ON SIMULATION OF FIRST-ORDER AUTO-REGRESSIVE PROCESSES WITH NEAR LAPLACE MARGINALS

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ABSTRACT

The focus of this paper is the modeling of a class of stationary non-Gaussian auto-regressive processes that often find applications in statistical signal processing. We propose a general simulation procedure for constructing a time series model with a near-Laplace marginal distributions. Our approach is based on a class of Monte Carlo rejection algorithms. A theoretical analysis of the average complexity of the proposed algorithms for simulating the time series model is included.

Index Terms— signal modeling, time series, Monte Carlo, rejection algorithms, Laplace distribution

1. INTRODUCTION

Simulation of correlated random processes with a given marginal distribution is a problem frequently encountered in signal processing studies (e.g., speech and image processing, analysis of communication systems). Of particular interest is the first-order stationary auto-regressive process $\{X_n\}$ of the form

$$X_n = \rho X_{n-1} + \varepsilon_n, \tag{1}$$

where $|\rho| < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables (innovation sequence). In generating $\{X_n\}$ with a given marginal distribution, we need to solve the challenging inverse problem of finding an appropriate distribution for the innovation process $\{\varepsilon_n\}$. The only trivial case is the Gaussian distribution, as it is known that if $X_n \propto N(0, v^2)$ then the innovation process must be also Gaussian, i.e., $\{\varepsilon_n\} \propto N(0, (1-\rho^2)v^2)$. On the other hand, in many applications $\{X_n\}$ is required to have a Laplacian distribution. In fact, it is well accepted that (1) closely describes speech signals if $\{X_n\}$ is distributed according to the Laplace density with $0.8 < \rho < 0.9$ [1]. Previously, time series modeling with the Laplace marginal distribution has been developed in [2]. Unfortunately, the resulting auto-regressive model experiences a singular behavior as the innovation process of the model has a non-zero probability mass at 0. Such singular behaviour is difficult to observe in real-world data sets. In [3], a *standard hyperbolic secant* density has been used as a model for the marginal density in (1), and the inverse transform method for simulating the resulting time series model has been suggested. However, the proposed formula is rather complex and has a limited scope of applications, since often one can compute the density function but not the corresponding distribution function and its inverse. Consequently, the inverse transform method is rarely applicable.

In this paper, we develop an approach for generating a process of the form (1) with a marginal density which is very close to the Laplace density. In fact, we choose a marginal density function that belongs to the class of *generalized hyperbolic secant distributions* [4], which consists of symmetric densities with exponentially decaying tails. We propose a Monte Carlo rejection method for simulating a process with the desired density. The method is based on establishing bounds on the density of the innovation process. These bounding densities are of simple form yielding efficient simulation algorithms. The method can be generalized to a larger class of densities with a limited knowledge about their shape. We develop two versions of the rejection method. The first one is based on some universal bounds which only require the knowledge of the second moment of the underlying density. The second rejection algorithm utilizes a bound derived specifically for our particular class of densities. A comparison of the efficiency two algorithms is made based on a detailed analysis of their average time behavior.

2. BACKGROUND AND PRELIMINARIES

In order to simulate the process in (1) with a given marginal density $f_X(\cdot)$, we begin with simulating X_0 according to $f_X(\cdot)$ followed by the generation of ε_1 with the distribution $f_\varepsilon(\cdot)$ from which we obtain X_1. This process is then iterated accordingly. Hence, in order to implement this simulation procedure one needs to determine the distribution $f_\varepsilon(\cdot)$. A solution to this inverse problem can be obtained in the transform domain by establishing a relationship between characteristic functions of $\{X_n\}$ and $\{\varepsilon_n\}$. Let $\varphi_X(t) = E\{e^{jtX_n}\}$ and $\varphi_\varepsilon(t) = E\{e^{jt\varepsilon_n}\}$ be the characteristics functions of $\{X_n\}$ and $\{\varepsilon_n\}$, respectively. Then from (1), it follows that

$$\varphi_X(t) = \varphi_X(\rho t)\, \varphi_\varepsilon(t). \tag{2}$$

Thus, for a given distribution of $\{X_n\}$ we can recover the distribution of $\{\varepsilon_n\}$ by finding the inverse Fourier transform of the function $\varphi_X(t)/\varphi_X(\rho t)$, provided that it defines a proper characteristic function. If X_n has the Laplace density then the distribution of $\{\varepsilon_n\}$ is not longer absolutely continuous and so neither is the distribution of the bivariate process (X_n, X_{n-1}) [2]. This undesirable property is shared by other important choices for the distribution of X_n, e.g., exponential and gamma distributions [5]. We next establish that if the marginal density is the generalized hyperbolic secant distribution then the joint distribution of (X_n, X_{n-1}) is indeed absolutely continuous. In the next section we propose Monte-Carlo rejection algorithms for constructing such a process.

