NONLINEAR FILTERING FOR CONTINUOUS-TIME SYSTEMS USING THE LINEAR FRACTIONAL TRANSFORMATION MODEL

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ABSTRACT

In this paper, we propose Bayesian filtering technique for continuous-time dynamical models with sampled-data measurements using the linear fractional transformation (LFT) model which transforms the nonlinear state space model into an exact equivalent linear model with a simple nonlinear feedback loop. The linear model is amenable to Euler discretization. Simulation results demonstrate that the proposed filtering technique gives better approximation and tracking performance than the unscented Kalman filter (UKF) which diverges for highly nonlinear problems.

Index Terms— nonlinear filtering, Bayesian estimation, continuous-time systems, sampled-data measurements, linear fractional transformation model

1. INTRODUCTION

The continuous-time state dynamics model with sampled-data measurements gives an accurate model for real systems that are continuous processes and for sensor data available at discrete sampling intervals only. Under linear assumption on the state space model and certain particular cases the conditional expectation of the state given a sequence of sampled-data measurements admits a closed form solution [1, 2]. In general, there exists no tractable method to compute the conditional mean and an approximation is inevitable.

In the literature there are two standard approximation approaches for nonlinear filtering. The extended Kalman filter (EKF) applies a local linearization to the nonlinear mapping around the state estimate [3, 4]. Using this method the nonlinear model is replaced by a linearized model based on the first-order Taylor series expansion. The unscented Kalman filter (UKF) [5] on the other hand, aiming at the direct approximation of the exact statistical model applies the unscented transformation [5] to compute the covariances involved and does not give a linearized model of the nonlinear mapping. It has been shown that in most applications UKF gives better approximation that EKF [6]. Despite the advantage of UKF over EKF, the two approaches work reasonably well under mildly nonlinear conditions only. The focus of some research has been in the direction of sequential Monte Carlo (SMC) methods for nonlinear Bayesian filtering applications [7], grid based methods [8] and other numerical solution to Kolmogorov forward equation [9].

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In nonlinear control, the linear fractional transformation (LFT) method (see e.g., [10, 11] and the references therein) is extensively employed in gain-scheduling control to describe nonlinear plants by an equivalent linear model with nonlinear feedback. The LFT approach gives an equivalent representation for any smooth nonlinear mapping [10, 12, 13]. Moreover, for highly nonlinear systems involving complex fractional terms the LFT model gives an exact equivalent representation. In this paper, we propose nonlinear Bayesian filtering using the LFT model for continuous-time dynamical models with sampled-data measurements. By applying the unscented transformation to the feedback loop only we derive a closed form solution to estimate the conditional mean of the state. Our simulation results show that the proposed filtering approach gives a better tracking performance than UKF which gives inconsistent estimates for highly nonlinear problems.

The paper is structured as follows: Section II reviews the unscented transformation method and the LFT model. We then state the main result of this paper, a closed form solution to Bayes recursion using the LFT model and give simulation results in Section IV to compare the performance of the proposed filter with UKF. We conclude with some final comments in Section V.

2. BACKGROUND

Consider the dynamical equation given by Itô differential equation as

$$dx(t) = f(x(t))dt + \tilde{B}(t)d\beta(t), \tag{1}$$

where $f(\cdot)$ denotes a nonlinear mapping and $\beta(t)$ denotes an independent Brownian motion process. The state $x(t) \in \mathbb{R}^n$ is a Markov process independent of $\beta(t)$. $\tilde{B}(t) \in \mathbb{R}^{n\times p}$ denotes the noise gain matrix. The dynamical model is alternatively given in terms of the white noise process $\tilde{w}(t)$ with covariance $\tilde{Q}(t)$ as

$$\frac{dx(t)}{dt} = f(x(t)) + \tilde{B}(t)\tilde{w}(t). \tag{2}$$

For measurements available at discrete sampling intervals, the measurement model is given by

$$z_k = g(x(t_k)) + D_k v_k, \tag{3}$$

where $g(\cdot)$ denote a nonlinear mapping, $z_k \in \mathbb{R}^m$ denotes the observation at time step k and the sampling instant $t_k = kT$ with T as the sampling interval. The measurement noise v_k with zero mean and covariance R_k is statistically independent of the state $x(t_k)$ and $D_k \in \mathbb{R}^{m\times q}$ denote the noise gain matrix.

