SINGLE AND MULTICHANNEL SAMPLING OF BILEVEL POLYGONS USING EXPONENTIAL SPLINES

Hojjat Akhondi Asl and Pier Luigi Dragotti

Imperial College London
Department of Electrical and Electronic Engineering
hojjat.akhondi-asl@imperial.ac.uk, p.dragotti@imperial.ac.uk

ABSTRACT

In this paper we present a novel approach for sampling and reconstructing any K-sided convex and bilevel polygon with the use of exponential splines [1]. It will be shown that with K+1 projections we are able to perfectly reconstruct a K-sided bilevel polygon from its samples. We will also investigate the multichannel sampling scenario, consisting of a bank of E-spline filters, each with a different delay parameter compared to the reference signal. We show how by retrieving the delay parameters, we can symmetrically sample and reconstruct a given bilevel polygon using exponential splines.

***Index Terms*—** Bilevel Polygon, Multichannel Sampling, Exponential Splines, FRI Signals, Projection-Slice Theorem

1. INTRODUCTION

Sampling theory plays a fundamental role in modern signal processing and communications. We all know that signals with bandlimited bandwidth can be sampled and reconstructed perfectly with Shannon's famous sampling theorem. Recently, it was shown [2, 3] that it is possible to sample and perfectly reconstruct some classes of non-bandlimited signals. Signals that can be reconstructed using this framework are called signals with Finite Rate of Innovation (FRI) as they can be completely defined by a finite number of parameters.

The results of [2, 3] apply only to 1-D signals while extensions to the multidimensional case were considered in [4, 5]. Maravic et al [5] considered some 2-D FRI signals, such as 2-D stream of Diracs and bilevel polygons using the Sinc and Gaussian sampling kernel. Shukla et al [4] proposed an algorithm, from the theory of complex moments, for sampling bilevel polygons with the use of B-splines as the sampling kernel. In [3], it was shown that exponential splines (E-splines), another important family of kernels, can be used as the sampling kernel to sample 1-D FRI signals. However, thus far, E-splines have not been considered to sample multidimensional signals.

The contribution of this paper is two-fold; First, we present an algorithm for sampling and perfectly reconstructing bilevel polygons using E-spline sampling kernels. Then we consider the case where a bank of E-spline filters is used to acquire a 2-D signal, where each filter has access to a delayed version of the input signal. It will be shown that, by registering the delay parameters from the relevant features of the image samples, it is possible to synchronize the different channels exactly so that perfect reconstruction of the original polygonal image and its delayed versions is achieved at the receiver. It is important to mention that generally, using a multichannel system to acquire a scene makes the system more robust to noise and sensor failure.

The paper is organised as follows: In Section II we will briefly discuss the sampling setup needed for sampling 2-D FRI signals. In Section III we will introduce our novel approach for sampling bilevel polygons using E-splines. In Section IV we will describe our method for sampling bilevel polygons in a multichannel system.

2. PRELIMINARIES AND PROBLEM SETUP

A general 2-D sampling setup for FRI signals is shown in Figure 1. Here, $g(x, y)$ represents the input signal, $\varphi(x, y)$ the sampling kernel, $s_{j,k}$ the samples and Tx, Ty are the sampling intervals. From the setup shown in Figure 1, the

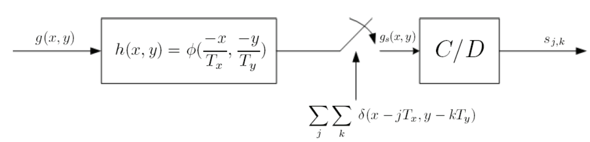


Fig. 1. 2-D sampling setup

samples $s_{j,k}$ are given by:

$$s_{j,k} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\, \phi(\frac{x}{T_x} - j, \frac{y}{T_y} - k)\, dx\, dy \quad (1)$$

where the kernel $\varphi(x, y)$ is the time reversed version of the filter response. $\varphi(x, y)$ can easily be produced by the tensor product between $\varphi(x)$ and $\varphi(y)$, that is $\varphi(x, y) = \varphi(x) \otimes \varphi(y)$. As mentioned before, $\varphi(x, y)$ is chosen to be an exponential reproducing kernel. The notion of exponential reproducing kernels is quite recent and were developed by Unser et al [1]. A function $\hat{\beta}_{\vec{\alpha}}(\omega)$ with Fourier transform