A random variable X with the generalized hyperbolic secant distribution [4] is defined by the two-parameter density

$$f_X(x;a;\alpha) = \frac{\alpha c_0(a)}{a + e^{\alpha x} + e^{-\alpha x}},\ -\infty < x < \infty \qquad (3)$$

where $a \geq 0$, $\alpha > 0$, and $c_0(a)$ is the normalizing constant. The density $f_X(x;a;\alpha)$ is symmetric with exponential tails, and possesses all moments. A quick inspection of (3) reveals that $f_X(x;a;\alpha)$ is bounded by the Laplace density, i.e., we have

$$f_X(x;a;\alpha) \leq 2c_0(a)\frac{\alpha}{2}e^{-\alpha|x|}, \quad -\infty < x < \infty. \qquad (4)$$

A careful examination of the constant $c_0(a)$ shows that $c_0(a)$ is the smallest for $a = 0$ and it is equal to $c_0(0) = 2/\pi$. Hence, one may conclude that the generalized hyperbolic secant distribution with $a = 0$ is the most similar to the Laplace distribution. This defines the so-called standard hyperbolic secant density function

$$f_X(x;\alpha) = \frac{2\alpha/\pi}{e^{-\alpha x} + e^{\alpha x}}\ , -\infty < x < \infty. \qquad (5)$$

This density was also used in [3] without the above reasoning. Fig. 1 compares the density $f_X(x;\pi/2)$ with the Laplace density with $\lambda = \sqrt{2}$. In this paper, we shall focus on the density in (5), however, our approach extends readily to the density in (3) and other more general models.

For our future developments we need to determine the characteristic function of a random variable X with the density defined in (5). Using the identity on page 537 in [6] we can obtain that

$$\varphi_X(t) = \left(ch\left(\frac{\pi t}{2\alpha}\right)\right)^{-1}, \qquad (6)$$

where $ch(x) = \frac{e^x + e^{-x}}{2}$ is the hyperbolic cosine. Then, we get that $\varphi_\varepsilon(t) = ch\left(\frac{\pi\rho t}{2\alpha}\right)/ch\left(\frac{\pi t}{2\alpha}\right)$, $|\rho| < 1$. Next, by the result on page 539 in [6], we obtain the desired density function of ε_n

$$f_\varepsilon(x) = \frac{2\alpha}{\pi}\frac{\cos\left(\frac{\pi\rho}{2}\right)ch(\alpha x)}{\cos(\pi\rho) + ch(2\alpha x)}. \qquad (7)$$

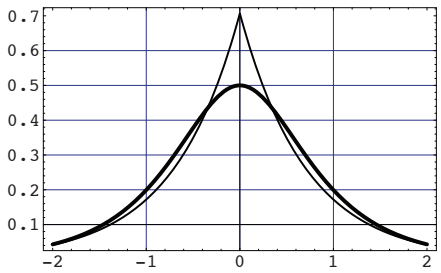

Fig. 1. The Laplace($\sqrt{2}$) density and the standard hyperbolic secant density (thick line) with $\alpha = \pi/2$.

3. SIMULATION ALGORITHMS

In this section we develop two rejection algorithms for simulating $\{\varepsilon_n\}$. The first strategy relies on the universal rejection algorithm proposed in [7]. Here we merely utilize the fact that the density $f_\varepsilon(x)$ is uni-modal and monotonic for $x \geq 0$. In the second direct rejection procedure we find explicit envelope densities tailored to the shape of $f_\varepsilon(x)$. The rejection strategy is much more flexible than the inverse method used in [3] as it can be extended to other probabilistic models with the minimal knowledge of underlying distributions [7].

