A NEAR OPTIMUM DETECTION IN ALPHA-STABLE IMPULSIVE NOISE

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ABSTRACT

Alpha stable distribution has gained much attention due to its generality to represent heavy-tailed and impulsive interference. In such non-Gaussian interference, the detection key is to evaluate the zero-memory nonlinearity (ZMNL) function of locally optimal (LO) detector. Unfortunately, there is no closed form expression for the probability density function (PDF) of alpha-stable distributions. Hereby, sub-optimum ZMNL function is adopted as an unavoidable approximation, such as classical Cauchy and Gaussian-tailed ZMNL (GZMNL). In this paper, an algebraic-tailed ZMNL (AZMNL) with a concise form is proposed. Based on such ZMNL, derived detector has near optimal performance in various impulsive noise environments. Furthermore, using Bi-parameter CGM (BCGM), a concise approximate expression for PDF of symmetric α-stable (SαS) distribution, the test threshold can be evaluated according with preset false alarm ratio easily.

Index Terms— Signal detection, Impulsive noise, α-stable distribution

1. INTRODUCTION

Signal detection, which detects the presence of a signal in noisy observations, is a classical problem that has to be implemented in a variety of applications, such as ones being in radar, sonar and communications. The signal detection problems usually are viewed as problems of hypothesis testing in statistical inference [1] in which the generalized likelihood ratio test (GLRT) is the most widely accepted method of solution. In most of previous work on detection, it has been assumed that the signal is embedded in Gaussian noise and the detectors are designed accordingly, since the Gaussian noise assumption has been generally justified with the central limit theorem and with the analytical convenience of the Gaussian probability density function (PDF) which leads to linear and hence tractable equations. However, there are many cases, in which the noise is decidedly non-Gaussian. Non-Gaussianity often results in significant performance degradation for detector designed under the Gaussian assumption [1, 2].

In a detection problem, optimal processing is feasible only if the noise PDF is analytically known and tractable. Unfortunately, there are no closed forms for the probability densities of α-stable distribution except for three special cases, say, Gaussian, Cauchy and Pearson distribution [3]. Therefore, the general approach to handle such impulsive input is to develop a limiter to clip the noise. In fact, such limiter essentially is equivalent to a zero-memory nonlinearity (ZMNL) function (optimum or sub-optimum) derived from Neyman-Pearson lemma [1]. Since closed PDF of α-stable distribution does not exist, deducing analytical optimum ZMNL function is not an easy task. To overcome such difficulty, a tractable approximation for ZMNL function is unavoidable. Several sub-optimum ZMNL have been proposed in prior work, such as hole-puncher, Cauchy, Limiter plus integrator (i.e. soft-limiter) [2] and Gaussian-tailed [4] ZMNL. However, these ZMNL based detectors are not always valid in various impulsive noise environments. In this paper, we suggest an algebraic-tail ZMNL (AZMNL) and develop a concise representation of AZMNL by the mean square error (MSE) criterion in [5]. Simulation results illustrate that proposed method achieves near optimum performance in various impulsive interference.

In a radar or sonar system, to achieve data adaptive detection, the threshold must be adjusted according to the interference to maintain constant false alarm ratio (CFAR). This requires evaluating the test threshold according preset false alarm ratio (FAR). An analytical PDF is expected in solving such problem as it is the same as in designing a detector. Recently, a novel and concise approximate PDF of α-stable distribution, BCGM [6], has been presented. Using such an absolute analytical expression, the relation between test threshold and FAR can be evaluated handily, which indicates that the detector is adaptive in CFAR sense.

In the next section we review the basic detection problem. Section 3 presents the algebraic-tail ZMNL (AZMNL). In section 4, we show how to evaluate the test threshold to maintain CFAR in terms of BCGM. Final section shows experimental results.

