TDOA ESTIMATION FOR CYCLOSTATIONARY SOURCES: NEW CORRELATIONS-BASED BOUNDS AND ESTIMATORS

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ABSTRACT

We consider the problem of Time Difference of Arrival (TDOA) estimation for cyclostationary signals in additive white Gaussian noise. Classical approaches to the problem either ignore the cyclostationarity and use ordinary cross-correlations, or exploit the cyclostationarity by using cyclic cross-correlations, or combine these approaches into a multicycle approach. Despite contradicting claims in the literature regarding the performance-ranking of these approaches, there has been almost no analytical comparative performance study. We propose to regard the estimated (ordinary or cyclic) correlations as the "front-end" data, and based on their asymptotically Gaussian distribution, to compute the asymptotic Cramér-Rao bounds (CRB) for the various combinations (ordinary/single-cycle/multi-cycle). Using our Cyclic-Correlations-Based CRB (termed "CRBCRB"), we can bound the performance of any (unbiased) estimator which exploits a given set of correlations. Moreover, we propose an approximate maximum likelihood estimator (with respect to the correlations), and show that it attains our CRBCRB asymptotically in simulations, outperforming the competitors.

Index Terms— TDOA, cyclic-correlations, multi-cycle.

1. INTRODUCTION

Estimation of time difference of arrival (TDOA) of a source signal to two spatially-separated sensors is a fundamental problem in passive emitter location systems. Classical approaches to this problem assume wide-sense stationarity (WSS) of the source signal, and use the peak-location of the (unfiltered or filtered) observations' cross-correlation (CC) as an estimate of the TDOA. The possible use of general pre-filters spans a family of "generalized CC" (GCC) estimators, introduced by Knapp and Carter [1] in 1976. Knowledge of the power spectral distributions of the source and noise signals allows to construct pre-filters realizing (under a Gaussianity assumption) the maximum likelihood estimate (MLE), which is asymptotically efficient [1].

Digital communication signals exhibit inherent periodicity (due to their modulation, sampling and coding), which gives rise to periodically-varying statistical characteristics, and are therefore termed *cyclostationary*. The periodicity frequency is termed *cyclic frequency*. The Fourier coefficient (at the cyclic frequencies) of the time-varying correlation function (at each lag) are termed *cyclic correlations*.

A family of TDOA estimation algorithms for cyclostationary sources was introduced by Gardner in [2], [3] and used cyclic (rather than "stationary") cross-correlations. The principal motivation behind the use of such algorithms is their inherent immunity against interference and noise signals which are either not cyclostationary, or do not share the cyclic frequency of the cyclostationary signal-of-interest (SOI). One of the most successful cyclostationary algorithm is the "spectral coherence alignment" (SPECCOA) [2], [3]. Algorithms which exploit more than one cyclic frequency are called *multi-cycle* algorithms, e.g., the Multicycle Spectral Correlation Product Estimator (MCSCPE), by Gardner [4].

To date, the performance evaluation for cyclic TDOA algorithms in an interference-free environment was mainly restricted to empirical simulation studies. To the best of our knowledge, almost no analytic performance analysis appeared in open literature, with the recent exception of [5], in which some small-error analysis for a particular multi-cycle algorithm was presented. Nevertheless, there seems to be some disagreement as to which approach (single-cycle, multi-cycle, ordinary cross-correlation) is theoretically preferable in an additive white Gaussian noise (AWGN)-only environment.

For example, in ([2], p.1170) Chen and Gardner claimed, regarding single-cycle algorithms, that they *"... outperform conventional TDOA algorithms ... when the only corruption to the SOI is the receiver noise"*. Using empirical simulation results, they also maintained ([3], p.1193) that SPECCOA far outperforms conventional GCC in AWGN environments. The intuition behind those claims is that *"these signal-selective methods discriminate against not only interference but also noise (and , more generally, anything that is not cyclostationary with the cycle frequency being used"* ([3], p.1194).

