OPTIMAL VARIABLE FRACTIONAL DELAY FILTERS IN TIME-DOMAIN L-INFINITY NORM

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ABSTRACT

This paper presents an efficient implementation of the variable fractional-delay (VFD) filter, which is optimal in the time-domain L-infinity norm. The proposed filter has two stages. The first consist in computing the conventional Lagrange interpolator from the signal samples, but weighted using a set of fixed coefficients. And the second consists in multiplying the result of the previous step by a smooth function, which can be well approximated by a polynomial. The paper includes a numerical evaluation of this interpolator and a low-complexity, low-latency implementation based on multiplications.

Index Terms— Interpolation, Bandlimited signals, FIR digital filters, Timing circuits.

1. INTRODUCTION

The design of efficient variable fractional-delay (VFD) filters is the subject of numerous papers [1, 2, 3, 4]. The key problem in this kind of filters is to achieve the desired frequency response for any fractional delay with minimum complexity. The usual solution approach is based on the Farrow structure in any of its variants; (see [2, 3, 4] and references therein). Recently, an alternative design method has been presented in [4], which can be viewed as a modification of the conventional Lagrange interpolator; (see [5, Sec. II]). Basically, it is based on computing the Lagrange interpolant of the product of the signal with a fixed function, and then solving for the signal value. The purpose of this paper is to show that the optimal interpolator in the time-domain L^∞ norm, which was derived in [6], is also a modified Lagrange interpolator of the type in [4], and to propose an efficient VFD filter implementation based on it, which is division-free. The use of the time-domain L^∞ norm is motivated by the fact that bounded band-limited signals are, actually, polynomials but of infinite degree. This leads to simple and efficient interpolation structures. Besides, if the interpolation error is bounded in the norm L^∞, then it is also bounded in any norm L^p, $1 \leq p < \infty$.

The interpolation problem and the optimal solution in the L^∞ norm are stated in the next section. Then, in Sec. 3 it is shown that this optimal solution is actually a *weighted* Lagrange interpolator. Sec. 4 presents the efficient VFD filter based on the interpolator in Sec. 3. Finally, its performance is analyzed is Sec. 5.

2. PROBLEM STATEMENT AND OPTIMAL SOLUTION IN TIME-DOMAIN L^∞-NORM

Consider a signal $\mathrm{s}(t)$ with spectrum in $[-B/2, B/2]$ which is sampled with period T following $BT < 1$. The usual method to introduce a fractional time shift v in the sequence of samples $\mathrm{s}(pT)$ is based on an FIR filter whose coefficients depend on v. More precisely, the n-th sample $\mathrm{s}(nT)$ is time-shifted using the formula

$$\mathrm{s}((n+v)T) \approx \sum_{p=-P}^{P} \mathrm{s}((n+p)T)\mathrm{a}_p(vT), \qquad (1)$$

where the shift follows $-1/2 \leq v < 1/2$. For simplicity, this expression has been written in a non-causal way relative to the index n, i.e., the last sample available at the filter input has index $n+P$ and not index n. This last index is irrelevant for the selection of $\mathrm{a}_p(vT)$, given that it can be assumed the interpolated signal is $\mathrm{z}(t) \equiv \mathrm{s}(nT+t)$. Thus, for the selection of $\mathrm{a}(vT)$, it is allowed to set n to zero and replace vT with the time variable t, $|t/T| \leq 1/2$,

$$\mathrm{s}(t) \approx \sum_{p=-P}^{P} \mathrm{s}(pT)\mathrm{a}_p(t). \qquad (2)$$

The coefficients $\mathrm{a}_p(t)$ in this formula are usually optimized using frequency domain criteria, i.e., by minimizing the interpolation error for the set of phasors $\mathrm{e}^{j2\pi ft}$ with f in the band $[0, B/2]$. However, this kind of design does not give any bound on the interpolation error for the case in which $\mathrm{s}(t)$ cannot be regarded as a finite-energy signal. Besides, it does not suggest any efficient way to evaluate the coefficients $\mathrm{a}_p(v)$.

