BISPECTRUM ON FINITE GROUPS

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ABSTRACT

The algebraic theory of finite groups appears in signal processing problems involving the statistical analysis of ranked data and the construction of invariants for pattern recognition. Standard signal processing techniques involving spectral analysis are, in theory, possible for data defined on finite groups by using the Fourier transform provided by group representations. However, one such technique, the bispectrum, which is useful for analysing non-Gaussian data as well as for constructing geometric invariants, has not been explored in detail for finite groups. This paper shows how to construct the bispectrum on an arbitrary finite group or homogeneous space and explores its properties. Examples are given using the symmetric group as well as wreath-product groups.

***Index Terms*—** SYMMETRIC GROUP, BISPECTRUM, WREATH-PRODUCT

1. INTRODUCTION

The algebraic theory of finite groups is useful for handling signal processing problems involving the statistical analysis of ranked data [1] and the recognition of transformed copies of a template in images [2]. On each finite group, the Peter-Weyl theorem shows how the Fourier transform may be constructed from the group's irreducible unitary representations [3], making it possible, at least in theory, to apply signal processing techniques involving spectral analysis of data. However, one such technique, the bispectrum, has not been explored in detail for finite groups. The bispectrum is phase-sensitive but Gaussian-noise insensitive, making it useful for devising geometric invariants for pattern recognition as well as for analyzing non-Gaussian signals in Gaussian noise [4]. This paper shows how to construct the bispectrum on an arbitrary finite group or homogeneous space, describes some of its key properties, and provide examples using the symmetric group of permutations as well as wreath-product groups.

The theory of the bispectrum has been studied for functions defined on the real line (and $\mathbf{R}^n$) [5]. It has been extended from the Euclidean domains to handle functions whose domain is a sphere [6], which occur in analyzing geophysical data as well as in cardiac imaging. The sphere is a space on which the non-commutative group of three-dimensional rotations $SO(3)$ acts transitively, i.e., it is possible to map any point on the sphere to any other through an appropriate rotation. This result suggests that the concept of the bispectrum should carry over to finite groups as well as to their homogeneous spaces (sets on which they act transitively), as discussed further below. Ref. [7] proposes the "skew-spectrum" for graphs and compares it to the abstract bispectral theory contained in [8]. This paper elaborates on the bispectrum for finite groups and homogeneous spaces with relevant examples.

2. BACKGROUND

The bispectrum is defined for functions on the real-line as the Fourier transform of the function's triple correlation, which is obtained from integrating the function against two independently shifted copies of itself as follows:

$$T_f(x,y) = \int_{-\infty}^{\infty} f^*(t) f(t+x) f(t+y) dt.$$

If F is the Fourier transform of the function f, then it can be shown [5] that the bispectrum is the two-variable product spectrum:

$$B_f(u,v) = \mathcal{F}\{T_f\} = F(u)F(v)F(u+v)^*. \tag{1}$$

This equation shows that the bispectrum carries phase information, but yet remains invariant under translation of the function; translation by x has the effect $F(u) \mapsto F(u)e^{j2\pi ux}$, and

$$e^{j2\pi ux} e^{j2\pi vx} e^{-j2\pi(u+v)x} = 1 \tag{2}$$

Furthermore, it is known that for a large class of functions, including those that are bandlimited, the bispectrum is unique to the function up to a single unknown translation [5]. This makes the bispectrum useful for, among other applications, averaging translating copies of a function in Gaussian noise without averaging out the function. The function may be recovered from its bispectrum using a recursion of the form (shown here for simplicity for sampled spectra):

$$\hat{F}(n) = \frac{B_f(n-1,1)^*}{\hat{F}(n-1)\hat{F}(1)}$$

The bispectrum is useful for statistical signal processing primarily because it is derived from the third order cumulant $E[X_t X_{t+u} X_{t+v}]$, which vanishes identically for any zero mean Gaussian process [4].

