GENERATING MATRIX OF DISCRETE FOURIER TRANSFORM EIGENVECTORS

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ABSTRACT

This paper provides a novel method to obtain the eigenvectors of discrete Fourier transform (DFT), which are accurate approximations to the continuous Hermite-Gaussian functions (HGFs). The proposed method uses a generating matrix and an initial eigenvector. By multiplying the initial eigenvector with the generating matrix, we can derive a new eigenvector. Repeating this procedure we can acquire all the eigenvectors. Compare with the conventional $O(N^3)$ commutative matrix method, this new method can generate all the DFT eigenvectors with complexity reduced to $O(N^2 logN)$. The generating matrix can be further used to intensify the conventional commuting matrix. The simulation result shows that the Hermite-Gaussian like (HGL) eigenvectors of the strengthened commuting matrix outperform those of Santhanam's.

Index Terms─ Discrete Fourier transform, Hermite-Gauss functions, eigenvector

1. INTRODUCTION

The Fractional Fourier transform (FRFT) [1,2,3,8,9] has many possible applications in signal processing area, such as optimal filtering [1], data encryption [2], moving target indication via Radar system [3], etc. To develop a discrete version of the FRFT (DFRFT), recent efforts have focused on generating an orthogonal basis of eigenvectors for the DFT, since the FRFT has the same eigenvectors with the Fourier Transform, that is, Hermite-Gaussian functions. This can be done by developing a commutative matrix that shares a common basis of eigenvectors with the DFT. These commuting matrices includes the Dickinson-Steiglitz extended-tridiagonal matrix of the DFT [4], the Pei's new nearly tridiagonal matrix of the DFT [5], the Candan's matrix [6] and Santhanam's [7]. Although the almost tridiagonal matrix takes advantage of fast computation, the dense matrix like Candan's and Santhanam's can obtain HGL eigenvectors in higher accuracy and thus considered to be a better method since the eigenfunction of FRFT is Hermite-Gaussian function.

In this paper we propose a revolutionary method using generating matrix, whose time complexity is $O(N^2\log N)$ and the accuracy is comparable to the Santhanam's method, which can obtain the most accurate HGL eigenvectors. The generating matrix, like an operator, can transform an initial eigenvector into another eigenvector. By carefully choosing the generating matrix of DFT, we can transform an HGL eigenvector into another one, and thus we can acquire all the HGL eigenvectors by repeating this procedure.

The generating matrix is strongly connected to commutative matrix. In fact, the commutative matrix can be considered a special case of the generating matrix. So it is natural to use generating matrix to construct a new commutative matrix. We have experimented on many combinations to build the new commutative matrix and finally obtain one that can outperform the Santhanam's.

This paper is organized as followed. Section 2 will discuss the generating matrix of DFT and give an example. The new commutative matrix of DFT will be proposed in section 3. We will demonstrate the experiment result in section 4. The conclusion is given in section 5.

2. GENERATING MATRIX OF DFT EIGENVECTORS

Define the $N \times N$ DFT matrix $\mathbf{F}$ as

$$[\mathbf{F}]_{kn} = \frac{1}{\sqrt{N}} e^{\frac{-2\pi i}{N}kn}, 0 \le n, k \le N-1 \quad (1)$$

It is well known that

$$\mathbf{F}^2 = \mathbf{J}, \mathbf{F}^4 = \mathbf{I} \quad (2)$$

where

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \vdots & \dots & 0 & 1 & 0 \\ \vdots & 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \quad (3)$$

and $\mathbf{I}$ is the Identity matrix.

Suppose a matrix **A** satisfied that

$$\mathbf{JAJ}=\lambda\mathbf{A},\ \lambda \text{ is a constant.} \tag{4}$$

then $\mathbf{S_A}$, the generating matrix of DFT corresponding to **A** is defined as

$$\mathbf{S_A} = \lambda^{1/2}\mathbf{F}^{-1}\mathbf{AF} + \mathbf{A} \tag{5}$$

Property: If v is an eigenvector of DFT with eigenvalue λ_v, then ($\mathbf{S_A}\, v$) is an eigenvector of DFT with $\lambda^{1/2}\lambda_v$.