Conversely, in a more recent comparison (by Gisselquist, [6]) between stationary and cyclostationary TDOA estimators in a wideband cyclostationary interference environment, the single-cycle algorithms were actually found to perform *much worse* than the best stationary algorithms. Moreover, Gisselquist pointed out a few fundamental flaws in previous experiments, such as failing to apply sub-sample interpolation and comparing against non-optimal stationary methods.

The purpose of this paper is to take an analytic approach in trying to resolve these contradictions, by deriving analytical asymptotic bounds on the performance of cyclic correlation-based TDOA algorithms in an AWGN environment. The main difficulty in deriving analytical results for cyclostationary signals stems from the fact that the probability distribution of the cyclostationary data is often too involved to formulate. We circumvent this problem by recognizing that the front-end data for these TDOA algorithms are the estimated cyclic correlations. Thus, we may consider only the estimated cyclic correlations vector, rather than work with the raw data samples. Fortunately, the asymptotic distribution of this vector is (under some reasonable assumptions) jointly complex Gaussian, thanks to an extension of the Central Limit Theorem, [7], regardless of the original data distribution. As such, computation of its mean and covariance, which can both be derived from the moments of the raw-data, allows a tractable derivation of the associated Cramér-Rao bound (CRB).

The resulting bound, which we term the "Cyclic-coRrelations-Based CRB" (CRBCRB) will serve as a lower-bound on the performance attainable by any (unbiased) estimator which is based solely on the estimated cyclic correlations. Of course, it would not bound the performance of general (unbiased) estimators exploiting the full raw data at the sensors, since generally the estimated correlations are not a sufficient statistic with respect to the raw data. It is certainly conceivable that there might exist better estimators, which would exploit the raw data differently, in a way which is not based on the estimated cyclic correlations. Nevertheless, the CRBCRB would serve as a bound on the performance which can be attained from the estimated cyclic correlation alone, be it by SPECCOA, MCSCPE, GCC (which is a particular case of a cyclic correlation, with zero cyclic frequency), or any other correlations-based algorithm.

Moreover, we would be able to exploit the knowledge of the estimated cyclic correlations' asymptotic distribution, so as to propose an approximate MLE, (approximately) maximizing the likelihood of these estimated cyclic correlations (rather than the likelihood of the raw data). This estimator would asymptotically approach the CRBCRB, and would therefore offer the (asymptotically) optimal exploitation of the chosen estimated cyclic correlations.

2. SOME PRELIMINARIES

Consider the following baseband signal model:

$$\begin{aligned} x(t) &= \tilde{s}(t) + \tilde{v}_1(t) \\ y(t) &= \tilde{s}(t-D) + \tilde{v}_2(t) \end{aligned} \tag{1}$$

where $\tilde{v}_1(t)$ and $\tilde{v}_2(t)$ are stationary complex white Gaussian noise processes, uncorrelated with $\tilde{s}(t)$, which is the cyclostationary SOI, whose cycle-period is denoted T. We denote the *fundamental cyclic frequency* of $\tilde{s}(t)$ by $F_d \triangleq \frac{1}{T}$. D denotes the TDOA parameter which we wish to estimate.

Assume now that the received signals $x(t)$ and $y(t)$ are low-pass filtered and then sampled with sampling period $T_s = T/P$ (where $P \geq 1$ is some integer),

$$\begin{aligned} x[n] &= x(nT_s) = s[n] + v_1[n] \\ y[n] &= y(nT_s) = s_d[n] + v_2[n], \end{aligned} \tag{2}$$

where $s[n] = s(nT_s), s_d[n] = s(nT_s - D)$ and $v_1[n] = v_1(nT_s), v_2[n] = v_2(nT_s)$, such that $v_1(t)$, $v_2(t)$ and $s(t)$ are the filtered versions of $\tilde{v}_1(t)$, $\tilde{v}_2(t)$ and $\tilde{s}(t)$, respectively (note that $s(t)$ remains cyclostationary with the same cyclic frequency after the filter). If the receiver's bandwidth is $\frac{1}{T_s}$, then $v_1[n], v_2[n]$ are white. When the receiver's output is sampled at the symbol rate ($P = 1$), the resulting sequences $x[n], y[n]$ are stationary. At a higher ($P > 1$) sampling rate, the resulting sequences $x[n], y[n]$ are cyclostationary.