2.1. Group representations

This section states some necessary facts from the literature on group representation theory (see ([3][9][10] for details). A unitary representation of a finite group G is a homomorphism $D_\omega : G \to U(n_\omega)$ where $U(n_\omega)$ is the group of $n_\omega \times n_\omega$ unitary matrices. Two representations $D_{\omega_1}, D_{\omega_2}$ are equivalent if there exists a unitary matrix U such that $D_{\omega_1}(g) = UD_{\omega_2}(g)U^\dagger$, where $\dagger$ is conjugate transpose. A representation is irreducible if it not equivalent to a direct sum of smaller dimensional representations. Let $\{D_\omega\}_{\omega\in\Omega}$ represent a complete set of unitary irreducible representations of G, one from each equivalence class. Their dimensions are constrained by $\sum_{\omega\in\Omega} n_\omega^2 = |G|$, where $|G|$ is the number of elements in G. The character χ_ω of a representation D_ω is defined as the trace $\chi_\omega(g) = \operatorname{tr}[D_\omega(g)]$. If $< f_1, f_2 >= 1/|G| \sum_{g\in G} f_1(g) f_2(g)^*$ denotes the inner product on G, then $< \chi_{\omega_1}, \chi_{\omega_2} >= 0$ for $\omega_1, \omega_2 \in \Omega$, unless $\omega_1 = \omega_2$, in which case $< \chi_{\omega_1}, \chi_{\omega_1} >= 1$.

The (matrix-valued) Fourier transform on G is defined by coefficients

$$F(\omega) = \frac{1}{|G|}\sum_{g\in G} f(g) D_\omega(g)^\dagger.$$

A (left) translation of f by $h \in G$ is $f(g) \mapsto f(hg)$, and $f_1(g) = f_2(hg)$ if and only if $F_1(\omega) = F_2(\omega)D_\omega(h)$. The inverse Fourier transform is

$$f(g) = \sum_{\omega\in\Omega} n_\omega \operatorname{tr}[F(\omega)D_\omega(g)]$$

Note that we may choose $D_0(g) = 1$ to be the first irreducible representation for every finite group. Hence, $F(0)$ is the mean (DC) value of the function, just as it is in the ordinary Fourier transform on the real line.

3. FINITE GROUPS

In this section, the theory of the bispectrum on finite groups is developed. The left-invariant autocorrelation of f is

$$A_f(g') = \frac{1}{|G|}\sum_{g\in G} f^*(g) f(gg'). \tag{3}$$

As the name suggests, if $f_1(g) = f_2(hg)$, then $A_{f_1} = A_{f_2}$. Similarly, the triple correlation of f is

$$T_f(g_1, g_2) = \frac{1}{|G|}\sum_{g\in G} f^*(g) f(gg_1) f(gg_2). \tag{4}$$

Note that T_f is a function defined on $G \times G$, and that, like the autocorrelation, the triple correlation is invariant under left translation: $T_{f_1} = T_{f_2}$ if $f_1(g) = f_2(hg)$. Moreover, T is the sample third order cumulant, whose expected value is zero for zero-mean Gaussian noise.

The Fourier transformation of T_f requires Kronecker (tensor) products of the representation matrices $\{D_\omega\}_{\omega\in\Omega}$. The bispectrum is obtained by the formula:

$$B_f(\sigma,\delta) = \frac{1}{|G|}\sum_{g_1\in G}\sum_{g_2\in G} T_f(g_1,g_2) D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger.$$

for all $\sigma, \delta \in \Omega$. Inserting eq (4) above, and simplifying, and noting that the interior two summations (over g_1, g_2) yield

$$\begin{aligned}&\textstyle\sum_{g_1\in G}\sum_{g_2\in G} f(gg_1) f(gg_2)\left[D_\sigma(g_1)^\dagger \otimes D_\delta(g_2)^\dagger\right]\\ &\quad = [F(\sigma)\otimes F(\delta)]\,[D_\sigma(g)\otimes D_\delta(g)],\end{aligned}$$

we obtain that $B_f(\sigma,\delta)$ is

$$F(\sigma)\otimes F(\delta)\left[\frac{1}{|G|}\sum_{g\in G} f(g)^* D_\sigma(g)\otimes D_\delta(g)\right]. \tag{5}$$

