A NEW LOOK INTO THE ISSUE OF THE CRAMÉR-RAO BOUND FOR DELAY ESTIMATION OF DIGITALLY MODULATED SIGNALS

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ABSTRACT

The Cramér-Rao Bound (CRB) with its modifications is a well known lower bound on the variance of any unbiased estimator of an unknown parameter. Focusing on the estimation of the time-delay experienced by a digitally modulated signal, we first introduce an approximation of the Modified Cramér-Rao Bound (MCRB) as a function of the spectral properties of the modulated signal, irrespective of the specific format of the digital modulation. In particular, we show the equivalence of the MCRB for single carrier modulation and for multicarrier modulation. As a corollary, it is shown that multicarrier signals with uneven power distribution and/or non-contiguous frequency allocation bear a smaller MCRB than equally spaced and powered modulations.

Index Terms— Cramér-Rao bound, delay estimation, multicarrier.

1. INTRODUCTION

The Cramér-Rao Bound (CRB) [1]-[4] and the Modified CRB (MCRB) [3] are the most known and widely adopted lower bounds on the variance of estimators of a constant parameter. Nevertheless, when focusing on the estimation of the time-delay experienced by a digitally modulated signal, it is at times difficult to compare the CRBs of different modulations. Just to cite an example, in the navigation community, it is common practice to characterize the performance of the time of arrival (TOA) estimation accuracy of the direct sequence SS (DS/SS) modulations by the inverse of the normalized root mean square (RMS) bandwidth [5] (or, equivalently, of the normalized Gabor bandwidth [6]) of the spectra of the chip pulse signal. This is a direct consequence of the expression of the MCRB of the TOA estimation of a linearly modulated signal. Following such technique, the performance comparison of different DS/SS signal formats, collapses into the analysis of the shape of the power spectral density (PSD) of the ranging signal. The motivation of a similar analysis, that is particularly expedient when the signal format is different from the classical DS/SS, (as for multicarrier) is not easily available in the open literature and is at times given for granted without adequate formalization. A formalization of the problem that conducts to similar results is addressed in [4], but with specific assumptions on the ciclostationarity of the signal and on the piecewise constant domain of the parameter to estimate, while same CRB expression can be obtain as a sub case of the analysis proposed in [9] for the estimation of time-difference-of-arrival. In the following, we first formalize the analysis for a generic digitally modulated signal in a simple form, then we give a specific proof of the equivalence of the MCRB for a single carrier signal and a multicarrier signal with the same PSD, and we conclude with general remarks on the resulting MCRB for a multicarrier signal that leads us to the notion of *cognitive positioning*.

2. CRAMÉR-RAO BOUND AS FUNCTION OF SIGNAL SPECTRAL PROPERTIES

We focus on a generic bandpass modulation, whose format is

$$x_{BP}(t) = \Re\left\{\tilde{x}(t)e^{j(2\pi f_0 t+\varphi)}\right\} \tag{1}$$

where f_0 and φ are the carrier frequency and phase, respectively; and $\tilde{x}(t)$ is the complex envelope of the signal, that is the baseband equivalent of the bandpass signal with average transmitted real power P_x. The modulation format of $\tilde{x}(t)$ is generic and it is assumed that it uses a symbol sequence $\mathbf{c}$, modeled as an instance of a random sequence. Without loss of generality, the symbol sequence $\mathbf{c}$ can be thought as a random data sequence in a data communication application, or as the rangin code in radio navigation and positioning. Using the nomenclature of ranging codes (symbol=chip), the chip are modeled as binary iid random variables, so that the signal $\tilde{x}(t)$ turns out to be a *parametric random process*, for which each sample function is a signal $\overline{x}(t)$ with finite power P_x and chip rate $R_c = 1/T_c$. We recall that the PSD of a parametric random process $\tilde{x}(t)$ like ours is defined to be

$$S_{\tilde{x}}(f) \triangleq \lim_{T_{obs}\to\infty} \frac{E_{\mathbf{c}}\left\{|\overline{X}_{T_{obs}}(f,\mathbf{c})|^2\right\}}{T_{obs}} \tag{2}$$