$$\hat{\beta}_{\vec{\alpha}}(\omega) = \prod_{n=0}^{N} \frac{1 - e^{\alpha_n - j\omega}}{j\omega - \alpha_n}$$

is called E-spline of order N where $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ can be real or complex. The produced spline has a compact support and can reproduce any exponential in the subspace spanned by $(e^{\alpha_0 t}, e^{\alpha_1 t}, \ldots, e^{\alpha_N t})$ which is obtained by successive convolutions of lower order E-splines ((N+1)-fold convolution). Exponential spline kernels can therefore reproduce, with their shifted versions, real or complex exponentials. That is, in 2-D form, any kernel satisfying:

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}^{m,n} \varphi(x-j, y-k) = e^{\alpha_m x} e^{\beta_n y} \tag{2}$$

is an E-spline for a proper choice of the coefficients $c_{j,k}^{m,n}$ which can be found numerically [3]. Here, $m = 0, 1, \ldots, M$, $n = 0, 1, \ldots, N$, $\alpha_m = \alpha_0 + m\lambda_1$ and $\beta_n = \beta_0 + n\lambda_2$.

Before going any further, let us introduce an interesting property here: If we call $\tau_{m,n}$ to be:

$$\tau_{m,n} = \sum_j \sum_k c_{j,k}^{m,n} \, s_{j,k} \tag{3}$$

then by expanding $s_{j,k}$ and replacing the above property (assuming $T_x = T_y = 1$ for simplicity), we will obtain the exponential moments of the signal, i.e. :

$$\tau_{m,n} = < g(x,y), \sum_j \sum_k c_{j,k}^{m,n} \, \phi(x-j, y-k) > \tag{4}$$

$$= < g(x,y), e^{\alpha_m x} e^{\beta_n y} > \tag{5}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \, e^{\alpha_m x} e^{\beta_n y} \, dx \, dy \tag{6}$$

In the case of purely imaginary exponentials, we will have the discrete Fourier coefficients of the signal $g(x,y)$, that is:

$$\tau_{m,n} = G[\alpha_m, \beta_n] \tag{7}$$

where $G(u,v)$ represents the Fourier transform of the signal $g(x,y)$. This property is very handy when we are reconstructing FRI sampled signals using E-splines. In the next section we will present our work on how to sample bilevel polygons using E-splines.

3. A SAMPLING THEOREM FOR BILEVEL POLYGONS USING EXPONENTIAL SPLINES

Consider a non-intersecting, convex and bilevel K-sided polygon with vertices at points (x_n, y_n) where $n = 1, 2, \ldots, K$. The described polygon can be uniquely specified by its K vertices and since each vertex can be described by its x_n and y_n locations, then the polygon has a finite rate of innovation equal to 2K.

Reconstruction of sampled bilevel polygons with polynomial reproducing kernels have been looked at in [4, 5], however sampling methods for bilevel polygons with exponential splines have not been considered yet. In this section we will present our approach for reconstructing convex, bilevel polygons with the use of E-splines. At first we will see how the Fourier transform of bilevel polygons is represented.

S.Lee and R.Mittra [6] derived a general formula for the Fourier transform of any K-sided bilevel polygon where they showed that the Fourier transform is directly related to the location of the polygon's vertices (x_n, y_n) and expressed as:

$$G(u,v) = \sum_{n=1}^{K} e^{j(ux_n + vy_n)} \, \frac{p_{n-1} - p_n}{(u + p_{n-1}v).(u + p_n v)} \tag{8}$$