The cross-correlation $R_{yx}[n;\tau] \triangleq E\left\{y[n+\tau]x^*[n]\right\}$ is periodic in n with period P, and therefore admits a Fourier-series expansion (at each τ), in the cyclic correlations

$$R_{yx}^k[\tau] = \frac{1}{P}\sum_{n=0}^{P-1} R_{yx}[n;\tau] e^{-j2\pi(\frac{k}{P})n} \tag{3}$$

(for $k = 0, .., P-1$). The discrete-time cyclic frequencies (of $s[n]$) are $\alpha_k = \frac{k}{P}$ for $k = 0, .., P-1$. A standard estimate of the cyclic correlations from a finite number of samples N takes the form

$$\hat{R}_{yx}^{\alpha}[\tau] = \frac{1}{N}\sum_{n=0}^{N-1-\tau} y[n+\tau]x[n]^* e^{-j2\pi\alpha n}, \tag{4}$$

which under the so-called "mixing conditions" is asymptotically unbiased and mean-square consistent [7]. Moreover, if a finite number of estimated cyclic correlations (at various lags and cyclic frequencies) are concatenated into a single random vector, that vector would be (asymptotically) jointly complex multivariate Gaussian (CMVG).

The general distribution of a CMVG random vector $\mathbf{z}$ is characterized by its mean vector $\boldsymbol{\mu}_z \triangleq E[\mathbf{z}]$ and two covariance matrices: The Hermitian, positive semidefinite covariance $\mathbf{C} \triangleq E[(\mathbf{z}-\boldsymbol{\mu}_z)(\mathbf{z}-\boldsymbol{\mu}_z)^H]$ and the complex symmetric $\mathbf{J} \triangleq E[(\mathbf{z}-\boldsymbol{\mu}_z)(\mathbf{z}-\boldsymbol{\mu}_z)^T]$, *complementary covariance matrix*. Alternatively, we may denote $\mathbf{z} = \mathbf{x} + j\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are real-valued jointly Gaussian random-vectors, and define a real-valued Gaussian random-vector $\tilde{\mathbf{z}} = [\mathbf{x}^T, \mathbf{y}^T]^T$, whose mean is $\boldsymbol{\mu}_{\tilde{z}} \triangleq [\boldsymbol{\mu}_x^T, \boldsymbol{\mu}_y^T]^T$ and whose covariance is given by

$$\mathbf{C}_{\tilde{\mathbf{z}}} \triangleq \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{\mathbf{C}+\mathbf{J}\} & \mathrm{Im}\{\mathbf{J}-\mathbf{C}\} \\ \mathrm{Im}\{\mathbf{C}+\mathbf{J}\} & \mathrm{Re}\{\mathbf{C}-\mathbf{J}\} \end{bmatrix}. \tag{5}$$

3. ASYMPTOTIC MEAN AND COVARIANCE OF THE ESTIMATED CYCLIC CORRELATIONS

In order to obtain an asymptotic expression for the CRBCRB, all we need are the mean and covariance of the vector of concatenated estimated cyclic correlations. For each element of the estimated vector we have (from (4)),

$$\hat{R}^k_{yx}[\tau] = \hat{R}^k_{s_d s}[\tau] + \hat{R}^k_{s_d v_1}[\tau] + \hat{R}^k_{v_2 s}[\tau] + \hat{R}^k_{v_2 v_1}[\tau], \quad (6)$$

with obvious notations for the four components. Due to the assumption that the noise processes and the SOI are all mutually uncorrelated, the mean of $\hat{R}^k_{yx}[\tau]$ is readily given by

$$E\left\{\hat{R}^k_{yx}[\tau]\right\} = R^k_{s_d s}[\tau] = R^{kF_d}_{ss}(\tau \cdot T_s - D). \quad (7)$$

The covariance and the complementary covariance between any two elements can be easily expressed in terms of these elements' second joint-moments and their means.