where $\overline{X}_{T_{obs}}(f,\mathbf{c})$ is the Fourier transform of the generic sample function $\overline{x}(t)$ truncated in the time interval $[-T_{obs}/2; T_{obs}/2]$ and thus having finite energy, and where $E_{\mathbf{c}}\{\cdot\}$ denotes statistical expectation over the code chips. The power of the (band-pass) signal $x(t)$ is

$$P_x = \int_{-\infty}^{\infty} S_x(f)df = \frac{1}{2}\int_{-\infty}^{\infty} S_{\tilde{x}}(f)df \tag{3}$$

The baseband equivalent of the observed signal in a coherent receiver can be modeled as

$$\tilde{r}(t) = \tilde{x}(t-\tau) + \tilde{n}(t) \tag{4}$$

where τ is the group delay experienced by the ranging signal when propagating from the transmitter to the receiver in the time reference frame of the receiver, and $\tilde{n}(t)$ is complex-valued AWGN with two-sided power spectral density $2N_0$. The $\mathrm{MCRB}(\tau)$ can be calculated starting from the definition of the CRB, where the code sequence $\mathbf{c}$ is considered as a nuisance parameter:

$$MCRB(\tau) = \frac{N_0}{E_{\mathbf{c}}\left\{\int_{T_{obs}}\left|\frac{\partial\tilde{x}(t-\tau,\mathbf{c})}{\partial\tau}\right|^2 dt\right\}} = \frac{N_0}{\int_{T_{obs}} E_{\mathbf{c}}\left\{\left|\frac{\partial\tilde{x}(t-\tau,\mathbf{c})}{\partial t}\right|^2\right\}dt}$$
$$= \frac{N_0}{T_{obs}\int_{-\infty}^{\infty}\frac{1}{T_{obs}}E_{\mathbf{c}}\left\{\left|\frac{\partial\tilde{x}_{T_{obs}}(t-\tau,\mathbf{c})}{\partial t}\right|^2\right\}dt} \tag{5}$$

where the MCRB is expressed as a function of the signal $\tilde{x}_{T_{obs}}(t) = \tilde{x}(t)\cdot rect(t/T_{obs})$, that has finite energy. Using Parseval's relation we get

$$MCRB(\tau) = N_0\left[T_{obs}\int_{-\infty}^{\infty} 4\pi^2 f^2 \frac{E_{\mathbf{c}}\left\{\left|\tilde{X}_{T_{obs}}(f)\right|^2\right\}}{T_{obs}}df\right]^{-1} \tag{6}$$

We now adopt a crucial assumption that leads to an accurate approximation of the bound. Specifically, we assume T_{obs} very large, so that $E_{\mathbf{c}}\left\{\left|\tilde{X}_{T_{obs}}(f)\right|^2\right\}\big/T_{obs} \cong S_{\tilde{x}}(f)$. Under this hypothesis,

$$MCRB(\tau) = \frac{N_0}{T_{obs}4\pi^2\int_{-\infty}^{\infty} f^2 S_{\tilde{X}}(f)df} = \frac{B_{eq}T_c}{4\pi^2\cdot\frac{E_c}{N_0}\beta_{\tilde{x}}^2} \tag{7}$$

where we also let $T_{obs} = N\cdot T_c$ (N very large), and where $\beta_{\tilde{x}}$ is root mean square (RMS) bandwidth [5] (or, equivalently, the Gabor bandwidth [6]) of the complex signal, normalized to the complex signal power $\int_{-\infty}^{\infty} S_{\tilde{X}}(f)df = 2P_x$; and defined by

$$\beta_{\tilde{x}}^2 \triangleq \frac{\int_{-\infty}^{\infty} f^2 S_{\tilde{X}}(f)df}{\int_{-\infty}^{\infty} S_{\tilde{X}}(f)df}, \tag{8}$$

$E_c = P_x\cdot T_c$ is also the average signal energy per chip, and $B_{eq} = 1/2NT_c$ is the (one-sided) noise bandwidth of a closed-loop estimator equivalent to an open-loop estimator operating on an observation time equal to NT_c.

