GENERIC INVERTIBILITY OF MULTIDIMENSIONAL FIR MULTIRATE SYSTEMS AND FILTER BANKS

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ABSTRACT

We study the invertibility of M-variate polynomial (respectively : Laurent polynomial) matrices of size N by P. Such matrices represent multidimensional systems in various settings including filter banks, multiple-input multiple-output systems, and multirate systems. The main result of this paper is to prove that when $N - P \geq M$, then $\boldsymbol{H}(\boldsymbol{z})$ is generically invertible; whereas when $N - P < M$, then $\boldsymbol{H}(\boldsymbol{z})$ is generically noninvertible. As a result, we can have an alternative approach in design of the multidimensional systems.

Index Terms— Generic Invertible, Left Invertibility, Perfect Reconstruction, Multirate Systems, Generic Property.

1. INTRODUCTION

During the last two decades, one dimensional multirate systems in digital signal processing were thoroughly developed. In recent years, due to the high demand in multidimensional processing including image and video processing, volumetric data analysis and spectroscopic imaging, multidimensional multirate systems have been studied more extensively. One key property of a multidimensional multirate system is its perfect reconstruction, which guarantees that an original input can be perfectly reconstructed from the outputs.

In a multidimensional multirate system, a digital signal is split into several channels and processed with different sampling rates. The most popular multirate systems are filter banks. Using the polyphase representation in the z-domain [1], we can represent the analysis part as an $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ with entries in a Laurent polynomial ring $\mathbb{C}[z_1, z_2, ..., z_M, {z_1}^{-1}, ..., {z_M}^{-1}]$. Here M is the dimension of signals, N is the number of channels in the filter bank, and P is the sampling factor at each channel. An application of this setting may arise in multichannel acquisition. Here we collect data about unknown multidimensional signal $\boldsymbol{X}(\boldsymbol{z})$ as output of the analysis part. The acquisition system (filters $H_i(\boldsymbol{z})$ and sampling matrix $\boldsymbol{D}$) is fixed and known beforehand. The objective is to reconstruct $\boldsymbol{X}(\boldsymbol{z})$ with an $P \times N$ synthesis polyphase matrix $\boldsymbol{G}(\boldsymbol{z})$. The existence of a synthesis part becomes a purely mathematical question. The perfect reconstruction condition holds if and only if $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}_P$ where $\boldsymbol{I}_P$ is the $P \times P$ identity matrix.

Then it is a natural question to ask: When does the system have a high probability of the existence of an inverse? Rajagopal and Potter [2] and Zhou and Do [3] have investigated this question and made several conjectures. We investigate the systems by varying M, N and P. In the experiments, we found that when $M - N \geq P$, the existence of an inverse is "almost surely". On the other hand, when $M - N < P$, the nonexistence of an inverse is "almost surely". To precisely study this inverse existence problem, we employ the concept of "hold generically" [4].

2. GENERIC INVERTIBILITY

2.1. Generic Property

In [3], Zhou and Do made the following conjectures.

Conjecture 1 *Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ M-variate polynomial (resp. : Laurent polynomial) matrix with $N \geq P$. If $N - P \geq M$, then it is "*almost surely*" polynomial (resp. : Laurent polynomial) left invertible. Otherwise, it is "*almost surely*" polynomial (resp. : Laurent polynomial) left noninvertible.*

However, Zhou and Do did not give a precise definition of "almost surely". In order to have the appropriate language, we employ the concept of "hold generically".

Definition 1 (Generic) *[4] A property is said to* hold generically *for polynomials $f_1, .., f_n$ of degree at most $k_1, ..., k_n$ if there is a nonzero polynomial F in the coefficients of the f_i such that the property holds for $f_1, ..., f_n$ whenever the polynomial $F(f_1, ..., f_n)$ is nonvanishing.*

Lemma 1 *If a property of polynomials of degree at most k_1, ..., k_n in m variables is generic, then the coefficient space $\mathcal{C}$ of polynomials whose polynomials failed to satisfy the property is measure zero and nowhere dense.*