The Kronecker product $D_\sigma(g) \otimes D_\delta(g)$ is, in general, reducible, and hence we may write for suitable indices ω_1, ω_2, . . ., ω_k, all in Ω, that

$$D_\sigma(g)\otimes D_\delta(g) = C_{\sigma\delta}\left[D_{\omega_1}(g)\oplus\cdots\oplus D_{\omega_k}(g)\right] C_{\sigma\delta}^\dagger. \tag{6}$$

Here, $\oplus$ is the direct sum of matrices, the unitary matrix $C_{\sigma\delta}$ is the Clebsh-Gordan matrix for σ, δ, and the indices ω_1, . . ., ω_k depend on the selection of σ, δ. From matching the dimensions on both sides of (6), we see that $n_\sigma n_\omega = n_{\omega_1} + \cdots + n_{\omega_k}$. Using (6) in (5) yields the bispectrum formula for arbitrary finite groups:

$$B_f(\sigma,\delta) = F(\sigma)\otimes F(\delta) C_{\sigma\delta}\left[F(\omega_1)^\dagger\oplus\cdots\oplus F(\omega_k)^\dagger\right] C_{\sigma\delta}^\dagger. \tag{7}$$

The formula simplifies to (1) if G is Abelian, in which case the fundamental theorem of finite Abelian groups shows that G is a direct product of cyclic groups, each of which is represented by the complex exponentials $x \mapsto e^{j\omega x}$.

We now develop some of the properties of the bispectrum (7) for finite groups. By taking the trace on both sides of (6), we see that

$$\chi_\sigma(g)\chi_\delta(g) = \chi_{\omega_1}(g) + \chi_{\omega_2}(g) + \cdots + \chi_{\omega_k}(g).$$

Since characters are orthonormal, we can obtain the indices ω_1, . . ., ω_k (which are not necessarily distinct) and their multiplicities by computing inner products $< \chi_\sigma\chi_\delta, \chi_\omega >$, for all $\omega \in \Omega$. Moreover, if either σ or δ is one-dimensional, then it follows that $D_\sigma \otimes D_\delta$ is irreducible (because $<\chi_\sigma\chi_\delta, \chi_\sigma\chi_\delta>= 1$, see Section 2.1). In particular, if we take $\sigma = 0$, then the decomposition (6) is trivial and we have

$$B_f(0, \delta) = F(0)F(\delta)F(\delta)^\dagger.$$

It can be seen that, aside from multiplication by the scalar $F(0)$, this is the Fourier transform of the autocorrelation A_f in (3). A matrix of the form $FF^\dagger$ is positive semidefinite, and by using the well-known fact that every matrix has a polar decomposition, we see that $B_{f_1} = B_{f_2}$ for two functions f_1 and f_2 if and only if $F_1(\omega) = F_2(\omega)U(\omega)$ for some unitary matrix $U(\omega)$ (assuming that f_1, f_2 have a non-zero DC value).

Suppose now that all of the Fourier matrices $F_1(\omega)$ of a particular function f_1 are non-singular. In that case, we obtain by straightforward manipulation that $B_{f_1} = B_{f_2}$ if and only if $F_1(\omega) = F_2(\omega)U(\omega)$ for unitary matrices $U(\omega)$ and all $\omega \in \Omega$, and moreover for all $\sigma, \delta \in \Omega$ that these unitary matrices satisfy the decomposition

$$U(\sigma) \otimes U(\delta) = C_{\sigma\delta}\left[U(\omega_1) \oplus \cdots \oplus U(\omega_k)\right] C_{\sigma\delta}^\dagger. \qquad (8)$$

It is clear that from eq (6) that if $U(\omega) = D_\omega(h)$ for some $h \in G$, i.e., that f_1 is a left translation of f_2, then eq (8) is satisfied. It is shown furthermore in [8] that (8) is satisfied only if $U(\omega) = D_\omega(h)$, ensuring that the bispectrum is unique for a function with non-singular Fourier coefficients up to a left translation on G.