An alternative approach to this problem is to attempt to minimize the interpolation error uniformly in the time domain instead of in the frequency domain, assuming that $\mathrm{s}(t)$ is bounded by a constant A_s, $|s(t)| \le A_s$, but not necessarily of finite energy. Mathematically, this approach is much harder than the usual FIR filter design above but, luckily, it was already solved in [6]. This reference is not cited in the recent literature, despite containing fundamental results on this interpolation problem. In [6], it was shown that the optimal interpolator in the time-domain L^∞ norm is

$$s(t) \approx \sum_{p=-P}^{P} s(pT) \frac{\phi(t)}{\phi'(pT)(t-pT)} \quad (3)$$

where $\phi(t)$ is a specific band-limited signal with spectrum in $[-B/2, B/2]$. It was also demonstrated in [6] that $\phi(t)$ is closely related with a variational problem involving band-limited signals with zeros at the sampling points pT, $-P \le p \le P$. In short, let $\mathcal{S}$ denote the set of signals $s(t)$ fulfilling the following conditions:

- $s(t)$ is real and its spectrum lies in $[-B/2, B/2]$.
- $|s(t)| \le 1$ for any t.
- $s(pT) = 0$ for $-P \le p \le P$.

Then, for a fixed instant t', $t' \ne pT$, $-P \le p \le P$, there is a unique signal $b(t)$ in $\mathcal{S}$ whose value at t' is maximum over all signals in $\mathcal{S}$,

$$b = \arg\max_{s \in \mathcal{S}} s(t'). \quad (4)$$

Besides, it turns out that $b(t)$ is the same signal for any t' in $t_a < t' < t_b$, $t' \ne pT$, where t_a and t_b are two specific instants following $t_a < -PT$ and $t_b > PT$. The optimal kernel in (3) is then

$$\phi(t) = b'(t)L(t)/M(t), \quad (5)$$

where $M(t)$ is a polynomial of degree $2P$ with real roots

$$M(t) \equiv \prod_{p=-P}^{P-1} t - m_p, \quad (6)$$

Actually, m_p is the only root of $b'(t)$ in $[pT, (p+1)T]$.

Fig. 1 shows the signal $b(t)$ for $B = 0.8/T$ and $P = 8$. Note that since $b(t)$ is zero at the sampling points, this signal is not even detected by any interpolator whose input data is the set of samples at $t = pT$. Besides, it was demonstrated in [6] that $|b(t)|$ is a lower bound on the error of *any* interpolator whose input is the set of samples at pT, $-P \le p \le P$, and that this lower bound is attained by the interpolator in (3) with $\phi(t)$ given in (5). This is true for any t lying between t_a and t_b, where t_a is the largest t following $t < -PT$ and $|b(t)| = 1$, and t_b is the smallest t following $t > PT$ and $|b(t)| = 1$.

Fig. 2 also depicts $b(t)$, but enlarging a shorter time range. Note that the maxima of $|b(t)|$ increase only slightly as $|t|$ departs from zero. Thus, the interpolator delivers accurate approximations to the input signal for time ranges with length much larger than T. This is in contrast with the usual FIR designs in which the variation range of the fractional time shift is T.

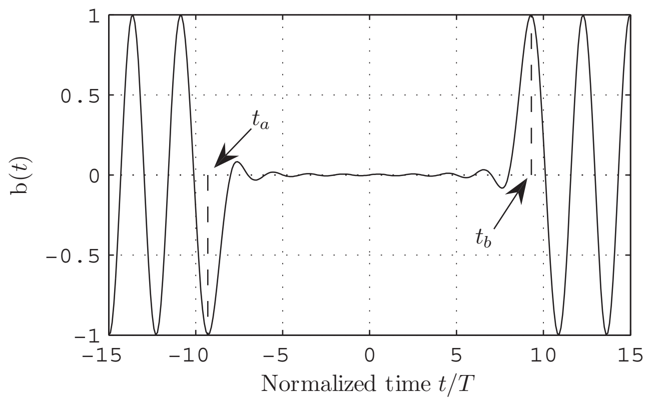

Fig. 1. Signal maximizing the interpolation error, $b(t)$.