Here p_n represent the gradients of the polygonal lines. The derived equation closely follows the 2-D harmonic retrieval data model [7], but since the equation has a time-varying amplitude, 2-D harmonic retrieval methods cannot simply be applied to retrieve the locations of the vertices of the polygon. With the use of Radon transform and the projection-slice theorem [8] we obtain an algorithm for retrieving the locations of the vertices of bilevel polygons from their samples. From projection-slice theorem we know that there is a direct relationship between the 2-D Fourier transform and the 1-D Fourier transform of a Radon projection i.e. :

$$G(\omega\cos(\theta), \omega\sin(\theta)) = \hat{R}_g(\omega, \theta) \tag{9}$$

where $G(u,v)$ is the Fourier transform of $g(x,y)$ and $\hat{R}(w,\theta)$ is the 1-D Fourier transform of the Radon transform of $g(x,y)$. With the help of this mapping, we can transform the Fourier coefficients of bilevel polygons, obtained from E-spline sampling kernel (see equation (3)), to the Radon domain, as follows:

$$\hat{R}_g(\omega,\theta) \times \omega^2 = \sum_{n=1}^{N} a_n \times e^{j\omega(cos(\theta)x_n + sin(\theta)y_n)} \tag{10}$$

where a_n is : $\frac{p_{n-1}-p_n}{(cos(\theta)+p_{n-1}sin(\theta)).(cos(\theta)+p_n sin(\theta))}$. Let us introduce $S(\omega,\theta) = \hat{R}_g(\omega,\theta) \times \omega^2$ to present the new mapped equation. Thus, the above equation can be rewritten as:

$$S(\omega,\theta) = \sum_{n=1}^{N} a_n \times e^{j\omega(cos(\theta)x_n + sin(\theta)y_n)} \tag{11}$$

At $\omega = 0$, $S(\omega,\theta) = 0$ so the minimum required spline order can be decreased by 1 as the first data sample is always zero. The mapped equation, at different projections, follows the data model used for the 1-D harmonic retrieval data model exactly, that is:

$$G(\omega,\theta) = \sum_{n=1}^{N} a_n . e^{j\omega z_n} = \sum_{n=1}^{N} a_n . (u_n)^{\omega} \tag{12}$$

where a_n is defined as before, $z_n = cos(\theta)x_n + sin(\theta)y_n$ and $u_n = e^{jz_n}$. Thus, by using a 1-D harmonic retrieval method, for example the Prony's method, we can find all the a_n's and the z_n's. By backprojecting z_n's according to their θ we are able to retrieve some information about the polygon's vertices. The question here is that how many projections will definitely guarantee us to perfectly reconstruct the polygon? Let us assume that the function $g(x, y)$ contains K Diracs, then, as Maravic points out in her paper [9], K+1 projections will entirely specify the signal, i.e. points that have K+1 line intersections from the back-projections correspond to the K Diracs. Any K-sided convex and bilevel polygon is completely specified by the location of its K vertices. If we think of the K vertices as Diracs then K+1 projections will guarantee us to perfectly retrieve the vertices of the bilevel polygon. By projections we mean line integrals at arbitrary angles $\tan^{-1}(\frac{n}{m})$, where m and n are the indices of the samples.

To reconstruct a set of K Diracs from its samples, we need at least 2K data points, thus a minimum 2-D spline order of 2K-1 is required. For bilevel polygons, as the first data sample is always zero, we need a minimum 2-D spline order of 2K-2 at each projection angle. Assuming that the input signal is sampled at a rate $T_x = T_y = T$ with an E-spline order of 2K-2, then 3 immediate projections will be available at the angles 0, 90 and 45 degrees. Since K $\geq$ 3, more projections will be needed, thus, a higher spline order is necessary for the retrieval of all the parameters z_n. The next immediate angles are at $tan^{-1}(2)$ and $tan^{-1}(\frac{1}{2})$, therefore for K = 3 and 4 for example, a minimum spline order of 2(2K-2) = 8 and 12 is required respectively. Thus, the minimum spline order required for a perfect reconstruction of a given K-sided bilevel polygon is $N = p.(2K - 2)$ where $p = max(m, n)$ needed in order to produce at least K+1 projections. From the K+1 projections, by using Prony's method, all the set of parameters are retrieved, then normalized by dividing to $\sqrt{(m^2 + n^2)}$. Finally all the retrieved parameters are back-projected. Points that have K+1 line intersections correspond to the K vertices of the polygon. Figure 2 shows an example of the sampling process where the input signal, corresponding samples and the reconstructed signal are shown.