Proof By the definition of hold generically, there exists a nonzero polynomial F in the coefficients of the f_i such that the property fails to satisfy for $f_1, ..., f_n$ for which the polynomial $F(f_1, ..., f_n)$ is vanishing. Let R_i be the set of M-variate polynomials of degree less than or equal to k_i. By Gunning in [5, p.9], $\lambda_l(\{(f_1, ..., f_n) \in \prod_{i=1}^n R_i \mid F(f_1, ..., f_n) = 0\}) = 0$ where $l = \binom{k_1+m}{m} + ... + \binom{k_n+m}{m}$ is the dimension of the coefficient space. Thus, the coefficient space $\mathcal{C}$ of polynomials whose polynomials failed to satisfy the property is measure zero. To show the set is nowhere dense, it is equivalent to show that the closure of the set contains no open set. Suppose it contains an open ball $B(\epsilon)$ with some radius $\epsilon > 0$. Since $F^{-1}(\{0\})$ is a closed set, $\overline{\mathcal{C}}$ is also in $F^{-1}(\{0\})$. Thus, $F^{-1}(\{0\})$ contains the open ball $B(\epsilon)$. However, this contradicts the fact that $F^{-1}(\{0\})$ is measure zero. Therefore, the coefficient space of polynomials whose polynomials failed to satisfy the property is nowhere dense.

The immediate consequence is that if $f_1, ..., f_n$ are drawn independently from a probability distribution with respect to the Lebesgue measure, the property of $f_1, ..., f_n$ holds with probability one. Furthermore, suppose $\tilde{f}_0, ..., \tilde{f}_n$ satisfies the property. Since the coefficient space $\mathcal{C}$ of polynomials whose polynomials failed to satisfy the property is nowhere dense, there exists an open ball $B(\epsilon)$ around $\tilde{f}_0, ..., \tilde{f}_n$ for some $\epsilon > 0$ such that the property is satisfied within the open ball $B(\epsilon)$. This shows that the system with the property is robust [6].

2.2. Generically Invertible when $N - P \geq M$

To prove our main theorem in this section, we need to employ the resultant of the polynomials.

Theorem 1 (Resultant) *If we fix positive degrees $k_0, ..., k_n$, then there is a unique nonzero polynomial called the* resultant $\mathrm{RES}_{(k_0,...,k_n)} \in \mathbb{C}[\{u_{i,j}\}]$ *where the variables $u_{i,j}$ correspond to the coefficients of i-th polynomial. If $F_0, ..., F_n \in \mathbb{C}[x_0, ..., x_n]$ are homogeneous of degrees $k_0, ..., k_n$, then F_0, ..., F_n have a nontrivial common zero over $\mathbb{C}$ if and only if* $\mathrm{RES}_{(k_0,...,k_n)}(F_0, ..., F_n) = 0$.

Theorem 2 *If $N - P \geq M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically polynomial left invertible.*

Proof The strategy of this proof is to find a nonzero polynomial F such that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for every noninvertible matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k. Let $\boldsymbol{Z} = (z_0, ..., z_M)$. If $f(\boldsymbol{z}) = f_0(\boldsymbol{z}) + f_1(\boldsymbol{z}) + ... + f_l(\boldsymbol{z})$ is the decomposition of the polynomial $f(\boldsymbol{z})$ into sums of forms $f_i(\boldsymbol{z})$ of degree i, then the homogenization $\overline{f}(\boldsymbol{Z})$ of $f(\boldsymbol{z})$ of degree k is defined to be $\overline{f}(\boldsymbol{Z}) = z_0^k f_0(\boldsymbol{z}) + z_0^{k-1} f_1(\boldsymbol{z}) + ... + z_0^{k-l} f_l(\boldsymbol{z})$. Let $\boldsymbol{h}_i(\boldsymbol{Z})$ be the ith row of an $N \times P$ matrix $\overline{\boldsymbol{H}}(\boldsymbol{z})$. Let $t_i(\boldsymbol{Z})$ be the determinant of the $P \times P$ submatrix containing $\boldsymbol{h}_i(\boldsymbol{Z}), \boldsymbol{h}_{i+1}(\boldsymbol{Z}), ..., \boldsymbol{h}_{i+P-1}(\boldsymbol{Z})$. Define ϕ to be a function such that $\boldsymbol{H}(\boldsymbol{z}) \mapsto (t_1(\boldsymbol{Z}), t_2(\boldsymbol{Z}), ..., t_{M+1}(\boldsymbol{Z}))^T$. Rajagopal and Potter in [2, 7] show that if $\boldsymbol{H}(\boldsymbol{z})$ is noninvertible and $N \geq P$, then the $P \times P$ maximal minors of $\boldsymbol{H}(\boldsymbol{z})$ have a common zero. Suppose $(\tilde{z}_1/\tilde{z}_0, \tilde{z}_2/\tilde{z}_0, ..., \tilde{z}_M/\tilde{z}_0)$ is a solution of the maximal minors of $\boldsymbol{H}(\boldsymbol{z})$ where $\tilde{z}_0 \neq 0$. Then $(\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_M)$ is a nonzero solution of maximal minors of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$. Since $\{t_1, ..., t_{M+1}\}$ is a part of the subset of the set of maximal minors of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$, this implies that $\phi(\boldsymbol{H}(\boldsymbol{z}))$ have a nontrivial common zero. Therefore, by the property of the resultant shown in Theorem 1, we know $\mathrm{RES}_{(Pk,...,Pk)}$ $\circ\phi(\boldsymbol{H}(\boldsymbol{z})) = 0$ for all noninvertible matrices $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k. The $\mathrm{RES}_{(Pk,...,Pk)}$ and t_i are polynomials, so is $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi$. Last but not least, we need to show $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi$ is not a zero function. Let