The filtering problem involves estimating the state $x(t_k)$ at time t_k given a sequence of observations $Z_k = \{z_1, \ldots, z_k\}$. The estimate given by recursive Bayesian estimation is obtained in a two step process. The predicted state $m(t_k|t_{k-1})$ at time t_{k-1} is given by the Kolmogorov forward partial differential equation. On arrival of measurement z_k at time t_k, the optimal filtering estimate is given by applying Bayes rule.

Under linear assumption on the mappings $f(\cdot)$ and $g(\cdot)$, (2)-(3) are given by

$$\dot{x}(t) = \tilde{A}(t)x(t) + \tilde{B}(t)\tilde{w}(t), \tag{4}$$

$$z_k = C_k x(t_k) + D_k v_k. \tag{5}$$

This case has been treated in detail in [3, 4, 1]. In the general case there exists no tractable method to compute the conditional expectation $\mathbb{E}\{x(t_k)|Z_k\}$. Only under certain regularity conditions the solution can be obtained in an analytical manner [1, 2]. Nonlinear filtering therefore involves an approximation to evaluate the estimate. In the literature there are two standard approximation approaches. EKF uses the first-order Taylor series approximation of $f(\cdot)$ and $g(\cdot)$ around some estimate [3, 4]. On the other hand, UKF applies the unscented transformation [5] based on the statistical linear regression technique to evaluate the conditional mean. The expected value given by UKF has a higher order accuracy compared to the expected value given by EKF. However, these approximation techniques are more suited to the class of mildly nonlinear problems only to give estimates with reasonably good accuracy.

A transformation applied extensively in gain scheduling control to describe nonlinear plants is the LFT model [10, 11]. It is well known that any smooth nonlinear mapping admits an LFT representation [10, 12, 13]. This property makes the LFT model amenable to the most general class of nonlinear problems. The intrinsic linear structure ensures efficiency and ease of implementation which has been demonstrated in [13] by applying it to a highly nonlinear control problem. Unlike UKF which does not give a linear model for $f(x)$, the LFT model admits Euler discretization. Localizing the unscented transformation to the feedback path only, the approximation is more accurate than applying UKF to the continuous-time dynamical model. This motivates recursive Bayesian filtering using the LFT model for a general class of nonlinear problems with continuous-time dynamical equation and sampled-data measurements.

2.1. The Unscented Transformation Method

Suppose $x \in \mathbb{R}^n$ denotes a random variable where $x = [\,x(1),\, x(2),\, \ldots,\, x(n)\,]^T$ with mean $\bar{x} = [\,\bar{x}(1),\, \bar{x}(2),\, \ldots,\, \bar{x}(n)\,]^T$ and covariance R_x. Suppose a second random variable y depends on x through the mapping $y = f(x)$, where $f(x) = [\,f_1(x),\, f_2(x),\, \ldots,\, f_m(x)\,]^T$ is smooth and nonlinear. The random variable y can be expressed in the exact statistical form

$$f(x) = R_{yx}R_x^{-1}(x - \bar{x}) + f(\bar{x}) + e, \tag{6}$$

where $R_{yx}R_x^{-1}(x - \bar{x}) + f(\bar{x})$ is an affine function of x and R_{yx} is the cross-covariance of y and x. The error $e = y - R_{yx}R_x^{-1}(x-\bar{x}) - f(\bar{x})$ is a random quantity and is uncorrelated to x. The procedure for the unscented transformation applied by UKF to approximate the covariance and means is as follows. Regression points x_i, $i = 1, \ldots, p$ are selected around $\bar{x}$ in a manner such that the sample mean and covariance of the points are identical to the mean and covariance of x. Then the mean and covariance of the random variable y and the cross-covariance of y and x are approximated by the distribution of the regression points x_i and y_i, $i = 1, 2, \ldots, p$ as,

$$\bar{y} = \frac{1}{p}\sum_{i=1}^{p} y_i, \qquad R_y = \frac{1}{p}\sum_{i=1}^{p}(y_i - \bar{y})(y_i - \bar{y})^T, \tag{7}$$

$$R_{yx} = \frac{1}{p}\sum_{i=1}^{p}(y_i - \bar{y})(x_i - \bar{x})^T. \tag{8}$$

Details of the unscented transformation are given in [5].