In order to find the second joint moments, we first need to specify the statistical model for the SOI (as mentioned earlier, the noise processes $v_1[n]$ and $v_2[n]$ are always assumed to be zero-mean, white circular complex-Gaussian; we shall further assume that their variances are σ_1^2 and σ_2^2, respectively). To this end, we shall employ the standard Pulse Amplitude Modulation (PAM) model for the SOI:

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} a[k]p(t-kT) \quad (8)$$

where $p(t)$ is a square-integrable pulse and $\{a[k]\}$ is a zero-mean independently and identically distributed (i.i.d) symbols-sequence, with values drawn from a circular complex constellation with unit variance, i.e., $\sigma_a^2 = E\left\{|a[k]|^2\right\} = 1$.

PAM signals with root raised-cosine (RRC) pulse shape (with roll-off parameter R) are *strictly band-limited* with single-sided bandwidth B, as defined in [8]. For maximal roll-off ($R=1$) we get $B = \frac{1}{T}$, so if $T_s \leq \frac{1}{4}T$ (or $P \geq 4$), the low-pass filtering has no effect; Consequently, $s(t) = \tilde{s}(t)$, and the discrete-time cyclic correlation equals the sampled continuous time cyclic correlation $R^k_{ss}[m] = R^{kF_d}_{ss}(mT_s)$, which for PAM signals is given by (see [9], p.660):

$$R^{\alpha}_{ss}(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} p(t+\tau)p^*(t)e^{-j2\pi\alpha t}dt = \\ = \frac{1}{T}\int_{-\infty}^{\infty} P(f)P^*(f-\alpha)e^{j2\pi\tau f}df, \quad (9)$$

where $P(f)$ is the Fourier transform of the $p(t)$. Since for RRC pulse-shapes $P(f)$ vanishes for $|f| \geq \frac{1+R}{2T}$, the product $P(f)P^*(f-\frac{k}{T})$ will be nonzero only for $k = 0, \pm 1$, which means that PAM signals with RRC pulse shape will only have three cyclic frequencies $\alpha_k = kF_d, \;\; k = 0, \pm 1$.

It is shown in [10] (the derivation is omitted due to lack of space) that under these model assumptions the covariance $C^{k_1,k_2}_{yx,yx}[\tau,\ell] \triangleq cov\left\{\hat{R}^{k_1}_{yx}[\tau], \hat{R}^{k_2}_{yx}[\ell]\right\}$ is given by

$$C^{k_1,k_2}_{yx,yx}[\tau,\ell] = C^{k_1,k_2}_{s_d s, s_d s}[\tau,\ell] + R^{(k_1-k_2)F_d}_{ss}([\tau-\ell]T_s) \cdot \\ \cdot \frac{1}{N}\left(\sigma_1^2 e^{j\frac{2\pi}{P}(k_1-k_2)(\ell - D/T_s)} + \sigma_2^2 e^{j\frac{2\pi}{P}k_2(\tau-\ell)}\right) \\ + \frac{1}{N}\sigma_1^2\sigma_2^2\delta[\tau-\ell]\delta[k_1-k_2], \quad (10)$$

where $C^{k_1,k_2}_{s_d s, s_d s}[\tau,\ell]$ is the signal-term covariance (see [11]):

$$C^{k_1,k_2}_{s_d s, s_d s}[\tau,\ell] = \frac{1}{N}\sum_{k=-1}^{1}\sum_{m=-\infty}^{\infty} R^{kF_d}_{ss}([m+\tau-\ell]T_s) \cdot \\ \cdot R^{(k-k_1+k_2)F_d}_{ss}(mT_s)^* \cdot e^{-j\frac{2\pi}{P}(k_1 m - k(\ell - D/T_s))}. \quad (11)$$