4. HOMOGENEOUS SPACES

The bispectrum theory extends beyond groups to their homogeneous spaces, which are defined as sets on which the group acts transitively. For example, the set $X_n = \{1, 2, \ldots, n\}$ is a homogeneous space for the group of all permutations of n digits, S_n. If we let $S_{n,n}$ denote the subgroup of S_n whose elements fix the last entry n in X_n, then we obtain a bijection between X_n and the coset space $S_n/S_{n,n} = \{S_{n,n}g : g \in S_n\}$. In fact, every homogenous space X of a finite group G can be represented as the coset space G/H of a subgroup H whose elements fix a particular element of X.

To any function $\tilde{f}$ defined on a coset space G/H there corresponds a unique function f on G that is constant on cosets Hg, and furthermore $f(g) = \tilde{f}(\pi(g))$, where $\pi : G \to G/H$ is the canonical coset map. Note that f is invariant under left translation by elements of H, i.e., $f(g) = f(hg)$ for $h \in H$. If $\{F(\omega)\}_{\omega\in\Omega}$ are the Fourier transform coefficents of f, then by the translation property of the coefficients we must have that $F(\omega) = F(\omega)D_\omega(h)$ for all $h \in H$. By integration over H, we obtain that $F(\omega) = F(\omega)P_H(\omega)$, where

$$P_H(\omega) = \frac{1}{|H|}\sum_{h\in H} D_\omega(h).$$

Note that P_H is a projection matrix: $P_H P_H = P_H$. By integrating both sides of (6) we see that Kronecker produts of projection operators decompose according to the Clebsh-Gordon formula ([11], pg 190):

$$P_H(\sigma) \otimes P_H(\delta) =$$
$$C_{\sigma\delta}\left[P_H(\omega_1) \oplus \cdots \oplus P_H(\omega_k)\right] C_{\sigma\delta}^\dagger \left[P_H(\sigma) \otimes P_H(\delta)\right] \quad .$$

This ensures that the bispectrum formula (7) is valid for functions whose domain is a homogeneous space as well as those defined on groups.

5. EXAMPLES

The theory of bispectrum on finite groups is illustrated with two specific examples, the symmetric group S_4 and the wreath-product group $Z_{3,4}$.

5.1. Symmetric group example

Due to Cayley's theorem, the bispectral theory for symmetric groups serves as a template for every finite group. Certain facts about representations of S_n are used below; see [9] for more details. Every symmetric group has $P(n)$ irreducible representations, where $P(n)$ is the number of arithmetic partitions of n. The trivial representation $D_0(g) = 1$ and the sign representation $D_1(g) = (-1)^{\tau(g)}$, where $\tau(g)$ is the number of pairwise transpositions needed to produce g, are two irreducible representations of S_n that are one-dimensional.

As a specific example, the bispectrum for the symmetric group S_4 is described in detail. Note that $P(4) = 5$, since there are 5 ways that positive numbers add up to 4; hence, there are 5 irreducible representations, the first 2 of which are D_0 and D_1 as described above. The next 3 representations, denoted D_2, D_3 and D_4, have dimensions 2, 3, and 3, respectively. Note that the squares of the dimensions add up to $|G|$, i.e., $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$.