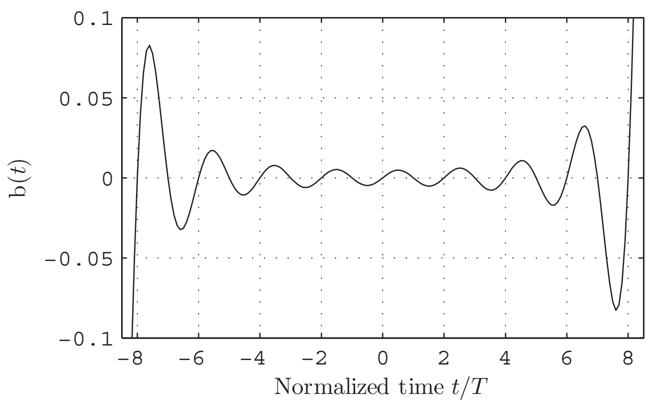

Fig. 2. Signal maximizing the interpolation error, $b(t)$.

3. THE OPTIMAL INTERPOLATOR VIEWED AS A WEIGHTED LAGRANGE INTERPOLATOR

The optimal interpolator in the previous section can be viewed as the conventional Lagrange interpolator, but applied to the product of $s(t)$ with a specific signal, [5, Sec. 2]. To see this note that (3) has the form of the conventional Lagrange interpolator with $\phi(t)$ in place of the Lagrange polynomial $L(t)$,

$$L(t) \equiv \prod_{p=-P}^{P} t - pT. \quad (7)$$

Actually, since $\phi(t)$ has a zero at pT for $-P \le p \le P$, it is possible to factor out $L(t)$, i.e.,

$$\phi(t) = L(t)g(t), \quad (8)$$

where the spectrum of $g(t)$ also lies in $[-B/2, B/2]$. Now, since $\phi'(pT) = L'(pT)g(pT)$ due to $L(pT) = 0$, Eq. (3) can be written as

$$s(t) \approx g(t) \sum_{p=-P}^{P} \frac{s(pT)}{g(pT)} \frac{L(t)}{L'(pT)(t-pT)}. \quad (9)$$

This formula shows that the conventional Lagrange interpolator is *optimal*, provided that its input samples are $\mathrm{s}(pT)/\mathrm{g}(pT)$ instead of $\mathrm{s}(pT)$, where $\mathrm{g}(pT)$ are fixed parameters, and provided the result is multiplied by the fixed function $\mathrm{g}(t)$.

4. EFFICIENT IMPLEMENTATION OF THE INTERPOLATOR WITH NO DIVISIONS

The formula in (9) is a Lagrange interpolator that is applied to the samples $\mathrm{s}(pT)/\mathrm{g}(pT)$ and then weighted using $\mathrm{g}(t)$. Thus, any implementation of the Lagrange interpolator can be modified to deliver (9), like the variable fractional delay (VFD) filter designs in [3, 4]. Conversely, the results in the sequel are applicable to the conventional Lagrange interpolator simply by setting $\mathrm{g}(t) = 1$. The implementation of (9) that involves the smallest number of arithmetic operations seems to be that in [4]. In this reference the factor $\mathrm{g}(t)$ in (9) is approximated using the square of a polynomial, which can be efficiently evaluated using Horner's algorithm. Regarding the summation in (9), the implementation in [4] involved one division per summation coefficient. However, the summation can be computed using only multiplications as shown in the sequel. For simplicity, the dependence on t will be omitted in the symbols defined.