Estimated cyclic correlations complementary covariance matrix $J^{k_1,k_2}_{yx,yx}[\tau,\ell] \triangleq cov\left\{\hat{R}^{k_1}_{yx}[\tau], \left(\hat{R}^{k_2}_{yx}[\ell]\right)^*\right\}$, which, due to the circularity of the noise distribution does not contain any noise-related terms, is given by

$$J^{k_1,k_2}_{yx,yx}[\tau,\ell] = \frac{1}{N}\sum_{k=-1}^{1}\sum_{m=-\infty}^{\infty} R^{kF_d}_{ss}([m+\tau+\ell]T_s - D) \cdot \\ \cdot R^{(k_1+k_2-k)F_d}_{ss}(-mT_s - D)e^{-j\frac{2\pi}{P}(k-k_2)(m+\ell)} \quad (12)$$

4. THE CYCLIC-CORRELATION-BASED CRB

The Fisher information matrix (with respect to D) is given by

$$I(D) = \frac{d\boldsymbol{\mu}^T_{\tilde{z}}(D)}{dD}\mathbf{C}^{-1}_{\tilde{\mathbf{z}}}(D)\frac{d\boldsymbol{\mu}_{\tilde{z}}(D)}{dD} + \\ 0.5 \cdot \mathrm{Tr}\left[\mathbf{C}^{-1}_{\tilde{\mathbf{z}}}(\mathrm{D})\frac{\mathrm{d}\mathbf{C}_{\tilde{\mathbf{z}}}(\mathrm{D})}{\mathrm{dD}}\mathbf{C}^{-1}_{\tilde{\mathbf{z}}}(\mathrm{D})\frac{\mathrm{d}\mathbf{C}_{\tilde{\mathbf{z}}}(\mathrm{D})}{\mathrm{dD}}\right],$$

where the observation vector $\tilde{\mathbf{z}}$ is the concatenation of the real and imaginary parts of the estimated cyclic correlations vector, and its mean and covariance are given by the expressions (7), (5), (10), (11) and (12) above.

To compute the information matrix (actually a scalar quantity in our case), we also need the derivative of the cyclic correlation $R^{\alpha}_{ss}(\tau - D)$ with respect to D - which can be computed for any pulse-shape, e.g., by using numeric integration in (9), multiplying the integrand by $-j2\pi f$.

Thus, the CRBCRB (on the mean-squared error) for estimating D from any combination of estimated cyclic-correlations can be readily obtained as $1/I(D)$, by substituting all of the relevant expressions in $I(D)$ above.

Fig.1 depicts the CRBCRB obtained from using 17 lags $8 \leq \tau \leq 8$) of the following correlation types: Single-cycle

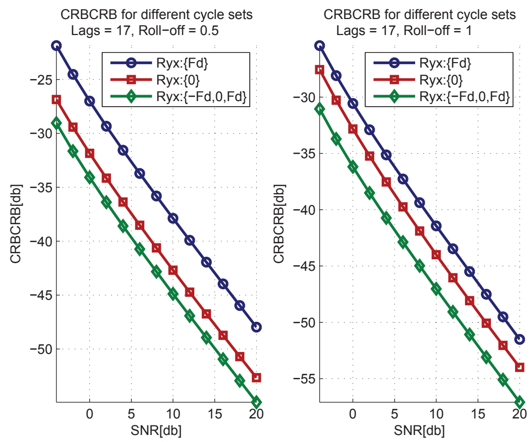

Fig. 1. Single-cycle and Multi-cycle CRBCRB vs. SNR for RRC Gaussian-symbols PAM signals with two different roll-offs.

with cyclic frequency $\alpha = F_d$ ($k = 1$); Single-cycle with cyclic frequency $\alpha = 0$ ($k = 0$) - this case is equivalent to the use of ordinary (stationary) cross-correlation; And multi-cycle with cyclic frequencies $\alpha = 0, F_d, -F_d$ ($k = 0, \pm 1$). The results are presented vs. the signal-to-noise ratio[1] for PAM signals with two roll-off values, with $P = 4$.