From (7)-(8), we conclude that the $\mathrm{MCRB}(\tau)$ depends on the PSD of the complex signal, independent of the type of modulation adopted. In particular, we see that signals with the same PSD have the same MCRB even if generated by different modulations. As a final remark, the approximation in (7)-(8) is particularly useful when a band-limitation is applied to the signal, since the $\mathrm{MCRB}(\tau)$ can be calculated by simply limiting the frequency integral in (8) to the band of interest.

2.1. Conventional DS/SS modulation

When the signal $\tilde{x}(t)$ is a classical DS/SS signal, with binary pseudorandom noise (PN) code, we have

$$\tilde{x}(t) = \sqrt{2P_x}\sum_{l=0}^{L_{PAM}-1}\gamma_l g(t - lT_c) \tag{9}$$

with $\gamma_l = \{\pm1\}$, iid, represent the product between the code element and the data symbols, $g(t)$ is a real-valued shaping pulse with energy T_c and $T_{obs} = L_{PAM}T_c$.

The PSD (2) is now calculated as

$$S_{\tilde{x}}(f) = \frac{2P_x\,|G(f)|^2}{T_c} \tag{10}$$

and thus, substituting (10) into (7), we obtain the conventional expression [3] of the $\mathrm{MCRB}(\tau)$ for a PAM signal

$$MCRB(\tau) = \frac{T_c^2}{4\pi^2\cdot 2L_{PAM}\cdot\frac{E_c}{N_0}\xi_g} = \frac{B_{eq}T_c\cdot T_c^2}{4\pi^2\cdot\frac{E_c}{N_0}\xi_g} \tag{11}$$

where

$$\xi_g = \frac{T_g^2\cdot\int_{-\infty}^{\infty} f^2|G(f)|^2 df}{\int_{-\infty}^{\infty}|G(f)|^2 df} \tag{12}$$

is the pulse shaping factor (PSF), an adimensional parameter related to the shape of the Fourier transform $G(f)$ of the pulse $g(t)$. T_g is the generic symbol spacing, that is, in this case $T_g = T_c$.

3. CRB FOR MULTICARRIER SIGNAL

We now tackle the problem of calculating the $\mathrm{MCRB}(\tau)$ for a multicarrier signal. In particular, we select the class of Filter Bank Multicarrier Modulation (FBMCM) or Filtered Multitone (FMT) [7]-[8] signal. The difference with respect to the well-know Orthogonal Frequency Division Multiplexing (OFDM) is that in the latter the spectra on the different subcarriers are overlapped to a large extent, and orthogonality is preserved in the time domain only if the observation (integration, correlation) time on each subcarrier is an exact multiple of the OFDM chip rate. In the FBMCM, the spectra on each subcarrier are strictly bandlimited and nonoverlapping, as in conventional single channel per carrier (SCPC), so that orthogonality is attained in the frequency domain and holds irrespective of the observation time. To be specific, the time-continuous transmitted FBMCM signal is

$$\tilde{x}(t) = \sqrt{\frac{2 \cdot P_x}{N}} \sum_{n} \sum_{k=0}^{N-1} p_k c_n^{(k)} g(t - nT_s) e^{j2\pi k(1+\alpha)t/T_s} \quad (13)$$