The formula in (9) can be written as

$$\mathrm{s}(t) \approx \mathrm{g}(t) \sum_{p=-P}^{P} \mathrm{s}(pT) \cdot \frac{1}{\mathrm{g}(pT)\mathrm{L}(pT)} \prod_{k=-P}^{p-1} (t - kT) \prod_{r=p+1}^{P} (t - rT). \quad (10)$$

Next, the factor $\mathrm{g}(t)$ can be approximated by the square of a polynomial in t, denoted $\tilde{\mathrm{g}}$, using the method in [4]. The factor

$$w_p \equiv [\mathrm{g}(pT)\mathrm{L}(pT)]^{-1} \quad (11)$$

is constant in t and can therefore be pre-computed. Finally, the products in (10),

$$b_p \equiv \prod_{k=-P}^{p-1} (t - kT), \quad c_p \equiv \prod_{r=p+1}^{P} (t - rT). \quad (12)$$

can be recursively computed using the formulas

$$b_p = \begin{cases} 1 \text{ if } p = -P \\ b_{p-1}(v - p + 1) \text{ if } -P < p \leq P \end{cases} \quad (13)$$

and

$$c_p = \begin{cases} 1 \text{ if } p = P \\ c_{p+1}(v - p - 1) \text{ if } -P \leq p \leq P - 1 \end{cases} \quad (14)$$

In summary, the interpolator in (10) can be written as

$$\mathrm{s}((n+v)T) \approx \tilde{\mathrm{g}} \sum_{p=-P}^{P} \mathrm{s}(pT) w_p b_p c_p, \quad (15)$$

and Eqs. (10) to (14) show that the computational burden is $\mathrm{O}(P)$. More precisely, if $Q \approx P/3$, the number of multiplications per interpolated value is $10.3P$. Thus, the interpolator roughly involves 5.15 multiplications per coefficient. The latency of this procedure is fixed by the recursive evaluation of (13) and (14). Nevertheless, these coefficients can be obtained with latency $1 + \lceil \log_2 P \rceil$ using the method in the following sub-section.

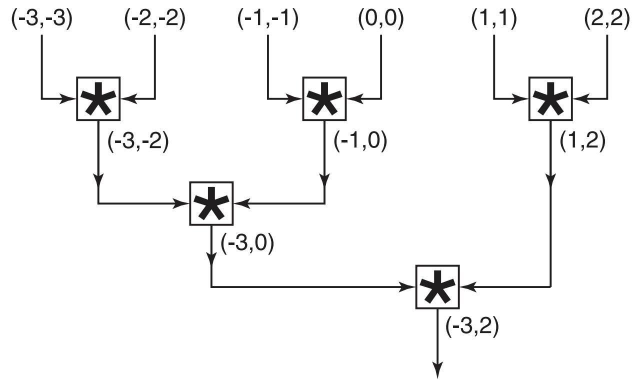

Fig. 3. .Tree structure for the computation of $(-P, P-1)$ with $P = 3$ The asterisk blocks implement the operation in (17).

4.1. Low-latency computation of b_p and c_p

For two integers p and q, $p \leq q$, let (p, q) denote the sequence of products $b_p, b_p b_{p+1}, \ldots, b_p b_{p+1} \ldots b_q$. Also, let $(p, q)[n]$ denote the n-th element of the sequence (p, q), that is,

$$(p, q)[n] = \prod_{k=p}^{p+n-1} b_k. \quad (16)$$

If $p \leq r < q$ the sequence (p, q) can be constructed from (p, r) and $(r+1, q)$ by concatenating (p, r) with $(r+1, q)$ but multiplied by the last element of (p, r), i.e., $(p, r)[r - p + 1]$. If the braces "$\{\cdot, \cdot\}$" denote concatenation, this operation can be concisely written as

$$(p, q) = \{(p, r), (p, r)[r - p + 1] \cdot (r + 1, q)\}, \quad (17)$$

where the product "$\cdot$" of the scalar $(p, r)[r - p + 1]$ with $(r + 1, q)$ is to be performed element-wise. The relationship (16) suggests the tree structure shown in Fig. 3 for the computation of $(-P, P - 1)$ with low latency. In this figure the asterisk block denotes the application of (16). This way it is possible to obtain all coefficients b_p with latency $1 + \lceil \log_2 P \rceil$.

The computation of c_p would be done in the same way.