We observe that the bound for a single-cycle method using cyclic ($\alpha = F_d$) correlation predicts significantly worse optimal performance than for a method using "ordinary" ($\alpha = 0$) correlation. For typical roll-off $R = 0.5$ the loss is $\sim 4.8[db]$ and for the highest roll-off $R = 1$ (not a very bandwidth-efficient signal) the loss is $\sim 2.6[db]$. The gain from using a multi-cycle approach (relative to ordinary correlations) for RRC with $R = 0.5, 1$ is $\sim 2.2[db], 3[db]$ (resp.). Thus, with the lower roll-off, the feature strength of the cyclic ($\alpha = \pm F_d$) correlations decreases, so that their potential contribution to improving the performance of the ordinary correlation becomes more marginal.

5. A NOVEL MULTI-CYCLE APPROXIMATE MLE

Having specified the asymptotic distribution of the estimated cyclic correlation values, an approximate MLE may be obtained by maximizing (with respect to D)

$$L(D) \stackrel{\triangle}{=} -\log|\mathbf{C}_{\tilde{\mathbf{z}}}(D)| - (\tilde{\mathbf{z}} - \boldsymbol{\mu}_{\tilde{z}}(D))^T \mathbf{C}_{\tilde{\mathbf{z}}}^{-1}(D)(\tilde{\mathbf{z}} - \boldsymbol{\mu}_{\tilde{z}}(D)). \quad (13)$$

Ignoring the usually negligible contribution of $\log|\mathbf{C}_{\tilde{\mathbf{z}}}(D)|$, the estimator reduces to the solution of a nonlinear Weighted Least Squares problem, which we solve using a Quasi-Newton search, following an initial guess obtained, e.g., from an ordinary CC-based TDOA estimate.

Under common regularity conditions, our MLE should asymptotically attain our CRBCRB (possibly outperforming any competing methods which use the same cyclic correlation values differently), as can indeed be observed in the supporting simulation results in Fig.2.

[1]Equal noise power was used for both sensors, $\sigma_1^2 = \sigma_2^2$.

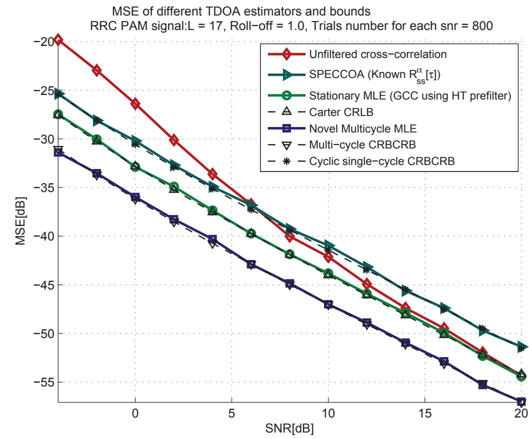

Fig. 2. MSE of different TDOA estimators for an RRC Gaussian-symbols PAM signal with roll-off $R = 1$ in AWGN, with $P = 4$ and observation length $N = 8192$ (2048 symbols). Each result reflects the average of 800 independent trials.

6. CONCLUSIONS

We derived a new lower-bound on the performance attainable by any unbiased TDOA estimator which is exclusively based on the estimated cyclic (and/or ordinary) correlations. We thereby showed analytically, that at least for the considered RRC PAM signal in AWGN, single-cycle cyclic correlations are less informative (admit a higher bound) than ordinary CC (single-cycle with zero cyclic frequency). We also proposed a novel correlations-based approximate MLE, whose performance approaches the predicted bound (asymptotically), outperforming other competitors.

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