2. THE DETECTION PROBLEM FORMULATION

Detection of weak deterministic signal can be formulated as a hypothesis-testing problem [1].

$$\begin{aligned} H_0 &: \quad \mathbf{x} = \mathbf{w} \\ H_1 &: \quad \mathbf{x} = A\mathbf{s} + \mathbf{w} \end{aligned} \tag{1}$$

where H_0 is the null hypothesis that indicates the presence of the signal and H_1 indicates the non-presence of the signal in the observation. $\mathbf{x}$ represents the received signal, A is the amplitude of the deterministic signal $\mathbf{s}$ and the $\mathbf{w}$ is IID noise subject to symmetric α-stable distribution (*SαS*). When $A \to 0$, the detected signal become a weak signal. The classical approach for weak signal detection is usually based on the Neyman-Pearson (NP) lemma [1], such as Locally Optimum (LO) detector [1]. Based on NP lemma, the test statistic of optimum detector is given by [2]

$$\Lambda_{NP} = \sum_{k=1}^{N} \ln\left\{\frac{f_{\alpha,\sigma}\left[x(k)\right]}{f_{\alpha,\sigma}\left[x(k) - As(k)\right]}\right\} \tag{2}$$

where $f_{\alpha,\sigma}\left[x(k)\right]$ is the *SαS*'s PDF. It compares to a preset threshold η. When $\Lambda_{NP} \geq \eta$, the detector decides that the signal s presents. For weak signal, the log-likelihood test for Locally Optimum (LO) detector is given by

$$\Lambda_{LO} = \sum_{k=1}^{N} s(k) g\left[x(k)\right] \underset{<_{H_1}}{\overset{\geq^{H_0}}{}} \eta \tag{3}$$

where $g(x)$ is the locally optimum ZMNL function defined as

$$g_{LO}(x) = -\frac{f'_{\alpha,\sigma}(x)}{f_{\alpha,\sigma}(x)} \tag{4}$$

Because $f_{\alpha,\sigma}(x)$ does not have a close form, its optimum ZMNL function does not have a simple analytical expression and needs to be evaluated numerically by FFT. There are two popular sub-optimal detectors to deal with α-stable interference, Cauchy and Gaussian-tailed ZMNL (GZMNL) based detector. As $\alpha = 1$, the interfering noise is Cauchy and the ZMNL function satisfies [2]

$$g_C(x) = \frac{2x}{x^2 + \sigma^2} \tag{5}$$

As to Gaussian-tailed ZMNL (GZMNL), it is defined as [4]

$$g_G(x) = \begin{cases} -\delta \exp(-(x+\delta)^2 / 2\sigma_r^2), & \text{if } |x| > \delta \\ x, & \text{if } |x| \leq \delta \end{cases} \tag{6}$$

where $\delta = 3\sigma_r + \text{median}(x)$ is the breakpoint and $\sigma_r = \sigma / 0.7$ controls the tail behavior. It is continuous and taking on exponentially decaying tails. Furthermore, it plays the role of transforming the stable noise to finite variance noise.

3. ALGEBRAIC-TAILED ZMNL

Most impulsive noise subjects to *SαS* with $1 < \alpha < 2$. Since the standard *SαS* density function can be represented by asymptotic series [2]

$$f_\alpha(x) = \frac{1}{\pi\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} \Gamma\left(\frac{2k+1}{\alpha}\right) x^{2k} \tag{7}$$

as $1 < \alpha < 2$, we have, near $x = 0$, $f_\alpha(x) \approx \frac{1}{\pi\alpha}\left[\Gamma(1/\alpha) - \frac{\Gamma(3/\alpha)}{2}x^2\right]$ and $f_\alpha{}'(x) \approx -\frac{\Gamma(3/\alpha)x}{\pi\alpha}$. Thereby, the optimum ZMNL function in the vicinity of $x = 0$ satisfies

$$g(x;\alpha) = -\frac{f_\alpha{}'(x)}{f_\alpha(x)} = \frac{\Gamma(3/\alpha)x}{\Gamma(1/\alpha) - \frac{\Gamma(3/\alpha)}{2}x^2} \approx \frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)}x \tag{8}$$

