 (Fig. 2-(b)), Fig. 2 clearly illustrates the relevance and efficiency of this new formulation of the SP. For this second example, δ-IMFs are obtained with Dirichlet boundary conditions.

VI. CONCLUSION

We give here some major theoretical contributions on the comprehension of the EMD. Indeed, we prove the SP's convergence towards a solution of a PDE. We also prove that the PDE's solution is a unique, which guarantees the uniqueness of the EMD decomposition. A mathematical characterization of basic modes is then brought out with δ-IMFs. Envelopes' interpolations problems and related issues are now eliminated, and as shown by our numerical results, the PDE-based EMD improves a lot the classical EMD.

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