2.2. The Linear Fractional Transformation (LFT) Model

From robust control theory it is known that any smooth nonlinear mapping $f(x)$ admits an equivalent representation known as the LFT model [10, 12, 13],

$$\begin{bmatrix} y \\ y_\Delta \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w_\Delta \end{bmatrix}, \tag{9}$$

$$w_\Delta = \Delta(x) y_\Delta, \tag{10}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n_\Delta}$, $C \in \mathbb{R}^{n_\Delta\times n}$ and $D \in \mathbb{R}^{n_\Delta\times n_\Delta}$. The auxiliary variables $w_\Delta \in \mathbb{R}^{n_\Delta}$ and $y_\Delta \in \mathbb{R}^{n_\Delta}$ introduced are related via the feedback connection $\Delta(x)$ which admits a simple structure of the form $\Delta(x) = \sum_{i=1}^{n} \Delta_i x(i)$.

The LFT model (9)-(10) is nonlinear in the feedback path only. Using this representation the approximation is localized to the feedback loop only for the estimation of the auxiliary random variable w_Δ in (10). Given the moments of x, the moments of w_Δ can be computed from the distribution of the regression points $w_{\Delta i} = \Delta(x_i)y_{\Delta i}$ by applying the unscented transformation as shown in (7) where $y_{\Delta i}$ is approximated as $y_{\Delta i} \approx Cx_i + D\bar{w}_\Delta$ with $\bar{w}_\Delta = \mathbb{E}\{w_\Delta\}$.

Suppose $y = f(x) + B_1 w$ denotes a random variable which depends on x with mean $\bar{x}$ and covariance R_x and $w \sim \mathcal{N}(\cdot; 0, R_w)$, independent of x. The equivalent representation for the mapping using the LFT model takes the form

$$y = Ax + B_1 w + B_2 w_\Delta, \tag{11}$$
$$y_\Delta = Cx + Dw_\Delta, \tag{12}$$
$$w_\Delta = \Delta(x)y_\Delta. \tag{13}$$

where $B_1 \in \mathbb{R}^{m\times n_w}$ and $B_2 \in \mathbb{R}^{m\times n_\Delta}$. Taking expectation in (11) and substituting the expected value of w_Δ, the expectation of y is $\bar{y} = A\bar{x} + B_2\bar{w}_\Delta$. Under the assumption that w_Δ and w are uncorrelated, the covariance of y and the cross-covariance with x are $R_y = AR_xA^T + B_1R_wB_1^T + B_2R_\Delta B_2^T + AR_{\Delta x}^T B_2^T + B_2R_{\Delta x}A^T$ and $R_{yx} = AR_x + B_2R_{\Delta x}$ respectively where $R_{\Delta w}$ denotes the cross-covariance of w_Δ and w.

3. BAYES RECURSION USING LFT MODEL

For smooth nonlinear mappings $f(\cdot)$ and $g(\cdot)$, the LFT model gives an exact representation for (2)-(3),

$$\dot{x}(t) = Ax(t) + \tilde{B}_1\tilde{w}(t) + \tilde{B}_2\tilde{w}_\Delta(t), \tag{14}$$
$$z_k = C_1x(t_k) + D_{11}v_k + D_{12}w_{\Delta k}, \tag{15}$$
$$z_\Delta(t) = C_2x(t) + D_{22}w_\Delta(t), \tag{16}$$
$$w_\Delta(t) = \Delta(x(t))z_\Delta(t), \tag{17}$$

where $A \in \mathbb{R}^{n\times n}$, $\tilde{B}_1 \in \mathbb{R}^{n\times n_w}$, $\tilde{B}_2 \in \mathbb{R}^{n\times n_\Delta}$, $C_1 \in \mathbb{R}^{m\times n}$, $D_{11} \in \mathbb{R}^{m\times n_v}$, $D_{12} \in \mathbb{R}^{m\times n_\Delta}$, $C_2 \in \mathbb{R}^{n_\Delta\times n}$, and $D_{22} \in \mathbb{R}^{n_\Delta\times n_\Delta}$. $w_\Delta(t) \in \mathbb{R}^{n_\Delta}$ and $z_\Delta(t) \in \mathbb{R}^{n_\Delta}$ denote auxiliary variables. The feedback connection admits a simple structure as indicated above.