$$\boldsymbol{T}(\boldsymbol{z}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ z_1^k & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 1 \\ z_M^k & z_{M-1}^k & \ddots & z_1^k \\ 0 & z_M^k & \ddots & \vdots \\ \vdots & \dots & \ddots & z_M^k \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

be an $N \times P$ matrix. Suppose $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi(\boldsymbol{T}(\boldsymbol{z})) = 0$. By Theorem 1, we know that t_i's have a nontrivial common zero. i.e. there exists $\tilde{\boldsymbol{Z}}$ a nonzero solution such that $t_{M+1}(\tilde{\boldsymbol{Z}}) = \tilde{z}_M^{Pk} = 0$. This implies $\tilde{z}_M = 0$. If $\tilde{z}_M = 0$, then $t_M(\tilde{z}_0, \tilde{z}_1, ..., \tilde{z}_{M-1}, 0) = \tilde{z}_{M-1}^{Pk} = 0$. Thus $\tilde{z}_{M-1} = 0$. Continuing the process, we can conclude $\tilde{z}_0 = \tilde{z}_1 = ... = \tilde{z}_M = 0$. This contradicts the assumption that $\tilde{\boldsymbol{Z}}$ is nontrivial. So $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi(\boldsymbol{T}(\boldsymbol{z})) \neq 0$. Therefore $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi$ is not zero function. By the definition of hold generically, we conclude that $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically polynomial left invertible matrix.

Theorem 3 *If $N - P \geq M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically Laurent polynomial left invertible.*

Proof If a polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left noninvertible, then $\boldsymbol{H}(\boldsymbol{z})$ is also polynomial left noninvertible. According to Theorem 2, this shows that $\mathrm{RES}_{(Pk,...,Pk)}$ $\circ\phi(\boldsymbol{H}(\boldsymbol{z})) = 0$ for all Laurent polynomial left noninvertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$.

2.3. Generically Noninvertible when $N - P < M$

Projective n-space $\mathbb{P}^n$ is the set of equivalence classes of $(n+1)$-tuples $(a_0, ..., a_n)$ of elements of $\mathbb{C}$, not all zero, under the equivalence relation given by $(a_0, ..., a_n) \sim (\lambda a_0, ..., \lambda a_n)$ for all nonzero $\lambda \in \mathbb{C}$.

Definition 2 (Height) *The* height *of a prime ideal* $\mathrm{ht}\, p$ *is the supremum of the lengths n of strictly descending chains* $p =$

$p_0 \supset p_1 \supset ... \supset p_n$ *of prime ideals. For an arbitrary ideal I,* $\operatorname{ht} I = \inf\{\operatorname{ht} p \mid I \subset p,\ p \text{ is prime ideal}\}$.

Lemma 2 *Given $\boldsymbol{H}(\boldsymbol{z})$ is $N \times P$ polynomial matrix in M variables of degree at most $k > 0$ and $N \geq P$. Let $V(\{m_i\}) := \{\boldsymbol{Z} \in \mathbb{P}^n \mid m_i(\boldsymbol{Z}) = 0 \quad \text{for all } i = 1, ..., \binom{N}{P}\}$ where m_i is a maximal minor of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$ with some ordering and $\overline{\boldsymbol{H}}(\boldsymbol{Z})$ is the homogenization of $\boldsymbol{H}(\boldsymbol{z})$ of degree k. Then $V(\{m_i\})$ is empty if and only if $\operatorname{ht}\langle m_i\rangle = M + 1$. Therefore if $V(\{m_i\})$ is empty, then $N - P \geq M$. In other words, if $N - P < M$, then $V(\{m_i\})$ is nonempty.*

Proof Since m_i is homogeneous, then the unit does not lie in $\langle m_i\rangle$. This implies that $\langle m_i\rangle \neq \mathbb{C}[x_0, ..., x_n]$. By [4, p.370] and the definition of radical ideal, $V(\{m_i\})$ is empty if and only if $\langle\sqrt{m}_i\rangle = \langle x_0, ..., x_M\rangle$. It is easy to see that $\operatorname{ht}\langle\sqrt{m}_i\rangle = M + 1$. Since $\operatorname{ht}\langle m_i\rangle = \operatorname{ht}\langle\sqrt{m}_i\rangle$, the height of $\langle m_i\rangle$ is also $M + 1$. Macaulay in [8, p.54] proved that $\operatorname{ht}\langle m_i\rangle \leq N - P + 1$.