which indicates that the ZMNL of *SαS* r.v. is linear near $x = 0$ with slope $k = \Gamma(3/\alpha)/\Gamma(1/\alpha)$. Moreover, the distributions of *SαS* density in tails have the approximate property as below [3]

$$\Pr(x > \lambda) \sim \sigma^\alpha \frac{C_\alpha}{2} \lambda^{-\alpha} \tag{9}$$

as $\lambda \to \infty$ and $1 < \alpha < 2$, where $C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}$. Hence we obtain

$$g(x \mid x \to \infty, \alpha) = -\frac{f_{\alpha,\sigma}{}'(x)}{f_{\alpha,\sigma}(x)} = \frac{\alpha+1}{x} \tag{10}$$

by $f_{\alpha,\sigma}(x \mid x \to \infty) = (\alpha\sigma^\alpha C_\alpha \lambda^{-(\alpha+1)})/2$ and $f_{\alpha,\sigma}{}'(x \mid x \to \infty) = (-(\alpha+1)\alpha\sigma^\alpha C_\alpha \lambda^{-(\alpha+2)})/2$, which indicates that the α-stable distribution has an approximate algebraic-tail as $(\alpha+1)/x$ [7]. Hence, we suggest a generic algebraic-tailed ZMNL (AZMNL) to approximate the true *SαS*'s ZMNL as a continued function

$$g_A(x) = \begin{cases} \frac{K(\alpha)}{x}, & |x| > \tau \\ kx, & |x| \leq \tau \end{cases} \tag{11}$$

where $k = \Gamma(3/\alpha)/\Gamma(1/\alpha)$, $\tau = \sqrt{K(\alpha)/k}$ and $K(\alpha)$ is a polynomial expression on α. To obtain the near optimum $K(\alpha)$, we adopt the mean square error (MSE) criterion suggested by S. Zozor et al. [5]. The criterion leads to the minimization of

$$\text{MSE}(K(\alpha)) = \int_{\mathbf{R}} \left(\frac{-f_{\alpha,\sigma}{}'(x)}{f_{\alpha,\sigma}(x)} - g_A(x)\right)^2 f_{\alpha,\sigma}(x)dx \tag{12}$$

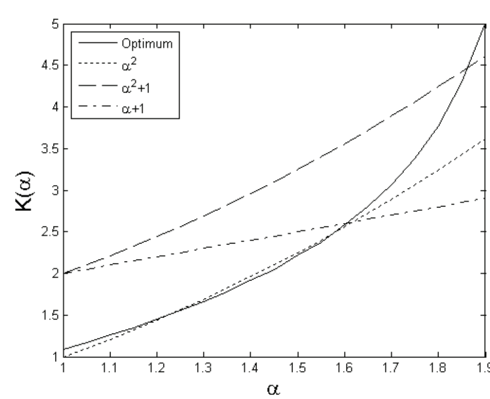


Fig.1 The comparisons between various optimum $K(\alpha)$

By such criterion, we find that the optimal function satisfies $K(\alpha) \approx 13.0859\alpha^4 - 68.4388\alpha^3 + 134.7758\alpha^2 - 115.9855\alpha + 37.6752$. However, a concise representation is always expected in real practice. To obtain a near optimal and concise expression, we compare the optimum $K(\alpha)$ with three expressions, namely $\alpha+1$, α^2+1 and α^2 as shown in Fig.5. From the results, it is easy to see that $K(\alpha)=\alpha^2$ is the closest expression to the optimal $K(\alpha)$ and reaches better approximation as $\alpha < 1.7$. Hence it can be viewed as a brief version of $K(\alpha)$.