Proof:

$$\begin{aligned}\mathbf{F}(\mathbf{S_A}v) &= \mathbf{F}(\lambda^{1/2}\mathbf{F}^{-1}\mathbf{AF}+\mathbf{A})v\\ &=(\lambda^{1/2}\mathbf{AF}+\mathbf{FA})v\\ &=(\lambda^{1/2}\mathbf{AF}+\mathbf{F}^{-1}\mathbf{JAJF}^2)v && \text{(By (2))}\\ &=(\lambda^{1/2}\mathbf{AF}+\mathbf{F}^{-1}\lambda\mathbf{AF}^2)v && \text{(By (4))}\\ &=(\lambda^{1/2}\mathbf{A}+\mathbf{F}^{-1}\lambda\mathbf{AF})\mathbf{F}v\\ &=\lambda^{1/2}(\mathbf{A}+\lambda^{1/2}\mathbf{F}^{-1}\mathbf{AF})\mathbf{F}v\\ &=\lambda^{1/2}\mathbf{S_A}\lambda_v v\\ &=\lambda^{1/2}\lambda_v(\mathbf{S_A}v) && \textbf{Q.E.D.}\end{aligned}$$

Thus given any eigenvector v, multiplied it with $\mathbf{S_A}$, we can obtain a new eigenvector $\mathbf{S_A}v$. And of course we can use $\mathbf{S_A}v$ to generate $\mathbf{S_A}(\mathbf{S_A}v)$, another eigenvector of DFT. Thus all the eigenvectors can be represented by

$$\mathbf{S}_\mathbf{A}^n v,\ n=0,1,2,...,N-1 \tag{6}$$

While given an initial HGL eigenvector, in order to make the new eigenvector HGL, we have to choose **A** carefully. Luckily, there is a convenient way to select it, inspired by continuous Hermite-Gaussian function. Recall that continuous Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \tag{7}$$

$$\begin{aligned}H_n'(x) &= (-1)^n[2xe^{x^2}\frac{d^n}{dx^n}e^{-x^2} + e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2}]\\ &= 2xH_n(x) - H_{n+1}(x)\end{aligned} \tag{8}$$

and the continuous Hermite-Gaussian functions

$$\psi_n(x) = \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}e^{-x^2}H_n(x) \tag{9}$$

thus

$$\begin{aligned}\psi_n'(x) &= \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}\frac{d}{dx}[e^{-x^2}H_n(x)]\\ &= \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}[-2xe^{-x^2}H_n(x) + e^{-x^2}H_n'(x)]\\ &= -2x\psi_n(x) + \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}e^{-x^2}[2xH_n(x) - H_{n+1}(x)]\\ &= \frac{-\sqrt{2(n+1)}}{\sqrt{(n+1)!2^{n+1}\sqrt{\pi}}}e^{-x^2}H_{n+1}(x)\\ &= -\sqrt{2(n+1)}\psi_{n+1}(x)\end{aligned} \tag{10}$$

So if we have an eigenvector v approximated to n_{th} order Hermite-Gaussian function, and choose $\mathbf{S_A}$ close to a differentiator, we can obtain a new HGL eigenvector. It is hard to find a proper **A** so that $\mathbf{S_A}$ is close to a differentiator. In practice, we can choose $\lambda^{1/2}\mathbf{F^{-1}AF}$ instead, and treat **A** in (5) as error.

To make $\lambda^{1/2}\mathbf{F^{-1}AF}$ close to a differentiator, recall that in continuous Fourier transform we have:

$$\begin{aligned}&F\{\frac{d}{dt}x(t)\} = j\omega X(\omega)\\ &\Rightarrow \frac{d}{dt}x(t) = F^{-1}\{j\omega F\{x(t)\}\}\end{aligned} \tag{11}$$

So in the discrete case, we can choose:

$$\mathbf{A} = \mathbf{D} = \begin{pmatrix} 0 & & & & & & \\ & 1 & & & 0 & & \\ & & 2 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & -2 & \\ & & & & & & -1 \end{pmatrix} \tag{12}$$

And by (4), λ=-1. So we use $\lambda^{1/2}=-j$ since there is a minus sign in (10).

Example:

Assume N=3, and by (12)

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

And by (5), choosing $\lambda^{1/2}=-j$

$$\mathbf{S_D} = \lambda^{1/2}\mathbf{F}^{-1}\mathbf{DF} + \mathbf{D}$$

$$= \begin{pmatrix} 0 & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}} & 1 & \frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -1\end{pmatrix}$$

Suppose the initial eigenvector

$$v = \begin{bmatrix} 1+\sqrt{3}\\ 1\\ 1\end{bmatrix}$$

One can easily verify that

$\mathbf{S}v = \begin{bmatrix} 0\\ 2\\ -2\end{bmatrix}$, which is also an eigenvector of DFT with eigenvalue $-j$, and

$\mathbf{S}^2 v = \frac{1}{\sqrt{3}} \begin{bmatrix} -4 \\ 2+2\sqrt{3} \\ 2+2\sqrt{3} \end{bmatrix}$, which is also an eigenvector of DFT with eigenvalue -1.