where the input stream of the ranging chips c_i (at the rate $R_c = 1/T_c = N/T_s$) is parallelized into N substreams with a *FBMCM symbol rate* $R_s = R_c/N = 1/(NT_c) = 1/T_s$ where T_s is the time duration of the "slow" ranging chips in the parallel substreams. We have used a "polyphase" notation for the k-th ranging subcode in the k-th substream as $c_n^{(k)} \triangleq c_{nN+k}$ where k, $0 \le k \le N-1$, is the subcode index for the subcarrier index, whilst n is a time index that addresses the n-th FBMCM block period of time length $T_s = NT_c$. The coefficients p_k introduce into the FBMCM signal model the possibility of an uneven power distribution on the different subcarriers. The substreams are then modulated onto a raster of evenly-spaced subcarriers with frequency spacing f_{sc}. Each subcarrier is spectrally shaped with a conventional square-root raised cosine (SRRC) Nyquist pulse $g(t)$ with roll-off factor α, signaling interval $T_s = NT_c$, and energy $E_g = T_s$. In the frequency domain, we have $G(f) = T_s\sqrt{G_N(f)/T_s}$ where $G_N(f)$ is the conventional Nyquist's raised-cosine spectrum with $G_N^{(0)} = T_s$ and bandwidth $(1+\alpha)/T_s$. The center frequency for the $k-$th subcarrier is $f_k = kf_{sc} = k(1+\alpha)/T_s$ to avoid spectral overlapping between the sub-channels. The resulting modulated signals are added together to give the (baseband equivalent of the) overall ranging signal.

The MCRB can be thus easily calculated from (7). In particular, assuming statistical independence of the code $\mathbf{c}$ the PSD of the FBMCM signal (13) is

$$S_{\tilde{x}}(f) = \frac{2 \cdot P_x}{N} \sum_{k=0}^{N-1} p_k^2 S\left(f - k\frac{(1+\alpha)}{T_s}\right) \quad (14)$$

where $S(f) = E\{|c_k^{(m)}|^2\}|G(f)|^2/T_s = G_N(f)$ is the PSD of each sub-stream with rate $1/T_s$. Substituting (14) into (7) after lengthy computations it is found

$$MCRB(\tau) = \frac{T_c^2}{8\pi^2 \frac{E_c}{N_0}\frac{N_m}{N}\left[\xi_g + \frac{(1+\alpha)^2}{N}\sum_k k^2 p_k^2\right]} \quad (15)$$

where N_m indicates the observation time $T_{obs} = N_m T_s$, $E_c = P_x \cdot T_c$, ξ_g is the PSF of the SRRC Nyquist pulse $g(t)$ defined as in (12) with $T_g = T_s$ and p_k^2 is the relative power weight of carrier k ($0 \le p_k < 1$), that satisfy $\sum_k p_k^2 = N$.

The term $\sum_k k^2 p_k^2$ in (15) depends on the position of the subcarriers with respect to the carrier frequency and plays a key role in the evaluation of the MCRB. In particular, when the spectrum is flat ($p_k = 1, \ \forall k$), the number of subcarriers N is odd and the carrier frequency is located at the *center* of the spectrum, the MCRB reduces to the final form

$$MCRB(\tau) = \frac{T_c^2}{8\pi^2 \frac{E_c}{N_0}\frac{N_m}{N}\xi_g\left[1 + \frac{(1+\alpha)^2}{12\xi_g}(N^2-1)\right]} \quad (16)$$

It is easy to verify that when $N = 1$, (16) reduces to the conventional expression (11) of a monocarrier signal.

Special attention has to be devoted to the case where the carrier frequency is located at the *edge* of the spectrum. Details are not reported here for lack of space, but we find that when $N \gg 1$, the following relation between the symmetric and the asymmetric case is found:

$$MCRB\big|_{\text{asymmetric}}(\tau) \cong MCRB\big|_{\text{symmetric}}(\tau)/4 \quad (17)$$

thus privileging *band-edge demodulation* when estimation accuracy is required.