3.1. Universal Rejection Method

The universal rejection algorithm can be applied to a large class of densities since the only a priori information required is expressed in terms of general properties of densities in the class. This includes such properties such as smoothness, unimodality, monotonicity and log-concavity [7]. In particular, we can utilize this theory by observing that the density $f_\varepsilon(x)$ in (7) is uni-modal and monotonic for $x \geq 0$. Next, by using Theorem 3.2 of Chapter 7 in [7], it can be shown that $f_\varepsilon(x) \leq h(x)$, where $h(x) = \min\left(f_\varepsilon(0), \frac{3\sigma_\varepsilon^2}{2|x|^3}\right)$, $\sigma_\varepsilon^2 = E\varepsilon^2$. The area under $h(x)$

$$A = 3\left(\frac{3}{2}\right)^{1/3}(\sigma_\varepsilon f_\varepsilon(0))^{2/3} \qquad (8)$$

gives the average number of iterations required in the rejection algorithm. The algorithm, however, requires a procedure for generating a random variate W with the envelope density $g(x) = h(x)/A$. The form of $g(x)$ allows us to apply the concept of the ratio of uniforms. Let U, V be independent random variables with uniform $[0,1]$ and uniform $[-\delta,\delta]$ distributions, respectively. Then, we can show (see the Appendix) that the density of a random variable $Z = V/U^p, p > 0$ is given by

$$f_Z(x) = \begin{cases} \frac{1}{2(p+1)\delta} & \text{when } |x| \leq \delta \\ \frac{\delta^{1/p}}{2(p+1)}\frac{1}{|x|^{1+1/p}} & \text{when } |x| \geq \delta. \end{cases} \qquad (9)$$

This result can be directly applied to $g(x)$ by noting that $g(x) = f_Z(x)$ if we choose $p = 1/2$ and $\delta = \left(\frac{3\sigma_\varepsilon^2}{2f_\varepsilon(0)}\right)^{1/3}$. As a result, we obtain a very efficient method for simulating W, i.e.,

$$W = \left(\frac{3}{2}\right)^{1/3} \left(\frac{\sigma_\varepsilon^2}{f_\varepsilon(0)}\right)^{1/3} \frac{V}{U^{1/2}}, \tag{10}$$

where U and V are independent uniform $[0,1]$ and uniform $[-1,1]$ random variates, respectively. Since $\sigma_\varepsilon^2 = \left(\frac{\pi}{2\alpha}\right)^2 (1-\rho^2)$ and $f_\varepsilon(0) = \frac{2\alpha}{\pi}\cos(\pi\rho/2)/[1+cos(\pi\rho)]$ therefore $\sigma_\varepsilon^2/f_\varepsilon(0) = \left(\frac{\pi}{2\alpha}\right)^3 \psi(\rho)$ where $\psi(\rho) = [1+cos(\pi\rho)](1-\rho^2)/cos(\pi\rho/2)$. We can now give the corresponding rejection algorithm for simulation of $\{\varepsilon_n\}$.

Algorithm I

Step 1. Generate three independent uniform $[0,1]$ random variates U_0, U_1, U_2.

Step 2. Set

$$W = \frac{\pi}{2\alpha}\left(\frac{3}{2}\psi(\rho)\right)^{1/3} \frac{2U_1 - 1}{\sqrt{U_2}},$$

Step 3. If

$$U_0 < \begin{cases} \frac{f_\varepsilon(W)}{f_\varepsilon(0)} & \text{when } |W| \le \frac{\pi}{2\alpha}\left(\frac{3}{2}\psi(\rho)\right)^{1/3} \\ \frac{2|W|^3 f_\varepsilon(W)}{3\sigma_\varepsilon^2} & \text{when } |W| > \frac{\pi}{2\alpha}\left(\frac{3}{2}\psi(\rho)\right)^{1/3} \end{cases}$$

then exit with $\varepsilon = W$; else go to **Step 1**.

The efficiency of the rejection algorithm is determined by the acceptance probability in **Step 3** [7]. In fact, the expected number of iterations equal to the constant A in (8). Plugging the appropriate expressions for σ_ε^2 and $f_\varepsilon(0)$ into (8) we can obtain the average efficiency of the universal rejection algorithm as

$$A_{UR}(\rho) = 3\left(\frac{3}{2}\right)^{1/3}\left((1-\rho^2)^{1/2}\frac{\cos(\pi\rho/2)}{1+\cos(\pi\rho)}\right)^{2/3}. \tag{11}$$

Note that $A_{UR}(\rho) \ge 3(3/2)^{1/3} = 2.16337\ldots$ and it tends to ∞ as $|\rho| \to 1$. The latter fact is not surprising as the case $|\rho| \to 1$ corresponds to the non-stationary auto-regressive process. For the practical range $|\rho| \le 0.9$ the constant $A_{UR}(\rho)$ varies between 2.17 and 4.29.