4. CALCULATE THE TEST THRESHOLD

Let us consider a LO detector with ZMNL function $g_z(x)$. Under H_0 assumption and for IID multi- sample case, the mean of $T(x)$, $E\left(T(x);H_0\right)=0$. And the variation of $T(x)$ satisfies

$$\mathrm{var}\left(T(x);H_0\right)=v^2\sum_{n=0}^{N-1}s^2[n] \tag{13}$$

$$v^2=\int_{-\infty}^{\infty} g_z^2(x) f_{\alpha,\sigma}(x)dx \tag{14}$$

where N is the length of testing block, $g_Z(x)$ is ZMNL. For the fixed detection threshold η, the false alarm ratio and detection threshold satisfy the following relations

$$P_{FA}=Q\left(\frac{\eta}{v\sqrt{N}}\right) \tag{15}$$

$$\eta = v\sqrt{N}Q^{-1}(P_{FA}) \tag{16}$$

Hereby, we can evaluate the detection threshold η with respect to preset P_{FA} by equation (15).

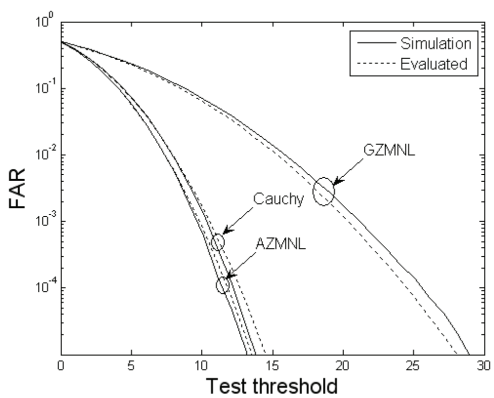


Fig.2 FAR curves as functions of test threshold η

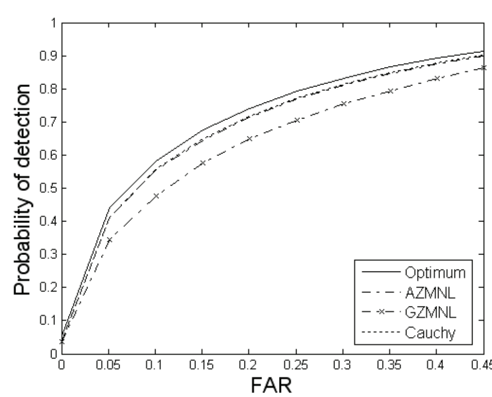
Via equation (13) and (14), the general relation between false alarm ratio and test threshold can be developed for sub-optimum detector. In addition, we adopt the BCGM as the approximation PDF of SαS interference to decrease the computation complexity in calculation on equation (14). The BCGM is defined as [6]

$$f_{\alpha,\sigma}(x)=(1-\varepsilon)\cdot\frac{1}{2\sqrt{\pi}\sigma}\exp\left(-\frac{x^2}{4\sigma^2}\right)+\frac{\varepsilon\sigma}{\pi(x^2+\sigma^2)} \tag{17}$$

where $\varepsilon=\frac{2\Gamma(-p/\alpha)-\alpha\cdot\Gamma(-p/2)}{2\alpha\cdot\Gamma(-p)-\alpha\cdot\Gamma(-p/2)}$ is the mixture ratio, $p=-0.25$, σ is the scale exponent of $S\alpha S$ distribution. Using such a concise expression, the variations of test statistic can be evaluated by equation (14) more easily, which is significant in a practical system. And then the false alarm ratio (FAR) according various value of test threshold η can be evaluated by equation (15). In order to examine the evaluated false alarm ratio, we perform Monte-Carlo simulations. When noise subjects to standard $S\alpha S$ with $\alpha=1.5$, we chose $N=20$ samples per block and perform 10^6 test simulations. For each sub-optimum ZMNL: AZMNL, GZMNL and Cauchy based detector, the real false alarm ratio calculated by test simulation is compared with the evaluated one by equation (15) concerning various test threshold. The experimental results are shown in Fig.2 in which we can observe that the curves obtained by test simulations are very close to the results evaluated by equation (15). Only minor error appears in the region of lower FAR, which relates the limited number of simulations. Consequently, the test threshold can be evaluated with preset FAR via equation (16). This indicates that the corresponding detector has the ability to adjust its test threshold according preset FAR and noise parameters adaptively.