In summary, the generating matrix of DFT is described in (5), and the special generating matrix that can produce HGL is $-j\mathbf{F}^{-1}\mathbf{DF}+\mathbf{D}$, where **D** is described in (12). Note that by conventional method, the complexity of obtaining all N HGL eigenvectors is $O(N^3)$ by applying QR algorithm on commutative matrix. Our new method, however, reduces the complexity to $O(N^2 \log N)$, since **D** is diagonal and we can use FFT to implement **F**. Thus calculating a new eigenvector costs $O(N \log N)$, so obtaining all N eigenvectors costs $O(N^2 \log N)$.

3. THE NEW COMMUTATIVE MATRIX OF DFT

In this section, we will focus on $\mathbf{S}= \mathbf{S}_\mathbf{D}=-j\mathbf{F}^{-1}\mathbf{DF}+\mathbf{D}$, where **D** is the diagonal matrix described in (12). We want to show that

$$\mathbf{FSS}^T = \mathbf{SS}^T\mathbf{F} \tag{13}$$

In other words, $\mathbf{SS}^T$ is a new commutative matrix of DFT. Thus the eigenvectors of DFT can be calculated by utilizing QR algorithm to $\mathbf{SS}^T$. Since the $\mathbf{SS}^T$ is a full matrix, the complexity of this method is $O(N^3)$, as the conventional ones. We will see in section 4 that the new matrix combined with the conventional one can have some improvement. To prove (13), we shall first show the following lemma:

Lemma 1:

$$\mathbf{S}^T = j\mathbf{F}^{-1}\mathbf{DF}+\mathbf{D} \tag{14}$$

Proof:

$$\mathbf{S}^T = (-j\mathbf{F}^{-1}\mathbf{DF}+\mathbf{D})^T$$
$$= -j\mathbf{F}^T\mathbf{D}^T(\mathbf{F}^{-1})^T+\mathbf{D}^T$$
$$= -j\mathbf{FDF}^{-1}+\mathbf{D}$$
$$= -j\mathbf{F}^{-1}\mathbf{JDJF}+\mathbf{D}$$
$$= j\mathbf{F}^{-1}\mathbf{DF}+\mathbf{D}$$

Q.E.D.

Lemma 2:

$$\mathbf{FSF}^{-1} = -j\mathbf{S} \tag{15}$$
$$\mathbf{FS}^T\mathbf{F}^{-1} = j\mathbf{S}^T \tag{16}$$

Proof:

$$\mathbf{FSF}^{-1} = \mathbf{F}(-j\mathbf{FDF}^{-1}+\mathbf{D})\mathbf{F}^{-1}$$
$$= (-j\mathbf{D}+\mathbf{FDF}^{-1})$$
$$= -j\mathbf{D}-\mathbf{F}^{-1}\mathbf{DF}$$
$$= -j(\mathbf{D}+(-j)\mathbf{F}^{-1}\mathbf{DF}) = -j\mathbf{S}$$
$$\mathbf{FS}^T\mathbf{F}^{-1} = \mathbf{F}(j\mathbf{FDF}^{-1}+\mathbf{D})\mathbf{F}^{-1}$$
$$= (j\mathbf{D}+\mathbf{FDF}^{-1})$$
$$= j\mathbf{D}-\mathbf{F}^{-1}\mathbf{DF}$$
$$= j(\mathbf{D}+j\mathbf{F}^{-1}\mathbf{DF}) = j\mathbf{S}^T$$

Q.E.D.

And we can now prove (12).

Proof:

$$\mathbf{FSS}^T\mathbf{F}^{-1} = \mathbf{FSF}^{-1}\mathbf{FS}^T\mathbf{F}^{-1}$$
$$= (-j)\mathbf{S}\times j\mathbf{S}^T$$
$$= \mathbf{SS}^T$$
$$\Rightarrow \mathbf{FSS}^T = \mathbf{SS}^T\mathbf{F}$$

Q.E.D.

4. EXPERIMENT RESULT

In this section we will compare the error norms for HGL eigenvectors. First we focus on the generating matrix **S**. Figure 1 illustrates the comparison between our method and the Santhanam's[7], while N=31. The initial eigenvector of our method is chosen to be the same as the 0th order of Santhanam's HGL eigenvector in convenience.