Definition 3 (Weak-Zero) *[9] A point in $\mathbb{P}^n$ is said to be* weak-zero *if at least one of its coordinates is zero.*

Lemma 3 *[10] The polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial invertible if and only if the set $V(\{m_i\})$ contains only weak-zeros where $\boldsymbol{H}(\boldsymbol{z})$, V and m_i are same as above lemma.*

Theorem 4 *If $N - P < M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically Laurent polynomial left noninvertible.*

Proof The strategy of the proof is the same as above Theorem 2. We will find a nonzero polynomial F such that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for every Laurent polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$. If $N < P$, then every polynomial matrix is left noninvertible. Now consider $\boldsymbol{H}(\boldsymbol{z})$ is invertible. Let c_{ij} be a coefficient for the constant term of $h_{ij}(\boldsymbol{z})$ where $\boldsymbol{H}(\boldsymbol{z}) = (h_{ij}(\boldsymbol{z}))$. Define a function F_1 such that

$$\boldsymbol{H}(\boldsymbol{z}) \mapsto \prod_{i=1,...,N\, j=1,...,P} c_{ij}.$$

If $h_{ij}(z_1, ..., z_{N-P+1}, 0, ..., 0) = 0$ for some i, j, then it implies $c_{ij} = 0$. This shows that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ in (1). If $h_{ij}(z_1, ..., z_{N-P+1}, 0, ..., 0) \neq 0$ for all i, j, then $\boldsymbol{H}(z_1, ..., z_{N-P+1}, 0, ..., 0)$ is also invertible because there exists Laurent polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$ such that $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$ and $\boldsymbol{G}(z_1, ..., z_{N-P+1}, 0, ..., 0)$ is well-defined. We can now assume that $M = N - P + 1$. Define $t_i(\boldsymbol{Z})$ to be the same as Theorem 2. Let $t_j^{(i)} = t_j(z_0, ..., \overset{i\text{-th}}{0}, ..., z_M)$. Define θ_i to be a function such that $\boldsymbol{H}(\boldsymbol{z}) \mapsto (t_1^{(i)}, ..., t_M^{(i)})^T$ for $i = 0, ..., M$. By Lemma 2 and Lemma 3 and the fact that $\{t_1^{(i)}(\boldsymbol{Z}), ..., t_M^{(i)}(\boldsymbol{Z})\}$ is the subset of the set of maximal minors of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$, it implies that $\theta_i(\boldsymbol{H}(\boldsymbol{z}))$ have a nonzero common zero for some $i = 0, ..., M$. By the property of the resultant shown in Theorem 1, we know that given any Laurent polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$, so

		N			
		1	2	3	4
M=1	P 1	0	500	500	500
	P 2	0	0	500	500
	P 3	0	0	0	500
M=2	P 1	0	0	500	500
	P 2	0	0	0	500
	P 3	0	0	0	0
M=3	P 1	0	0	0	500
	P 2	0	0	0	0
	P 3	0	0	0	0

Table 1. Inversibility test for a random polynomial matrix generator with different N, P and M in 500 test cases

$\operatorname{RES}_{(Pk,...,Pk)} \circ \theta_i(\boldsymbol{H}(\boldsymbol{z})) = 0$ for some $i = 0, ..., M$. The $\operatorname{RES}_{(Pk,...,Pk)}$ and $t_j^{(i)}$ are polynomials, so is $\operatorname{RES}_{(Pk,...,Pk)} \circ \theta_i$. Similar to Theorem 2, we can show $\operatorname{RES}_{(Pk,...,Pk)} \circ \theta_i$ is not a zero function. Now let

$$F = F_1 \times \prod_{i=0}^{M} \operatorname{RES}_{(Pk,...,Pk)} \circ \theta_i. \qquad (1)$$

By previous discussion, $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for all Laurent polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$. This shows that if $N - P < M$, then a polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically Laurent polynomial left invertible.

Theorem 5 *If $N - P < M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically polynomial left noninvertible.*

Proof Similar proof from Theorem 3.