3.1 Minimization of the Cramér-Rao Bound

When introducing into the FBMCM signal model the possibility of an uneven power distribution on the different subcarriers, as in (13), a nice problem is the minimization of the MCRB (15) through maximization of $\sum_k k^2 p_k^2$ under the constraint $\sum_k p_k^2 = N$. When N is fixed, (15) is maximized for the *optimal power allocation scheme* that calls for a configuration wherein the power of the signal is concentrated at the edge of the bandwidth:

$$p_k = \sqrt{\frac{N}{2}}, \ k = \pm\frac{N-1}{2} \text{ and } p_k = 0 \text{ otherwise.} \quad (18)$$

This is in full agreement with the traditional theory of the maximization of the RMS bandwidth [7] (or of the Gabor bandwidth [8]) of the signal, that states that the more the power of the signal is concentrated at the edge of the band, the more accurate TDE is, even if the estimation is more prone to ambiguity errors. When comparing the optimal distribution with the *flat* one for $N \gg$, we can state that

$$MCRB\big|_{\text{optimal}}(\tau) \cong MCRB\big|_{\text{flat}}(\tau)/3 . \qquad (19)$$

This is the maximum gain that can be achieved over a flat spectrum by a signal that present (i) total bandwidth B, (ii) chip rate $R_c = 1/T_c$ and (iii) transmitted power P_x, while intermediate power configurations will achieve intermediate gains. On the contrary, when using the optimal configuration, the total number of chips transmitted is roughly N times smaller than the flat configuration (with $p_k = 1, \ \forall k$).

4. CRB COMPARISON BETWEEN MULTICARRIER AND SINGLE CARRIER SIGNALS

Multicarrier signals are substantially different from monocarrier signals. Nevertheless, as stated in Sec. 2, they should present the same MCRB, when having the same PSD. We consider a bandlimited PAM signal with the same spectral occupancy of the FBMCM (13). The signal format is as in (9), where here $g(t) = g_{SRRC}(t)$ is a SRRC with roll-off factor α, and signaling interval T_c. The shaping pulse is $G_{SRRC}(f) = T_c\sqrt{G_{N_c}(f)/T_c}$, where now $G_{N_c}(f) = T_c$ and the bandwidth is $(1+\alpha)/T_c$. The correspondent PSF is thus $\xi_{g_{SRRC}} = 1/12 + \alpha^2(1/4 - 2/\pi^2)$. As for the FBMCM of (13), P_x is the transmitted power, the total occupied band is $B = (1+\alpha)/T_c$ and the ranging chips c_n are transmitted at the rate $R_c = 1/T_c$. In order to observe the same time interval of the FBMCM signal, we should consider $T_{obs} = NT_s = N \cdot N_m T_c = L_{PAM} T_c \Rightarrow L_{PAM} = N_m \cdot N$. This ensures that the same number of code chips is transmitted as well. The resulting MCRB is

$$MCRB\big|_{PAM}(\tau) = \frac{T_c^2}{8\pi^2 \dfrac{E_c}{N_0} N_m N \left[\dfrac{1}{12} + \alpha^2\left(\dfrac{1}{4} - \dfrac{2}{\pi^2}\right)\right]} \qquad (20)$$

The large-N approximation of the MCRB (16) of FBMCM is

$$T_c^2\left[8\pi^2 \frac{E_c}{N_0} N_m N \frac{1+\alpha^2}{12}\right]^{-1} \qquad (21)$$

that is almost equal to (slightly better than) (20). The two coincide when $\alpha = 0$, in which case the two spectra are exactly the same. This is in perfect agreement with the fact that two signals with the same PSD have the same MCRB even if they are generated with different modulation formats.

5. CONCLUSIONS AND PERSPECTIVES

In this paper we formalized in a simple way an approximated expression of the MCRB for time-delay estimation as a function of the spectral properties of a modulated signal. In particular it is shown that the multicarrier signals have the same performance of single carrier signals when having the same PSD. As a corollary, it is also shown that i) multicarrier signals with uneven power distribution can be adopted as an easy implementation of an optimized signal that met the Gabor bandwidth criterion (thus minimizing the MCRB), and ii) multicarrier signals with even power distribution but uneven spectral allocation of the subcarrier may yield as well a boost in TDE performance. The latter consideration, when applied to the case of radio navigation, paves the way to what we may call *cognitive positioning system* (CPS), wherein the allocation of signal power for positioning is dynamically adjusted from time to time after an MRCB-minimizing criterion, and according to the current availability of unused frequency bands in a certain wide range.

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