3.2. Rejection Method Using Laplace Bounding Densities

We now develop a rejection algorithm that uses specific bounds tailored to the shape of $f_\varepsilon(x)$. Since $f_\varepsilon(x)$ has exponentially decaying tails, it is natural to seek for a bound in terms of a Laplace distribution. The following lemma gives such a bound.

Lemma 3.1 *For the density $f_\varepsilon(x)$ in (7) we have*

$$f_\varepsilon(x) \le \begin{cases} B_1(\rho)\frac{\alpha}{2}e^{-\alpha|x|} & |\rho| \le 1/2 \\ B_2(\rho)\frac{\alpha}{2}e^{-\alpha|x|} & 1/2 < |\rho| < 1, \end{cases} \tag{12}$$

where $B_1(\rho) = \frac{8}{\pi}\cos(\pi\rho/2)$, $B_2(\rho) = \frac{8}{\pi}\frac{\cos(\pi\rho/2)}{1-\cos^2(\pi\rho)}$.

The proof of this inequality is given in the Appendix. The proof of Lemma 3.1 reveals that one can obtain a tighter bound if we use a mixture of Laplace distributions. In fact, we have the following counterpart of (12).

$$f_\varepsilon(x) \le \begin{cases} \frac{2}{3}B_1(\rho)(\frac{3}{4}\frac{\alpha}{2}e^{-\alpha|x|} + \frac{1}{4}\frac{3\alpha}{2}e^{-3\alpha|x|}) & |\rho| \le \frac{1}{2} \\ \frac{2}{3}B_2(\rho)(\frac{3}{4}\frac{\alpha}{2}e^{-\alpha|x|} + \frac{1}{4}\frac{3\alpha}{2}e^{-3\alpha|x|}) & \frac{1}{2} < |\rho| < 1, \end{cases} \tag{13}$$

Hence, we have the mixture of Laplace(α) and Laplace(3α) distributions with the mixing probabilities $3/4$ and $1/4$, respectively.

As we have already noted the efficiency of the rejection algorithm is determined by the constants $B_1(\rho), B_2(\rho)$ appearing in bound (12). Note that $1.8006\ldots \le B_1(\rho) \le 2.546\ldots$ and $B_2(\rho)$ has a minimum at $\rho^* = 0.60817$ with $B_2(\rho^*) = 1.65397$. Also $B_2(0.8) = 2.2776$, $B_2(0.9) = 4.1716$. We will denote the constants $B_1(\rho), B_2(\rho)$ commonly as $B(\rho)$.

In Figure 2, we plot $B(\rho)$ along with the constant $A_{UR}(\rho)$ (see [11)] corresponding to the universal rejection algorithm. Also the constants $\frac{2}{3}B(\rho)$ corresponding to the bound based on the mixture of Laplace distributions are shown. It is interesting to note that the universal rejection method is more efficient than the rejection method based on the Laplace distribution for $|\rho| \le 0.3$. This fact can be easily explained by noting that $f_\varepsilon(x)$ is flat for small values of $|\rho|$ and this property is captured by the bounding curve $h(x)$. However, note that if we use the bound with the mixture of Laplace distributions then the universal rejection algorithm is less efficient than the rejection method based on the bound in (13) for all $-1 < \rho < 1$.

We are now in a position to propose the rejection algorithm using the Laplace bounding density. An algorithm that uses the bound based on the mixture of Laplace densities in (13) can be derived in an analogous way. In fact, in the algorithm given below, the random variate W should simulate the mixture $\frac{3}{4}\frac{\alpha}{2}e^{-\alpha|x|} + \frac{1}{4}\frac{3\alpha}{2}e^{-3\alpha|x|}$. This can be easily obtained by generating the uniform $[0,1]$ random variate U_3 and checking whether $U_3 \le 3/4$. If this is true then we set $W = -\operatorname{sign}(2U_1 - 1)\ln(U_2)/\alpha$, else $W = -\operatorname{sign}(2U_1 - 1)\ln(U_2)/3\alpha$.

Algorithm II

Step 1. Generate three independent uniform $[0,1]$ random variates U_0, U_1, U_2.

Step 2. Set $W = -\operatorname{sign}(2U_1 - 1)\ln(U_2)/\alpha$.