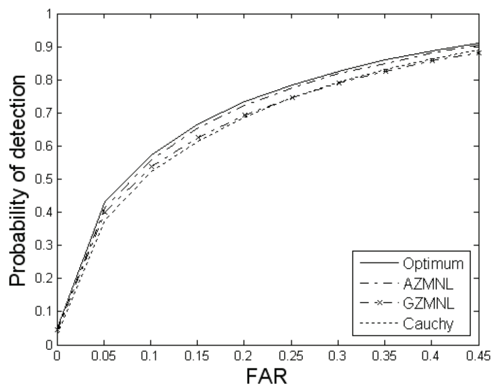
5. SIMULATIONS FOR WEAK SIGNAL DETECTION

The experimental simulations are performed to demonstrate the validity of suggested AZMNL based detector. Without loss of any generality, the noise is assumed as standard SαS noise and the detected signal is considered as known deterministic direct current signal in the following experiments.

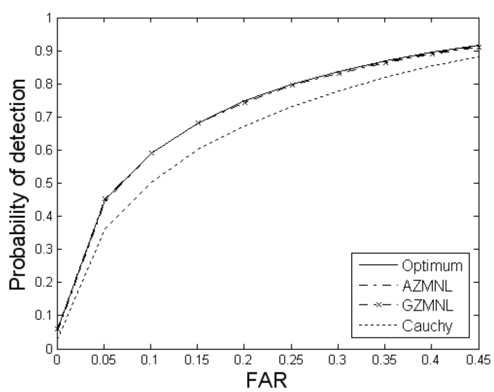


(a) $\alpha=1.2$

We evaluate the detection probability of AZMNL based detector, and make comparison with the optimum detector, Cauchy and GZMNL based detector. The results are based on 10000 test blocks of length N=20. The experiments are concerned with the corresponding receiver operating characteristics (ROCs) of detectors versus false alarm ratio with respect to $\alpha = 1.2, 1.5, 1.8$ as $A = 0.5$ (weak signal) respectively. The false alarm ratio is restricted within 0.001 to 0.45. The experimental results are shown in Fig.3.



(b) $\alpha = 1.5$



(c) $\alpha = 1.8$

Fig.3 Probability of detection versus false alarm ratio

From above results, one can clearly observes that AZMNL based detector has near optimal performance in weak signal case. The results also show that for small value of α, for example $\alpha = 1.2$ (the interference is close to Cauchy case), the Cauchy based detector performs much at one as AZMNL based detector, however, as the value of α increases, its performance degrades significantly, especially near Gaussian case (i.e. $\alpha = 1.8$). On the contrary, the GZMNL based detector renders near optimal performance for larger α, but becomes inferior in case of smaller α. This is due to the fact that Cauchy ZMNL is close to the optimal ZMNL near Cauchy distribution, and GZMNL is close to optimal ZMNL for larger α which indicates the input is near Gaussian r.v.. Compared with Cauchy and GZMNL, the proposed AZMNL can achieve good performance and is robust for operation in environments of stable interference of varying characteristic exponent α. Consequently, suggested detector is more efficient in impulsive circumstance.

6. CONCLUSION

In this paper, we have proposed a new AZMNL based detector. By means of BCGM, which is a concise approximation expression of PDF of *SαS* interference, the relation between test threshold and FAR is developed. Hence, an adaptive detector can be achieved by adjusting its testing threshold according with preset FAR and noise parameters. Various simulations have been provided to inspect the performances of new detector and the results illustrate the AZMNL based detector has near optimal performance and superiority to other detectors for weak signal detection. Although proposed method is available in the range 1<α<2, it is not a serious limit since noise with α value below 1 is too impulsive and hence is infrequently occurring in real world.

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