The bispectral coefficients in eq. (7) depend on the decomposition of Kronecker products $D_\sigma \otimes D_\delta$ into irreducible representations. Since there are 5 represenations, the decomposition may be written in a vector such as, for example, 01100; this particular vector indicates that only representations D_1 and D_2 are present. Using the character table for S_4 in [10], Table 1 shows the decompositions for various tensor products. It shows, in particular, that $B_f(3, 4)$ is the matrix

$$F(3) \otimes F(4)C_{34}\left[F(4)^\dagger \oplus F(3)^\dagger \oplus F(2)^\dagger \oplus F(1)^\dagger\right] C_{34}^\dagger.$$

Methods for calculating the coefficients of the 9×9 Clebsh-Gordan matrix C_{34} are discussed in [10] (pp 254-273).

5.1.1. *Permutations of* 4

As discussed in Section 4, the space $X_4 = \{1, 2, 3, 4\}$ is homogeneous for symmetric group S_4, and is in bijection with

Table 1. Decomposition of Tensor Products on $S4$

	0	1	2	3	4
0	10000	01000	00100	00010	00001
1	01000	10000	00100	00001	00010
2	00100	00100	11100	00011	00011
3	00010	00001	00011	10111	01111
4	00001	00010	00011	01111	10111

the coset space $S_4/S_{4,4}$, where $S_{4,4}$ is the subgroup of S_4 of permutations that fix the number 4. The 5 representations of S_4 span a 24-dimensional space, but when projected onto the cosets of $S_{4,4}$ span only a 4- dimensional space. Note that $S_{4,4}$ is isomorphic to S_3, the group consisting of all 6 permutations of 3 elements. Hence, the projected representations $D_\omega P_{S_{44}}(\omega)$, for $0 \leq \omega < 5$, have only 4 non-zero elements. We see in particular that $D_1 P_{S_{44}}$, the sign representation, must project to zero. Software for calculating the coefficients is available online [1].

5.2. Wreath-product group $Z_{3,4}$

The wreath-product group $Z_{3,4}$, discussed in [2], is a non-abelian group of automorphisms on the two-level tree with 3 branches at level 1 and 4 leaves per branch at level 2 (hence a total of 12 leaves). The elements of $Z_{3,4}$ may be described as vectors $[(j_0, j_1, j_2); i]$, where $0 \leq j_k < 4$ are cyclic shifts of the 4 leaves in each of the 3 branches of the tree, and $0 \leq i < 3$ is a cyclic shift of the 3 branches among themselves.

As discussed in [2], the irreducible representations of $\mathbf{Z}_{3,4}$ acting on the two-level tree are either 1-dimensional (there are 3 of these) or 3-dimensional (there are 3 of these). The Fourier transform is equivalent to a computation of the Radon transform from the leaves to the branch nodes, followed by the ordinary DFT on the branch nodes, as well as the DFT of the leaves in each branch (omitting the DC component). For example, let the leaves of the tree be represented by the vector $f = [1, 2, \ldots, 12]$. The branch node values are obtained from summing together 4 leaves each, and therefore produce the Radon transform values $s = [10, 26, 42]$. The 12-point spectrum F is obtained by combining the 3-point DFT of s, followed by the 3 non-DC DFT values of each of the 3 sets of 4 leaves, e.g., for the set $[1, 2, 3, 4]$, the non-DC DFT values are $[-2+2j, 2, -2-2j]$, and similarly for the other two sets. Reconstruction of f from F is obvious.

The bispectrum of f is obtained from computing the bispectrum of each of the 4 blocks of 3 similar coefficients in F, using (1). For example, the 3 elements $F(3), F(4), F(5)$ constitute the non-DC coefficients of the set $[1, 2, 3, 4]$ as described above. Shifting each index by 2 to account for their placement in X, the bispectral terms according to (1) include $F(3)F(3)F(4)^*$ and $F(3)F(4)F(5)^*$, as well as their complex conjugates.

[1] See "Symmetrica", available at http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/

6. CONCLUSIONS

This paper demonstrates the construction of the bispectrum for finite groups, and shows how it may be applied with illustrative examples of two different noncommutative groups. The results may be used in future research to develop phase-sensitive, and Gaussian noise insensitive, statistics for signal processing and pattern recognition.

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