4. MULTICHANNEL SAMPLING OF BILEVEL POLYGONS

In this section we investigate the scenario of multichannel sampling of bilevel polygons. A model of a multichannel system is shown in Figure 3 where the bank of E-spline filters $\varphi_1, \varphi_2, \ldots, \varphi_{N-1}$ receive different delayed versions of the original image $g_0(x, y)$. Here, the delays are denoted by $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{N-1}$.

Baboulaz [10] has looked at the case of multichannel sampling of a stream 1-D Diracs using exponential splines, with simple translation being the delay or the transformation parameter, which he proves that, unlike the polynomial reproducing kernels, we can truly distribute the acquisition of FRI signals with kernels reproducing exponentials. The reason is that, exponential splines can offer different kernels with the same order due to the arbitrary choice of the parameter λ in $\alpha_m = \alpha_0 + m\lambda$. The method discussed in [10] can be extended to the bilevel polygons' case where we only have translations x_0 and y_0 as the transformation parameters. The method is as follows:

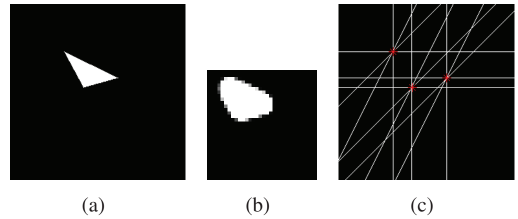


Fig. 2. (a) The original 3-sided polygon in a frame size of 256x256 (b) The 32x32 samples of the input signal (c) The reconstructed vertices with 3+1=4 back-projections, the crosses are the actual vertices of the polygon. [Not to scale]

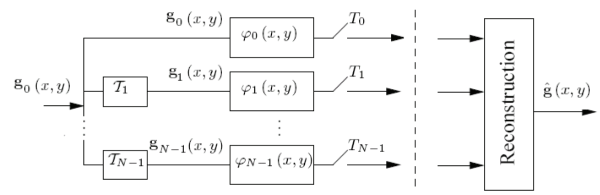


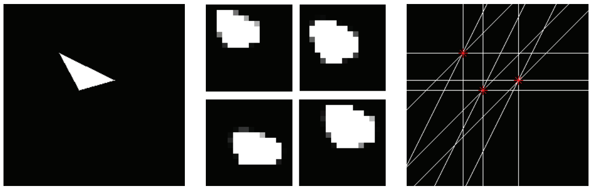
Fig. 3. A multichannel sampling scenario of FRI signals

Consider we have $g_0(x, y)$ as the reference image and $g_1(x, y)$ as its delayed version where $g_1(x, y) = g_0(x - x_0, y - y_0)$, which is just a translated version of the reference image. Bearing in mind the formula given in (6) for the exponential moments, with minimum spline order $[M, N]$ required, and leaving β_n intact, assume that one parameter is common between the sets α^0 and α^1, for example the first and the last parameter of these two sets, i.e. $\alpha^0_P = \alpha^1_0 = \alpha$. The exponential moments of the two sampled images at the corresponding parameters are:

$$\tau^0_{P,n} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\, e^{\alpha x} e^{\beta_n y}\, dx\, dy \tag{13}$$

$$\tau^1_{0,n} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x - x_0, y - y_0)\, e^{\alpha x} e^{\beta_n y}\, dx\, dy \tag{14}$$

which with simple rearranging leads to: $\frac{\tau^1_{0,n}}{\tau^0_{P,n}} = e^{\alpha x_0}\ e^{\beta_n y_0}$. By taking logarithms on both sides we will obtain a system of simple linear equations which we can solve for x_0 and y_0 using matrix equations. Therefore if one of the set of the parameters are common between the two acquisition devices then, not only we can exactly retrieve the shifts x_0 and y_0 but we



(a) (b) (c)

Fig. 4. (a) The reference image in a frame size of 256x256 (b) The 16x16 samples of the reference image (top-left) and the 16x16 samples of all other translated images with shifts $[x_0, y_0] : [50, 10]; [70, 70]; [20, 70]$ (c) The reconstructed vertices with 3+1=4 back-projections, the crosses are the actual vertices of the polygon. [Not to scale]

also can easily produce the rest of the samples of both images all together. This means that, by estimating the shifts, we can produce the higher order moments of the reference image from the lower order moments of its translated version and vice versa. It is important to mention that, in the case of having more than two sensors, we can just set one of the parameters of β_n common between the two sets. Finally we can run our algorithm described in section III to perfectly reconstruct the bilevel polygon.