2.4. Simulation and Applications

From Table 1, we used a random polynomial matrix generator to generate polynomial matrices with each entry of degree less than or equal to 4 and the random coefficients are from 1 to 100. In each value of N, P and M, we ran 500 samples to test the inversibility. We found out that they agreed with our theorems. These theorems lead to some applications. For image deconvolution from multiple FIR blur filters, Harikumar and Bresler in [6] show that perfect reconstruction is almost surely, when there are at least three channels. Since image is two dimension (i.e. $M = 2$) and the downsampling rate is just one (i.e. $P = 1$), by Theorem 3, we know that the perfect reconstruction is almost surely if number of channels is greater than two (i.e. $N \geq 3$). Therefore Harikumar and Bresler's image deconvolution is a special case of our main theorem. Another application is that we can have an alternative approach in designing multidimensional filter banks. We can freely design the analysis side first such that it satisfies the condition (i.e. $N - P \geq M$). Then , by Theorem 3 and Lemma 1, we can almost surely find a perfect reconstruction inverse for the synthesis polyphase matrix.

2.5. Fast Computation of Left Inverse

Another application is that we can improve the Laurent polynomial inverse algorithm [11]. Since if $N - P \geq M$ and $\boldsymbol{H}(\boldsymbol{z})$ is a polynomial matrix, then $\boldsymbol{H}(\boldsymbol{z})$ is generically Laurent polynomial left invertible by theorem 3. However, at the same time, the $\boldsymbol{H}(\boldsymbol{z})$ is generically polynomial left invertible by Theorem 2. Therefore instead of apply the Laurent Polynomial Inverse Algorithm in [11], we should simply apply the Polynomial Inverse Algorithm in [11] which is less expensive in term of time and storage. For a convenience sake, we denote our Laurent Polynomial Inverse Algorithm 2 in [11] to be LPIA and denote our Polynomial Inverse Algorithm 1 in [11] to be PIA.

Algorithm 1 (Faster Version) The computational algorithm for a Laurent polynomial left inverse matrix.
Input: $N \times P$ *Laurent polynomial matrix* $\boldsymbol{H}(\boldsymbol{z})$ *with* M *variables.*
Output: $P \times N$ *Laurent polynomial matrix* $\boldsymbol{G}(\boldsymbol{z})$*, if it exists.*
1. Multiply $\boldsymbol{H}(\boldsymbol{z})$ *by a common monomial* $\boldsymbol{z}^{\boldsymbol{l}}$ *such that* $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$ *are polynomial matrix.*
2. Call PIA with the input $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$*.*
3. If the output of PIA is $\boldsymbol{J}(\boldsymbol{z})$*, then output* $\boldsymbol{z}^{-\boldsymbol{l}}\boldsymbol{J}(\boldsymbol{z})$*.*
4. Otherwise call LPIA.

Example 1 *Compare the processing time between LPIA and Algorithm 1. Let* $\boldsymbol{H}(z_1, z_2)$

$$= \begin{pmatrix} 4z_1 & 7{z_1}^{-1}z_2^2 + 2 + 10{z_1}^{-1} \\ 1 + 10{z_1}^{-1} & 10z_1 + 3z_2 \\ 7z_1 + 9z_2 + 10{z_1}^{-1}z_2 + 10{z_1}^{-1} & 0 \\ 8{z_1}^{-1}z_2^2 + 10 + 4{z_1}^{-1} & 6{z_1}^{-1}z_2^2 \end{pmatrix}$$

be a Laurent polynomial matrix. Then we found out that the run time of LPIA and Algorithm 1 is 0.23 *sec and* 0.06 *sec respectively for using a desktop PC. This agrees that Algorithm 1 is faster than LPIA in this example.*

3. CONCLUSION

We shows that there is a sharp phase transition on the invertibility depending on the size and dimension of a given Laurent polynomial matrix. Specifically when $N - P \geq M$, the $N \times P$ polynomial (resp. : Laurent polynomial) of M-variate matrix is generically invertible; whereas when $N - P < M$, the matrix is generically noninvertible. Using this sharp phase transition property, we develop a fast algorithm to compute a particular left inverse for a given Laurent polynomial matrix.

These results suggest an alternative approach in designing multidimensional filter banks by freely generating filters for the analysis side first. If we allow an amount of oversampling (i.e. $N - P \geq M$), then we can almost surely find a perfect reconstruction inverse for the synthesis polyphase matrix. These results also have potential applications in multidimensional signal reconstruction from multichannel filtering and sampling.

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