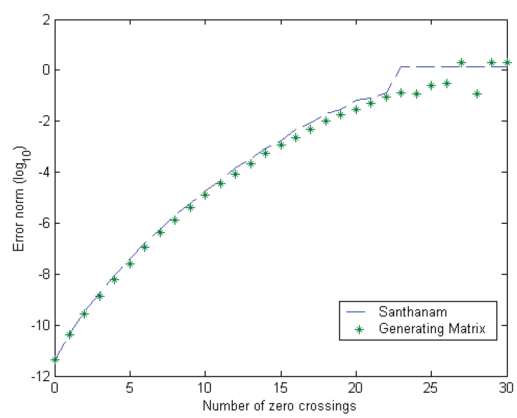

Fig. 1. Error norms for HGL eigenvectors of N=31

One can easily discover that our method is more accurate than the Santhanam's. But surprisingly, Santhanam's commutative matrix method surpass our method in accuracy while N>45, especially in low zero crossing region. Figure 2 illustrates this difference.

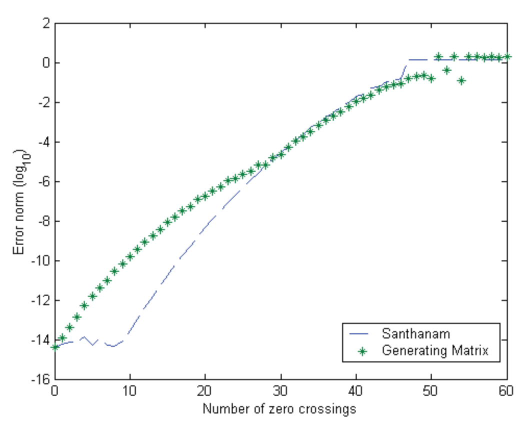

Fig. 2. Error norms for HGL eigenvectors of N=61

The reason of this phenomenon is that **S** is only an approximation of the differentiator. When N becomes larger, the error propagation becomes more severe.

Second, we use the new commutative matrix of DFT derived in section 3, added with the Santhanam's matrix. It is obvious that any linear combination of commutative matrix of DFT is still a commutative matrix of DFT. Figure 3 and 4 provide evidence that this new matrix outperform the Santhanam's.

5. CONCLUSION

In this paper we propose an innovative method to calculate the HGL eigenvectors, by using a generating matrix and an initial HGL eigenvector. The time complexity is deduced from $O(N^3)$, by implementing QR algorithm on commutative matrix, to $O(N^2 \log N)$, by applying FFT. The accuracy is comparable to the Santhanam's, and even better while the DFT length N is smaller than 45. A new commutative matrix of DFT based on the generating matrix is also proposed. The new matrix outperforms the Santhanam's matrix.

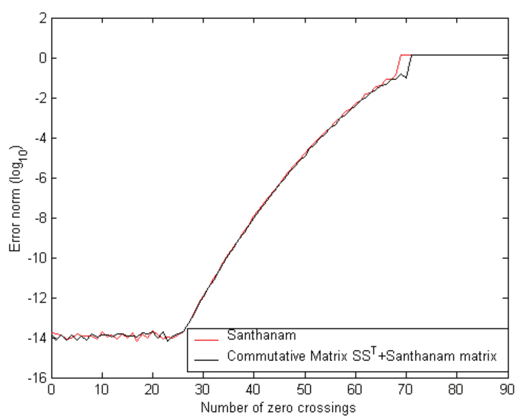

Fig. 3. Error norms for HGL eigenvectors of N=91

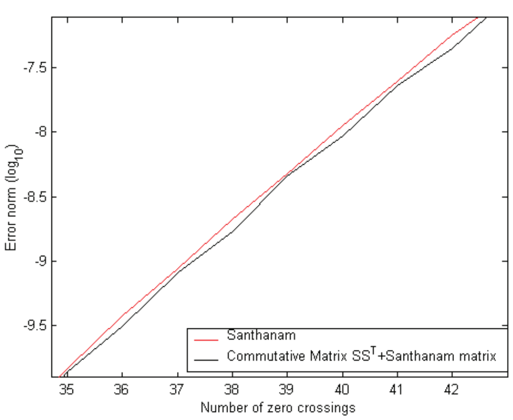

Fig. 4. The enlarged picture of Fig. 3, with the number of zero crossing from 35 to 42

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