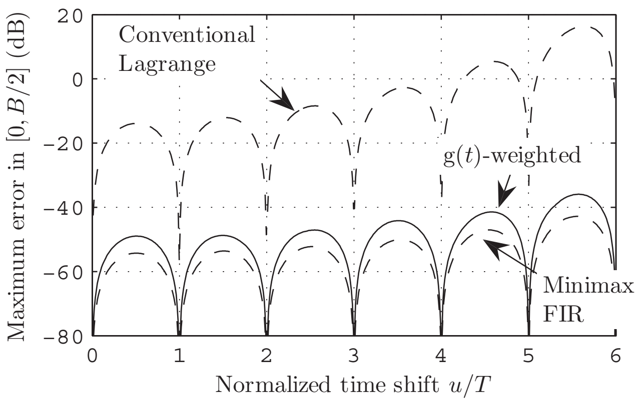

Fig. 4. Maximum error $\epsilon(t)$ in band $[0, B/2]$ versus time shift t for the conventional Lagrange and minimax FIR interpolators, and for the interpolator in (9), [g(t)-weighted interpolator].

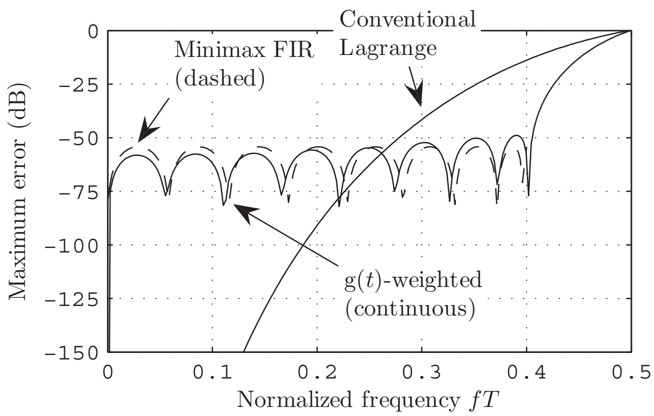

Fig. 5. Maximum error for time shift in $-T/2 \leq t < T/2$ versus frequency.

5. ERROR PERFORMANCE

For a given interpolator, let $\epsilon(t)$ denote the error function

$$\epsilon(t) \equiv \max_{f\in[-B/2,B/2]} |\mathrm{e}^{j2\pi ft} - \phi(f,t)|, \tag{18}$$

where $\phi(f,t)$ is the value delivered by the interpolator when its input is $\mathrm{s}(t) = \mathrm{e}^{j2\pi ft}$. Fig 4 shows $\epsilon(t)$ for the interpolator in (9), [g(t)-weighted], for the minimax FIR interpolator for each specific time shift t, and for the conventional Lagrange interpolator, ($B = 0.8/T$, $P = 8$). The curve for the interpolator in (9) is at most 7 dB above that of the minimax FIR interpolator. Observe the significant effect of weighting the Lagrange interpolator using the function g(t).

Next, let $\delta(f)$ denote the maximum error error in the interpolation of $\phi(f,t)$ with $|t/T| \leq 1/2$,

$$\delta(f) \equiv \max_{|t/T|\leq 1/2} |\mathrm{e}^{j2\pi ft} - \phi(f,t)|. \tag{19}$$

Fig. 5 shows $\delta(f)$ for the previously-mentioned interpolators. The effect of the weighting is to turn the maximally-flat response of the Lagrange interpolator into an almost equi-ripple response.

6. CONCLUSIONS

In this paper, the performance and the efficient implementation of the optimal interpolator in the time-domain L^∞ norm has been analyzed. It has been shown that this interpolator can be implemented by means of a simple modification of the conventional Lagrange interpolator. Its performance is close to that of the minimax FIR interpolator for every possible time shift, and it transforms the maximally-flat response of the Lagrange interpolator into an equi-ripple one, just by weighting the signal samples using fixed coefficients, plus a final scaling using a function of the time shift. Finally, an efficient evaluation procedure with low latency and based only on multiplications has been presented.

7. REFERENCES

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