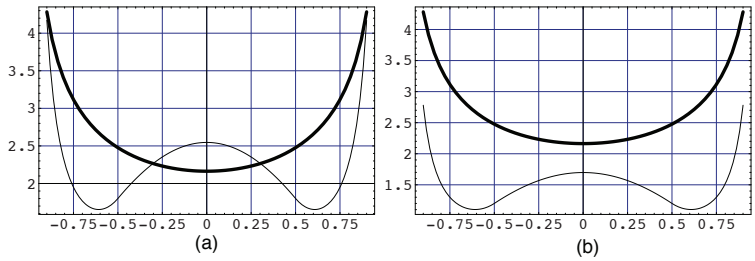

Fig. 2. The efficiency of rejection algorithms: (a) $B(\rho)$ (thin line) of ALGORITHM II (Laplace bound) and $A_{UR}(\rho)$ of ALGORITHM I (b) $\frac{2}{3}B(\rho)$ (thin line) of ALGORITHM II (Laplace mixture bound) and the $A_{UR}(\rho)$ of ALGORITHM I.

Step 3. If

$$U_0 < \frac{f_\varepsilon(W)}{B(\rho)\frac{\alpha}{2}e^{-\alpha|W|}}$$

then exit with $\varepsilon = W$; else go to **Step 1**.

It is worth mentioning that in **Step 2** we simulate the Laplace(α) random variate. Moreover, by noting that $e^{-2\alpha|W|} = U_2^2$, $e^{-4\alpha|W|} = U_2^4$ we can greatly simplify the ratio appearing in the acceptance decision in **Step 3**. As a result, the algorithm needs only two uniform random variates and a simple comparison step. The average number of iterations to produce a random variate ε is shown in Figure 2(a). The efficiency of the rejection algorithm employing the mixture of Laplace densities is depicted in Figure 2(b).

4. APPENDIX

Proof of (9)- It is known that the density function of a random variable $X = V/U$ is given by

$$f_X(x) = \int_{-\infty}^{\infty} |t| f_{VU}(tx, t)dt, \tag{14}$$

where $f_{VU}(v, u)$ is the joint density function of (V, U). In our case we have $Z = V/U^p$, $p > 0$, where V is uniform $[-\delta, \delta]$, whereas U is uniform $[0, 1]$. Moreover, V and U are independent random variables. Thus, the joint density function of (V, U^p) is

$$f_{VU^p}(v, u) = \frac{1}{2p\delta}u^{-(p-1)/p},\ 0 < u < 1, |v| \le \delta.$$

This and a direct application of (14) yield the proof of (9)

Proof of Lemma 3.1- For $x \ge 0$, $f_\varepsilon(x)$ in (7) can be written as

$$f_\varepsilon(x) = c_1 \frac{e^{-\alpha x} + e^{-3\alpha x}}{2c_2 e^{-2\alpha x} + e^{-4\alpha x} + 1}. \tag{15}$$

where $c_1 = \frac{2\alpha}{\pi}\cos(\pi\rho/2)$, $c_2 = \cos(\pi\rho)$. Note that for $|\rho| \le 1/2$, $0 \le c_2 \le 1$. Then, we can bound $f_\varepsilon(x)$ by $c_1\{e^{-\alpha x} + e^{-3\alpha x}\} \le 2c_1 e^{-\alpha x}$. A bound for $f_\varepsilon(x)$ in the case of $1/2 \le |\rho| < 1$ must be found in a different way since now $-1 < c_2 \le 0$. To do so, let $d(x) = 2c_2 e^{-2\alpha x} + e^{-4\alpha x} + 1$. The derivative $d^{(1)}(x) = 4\alpha(-c_2)e^{-2\alpha x} - 4\alpha e^{-4\alpha x}$ is equal to zero at $x^* = -\frac{1}{2\alpha}\ln(-c_2)$ where $d(x)$ reaches the minimum, as $d^{(2)}(x^*) = 8\alpha^2 c_2^2 > 0$. Since $d(x^*) = 1 - c_2^2$ we obtain that, for $1/2 \le |\rho| < 1$,

$$f_\varepsilon(x) \le \frac{c_1}{1 - c_2^2}\{e^{-\alpha x} + e^{-3\alpha x}\} \le \frac{2c_1}{1 - c_2^2}e^{-\alpha x}.$$

The case $x < 0$ can be treated in an analogous way.

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