Multichannel sampling aims to have sensors of lower order. Since we need less samples from each sensor, the support of the corresponding sampling kernels are also reduced. As an example, assume that we have multichannel system with 4 filter banks, where the reference image is a bilevel triangle in a frame size of 256x256, and the delayed images are a 2-D translated version of the reference image. If we want reconstruct each image independently, as was shown in Section III a 2-D spline order of $[M, N] = [8, 8]$ is required for each image, that is all the $\tau^i_{0:8,0:8}$ are required, but since we can sample the images symmetrically, the spline orders needed for each image can be reduced. Figure 4 illustrates an example of this scenario. α_3 and β_3 are chosen to be common between the set of parameters, thus we only have the following exponential moments from all the 4 filters: $\tau^0_{0:3,0:3}$, $\tau^1_{3:8,0:3}$, $\tau^2_{0:3,3:8}$ and $\tau^3_{3:8,3:8}$. In the Figure, the reference image, its 16x16 under-sampled version with an spline order of $[M, N] = [3, 3]$, the 16x16 samples of all the other under-sampled images with shifts $[x_0, y_0] : [50, 10]; [70, 70]; [20, 70]$ and spline orders $[M, N] = [5, 3]; [3, 5]; [5, 5]$ and the reconstructed reference image are all shown. Other images from other sensors could also be reconstructed but only the the reconstructed reference image is shown in the Figure.

5. CONCLUSION

In this paper we showed that with the use of projection-slice theorem and K+1 projections, a K-sided bilevel polygon can be perfectly reconstructed from its samples. We also showed that, by retrieving the delay parameters from the relevant features of image samples, we can symmetrically sample and reconstruct the reference image in a multichannel filter bank system. Investigating more complicated delay parameters, such as scaling, rotation and translation all-together and also examining the proposed sampling schemes under noisy conditions is an immediate future work for our research in this area.

6. REFERENCES

[1] M. Unser and T. Blu, "Cardinal Exponential Splines: Part I - Theory and Filtering Algorithms", IEEE Transactions on Signal Processing, vol. 53, pp. 1425, 2005.

[2] M. Vetterli, P. Marziliano and T. Blu, "Sampling Signals with Finite Rate of Innovation", IEEE Transactions on Signal Processing, vol. 50, p. 1417-1428, 2002.

[3] P.L. Dragotti, M. Vetterli and T. Blu, "Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon meets Strang-Fix", IEEE Transactions on Signal Processing, vol. 55, pp. 1741-1757, 2007.

[4] P. Shukla and P.L. Dragotti, "Sampling Schemes for Multidimensional Signals with Finite Rate of Innovation", IEEE Transactions on Signal Processing, vol. 55, pp. 3670-3686, 2007.

[5] I. Maravic and M. Vetterli, "Exact sampling results for some classes of parametric non-bandlimited 2-D signals," IEEE Transactions on Signal Processing, vol.52, no.1, pp. 175-189, 2004.

[6] S.W. Lee and R. Mittra, "Fourier transform of a polygonal shape function and its application in electromagnetics," IEEE Transactions on Antennas and Propagation, vol.31, no.1, pp. 99-103, 1983.

[7] F. Vanpoucke, M. Moonen, Y. Berthoumieu, "An Efficient Subspace Algorithm for 2-D Harmonic Retrieval," IEEE International Conference on Acoustics, Speech, and Signal Processing, vol.4, pp. 461-464, 1994.

[8] G. T. Herman, "Image Reconstruction from Projections: The Fundamentals of Computerized Tomography", Academic Press, New York, 1980.

[9] I. Maravic and M. Vetterli, "A sampling theorem for the Radon transform of finite complexity objects," IEEE International Conference on Acoustics, Speech, and Signal Processing, vol.2, pp. 1197-1200, 2002.

[10] L. Baboulaz, "Feature Extraction for Image Super-resolution using Finite Rate of Innovation Principles", PhD thesis, Imperial College London, 2008.