Under the assumption that the noise sequences $\{w(t)\}$ and $\{v(t_k)\}$ are mutually uncorrelated and independent of the state $x(t)$ and the auxiliary variable $w_\Delta(t)$ the following result holds.

Proposition 1 *Suppose the estimate of the state $x(t_{k-1})$ at time t_{k-1} given the data sequence Z_{k-1} is m_{k-1} with covariance of the error P_{k-1}. Then, the predicted state at time t_k conditional on the data up to time t_{k-1} is $m_{k|k-1}$ with covariance of the error in prediction $P_{k|k-1}$ where*

$$m_{k|k-1} = e^{AT}m_{k-1} + \int_0^T \left(e^{A\zeta}d\zeta\right)B_2\bar{w}_\Delta(t_{k-1}), \tag{18}$$

$$P_{k|k-1} = e^{AT}P_{k-1}e^{A^TT} + Q_k + R_{\Delta k-1} + e^{AT}\int_0^T \tilde{R}_{\Delta x}^T(t_{k-1})\cdot B_2^T e^{A^T\tau}d\tau + \int_0^T e^{A\tau}B_2\tilde{R}_{\Delta x}(t_{k-1})e^{A^TT}d\tau, \tag{19}$$

with

$$R_{\Delta k-1} = \int_0^T e^{A(t_k)\tau}\tilde{R}_\Delta(t_k)e^{A^T(t_k)\tau}d\tau. \tag{20}$$

Proposition 2 *Suppose the predicted state $x(t_k|t_{k-1})$ conditional on the data sequence Z_{k-1} at time t_{k-1} has the mean $m_{k|k-1}$ and is distributed with covariance $P_{k|k-1}$. Then, the conditional expectation of $x(t_k)|Z_{k-1}$ also conditional on the data z_k at time t_k is estimated as*

$$m_k = m_{k|k-1} + K_k(z_k - \eta_k), \tag{21}$$

and the covariance of the error in the estimate is

$$P_k = P_{k|k-1} - K_k(C_1P_{k|k-1} + D_{12}R_{\Delta x,k|k-1}), \tag{22}$$

with

$$\eta_k = C_1m_{k|k-1} + \int_0^T \left(e^{A\zeta}d\zeta\right)D_{12}\bar{w}_\Delta(t_k|t_{k-1}), \tag{23}$$

$$K_k = \left(P_{k|k-1}C_1^T + R_{\Delta x,k|k-1}^T D_{12}^T\right)\left(C_1P_{k|k-1}C_1^T + D_{11}R_kD_{11}^T + D_{12}R_{\Delta k|k-1}D_{12}^T + C_1R_{\Delta x,k|k-1}^T D_{12}^T + D_{12}R_{\Delta x,k|k-1}C_1^T\right)^{-1}, \tag{24}$$

where

$$R_{\Delta k|k-1} = \int_0^T e^{A\tau}\tilde{R}_\Delta(t_k|t_{k-1})e^{A^T\tau}d\tau. \tag{25}$$

4. SIMULATION RESULTS

In this section we present simulation results for the nonlinear benchmark model [13]. We consider tracking the kinematic state of the unstable nonlinear system $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T = [\xi(t), \dot{\xi}(t), \theta(t), \dot{\theta}(t)]^T$ where $(\xi(t), \dot{\xi}(t))$ denote the translational position and velocity of the oscillator and $(\theta(t), \dot{\theta}(t))$ denote the rotational position and velocity of the actuator respectively. The nonlinear state-space model is given by

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \frac{-x_1(t)+\epsilon x_4^2(t)\sin x_3(t)}{1-\epsilon^2\cos^2 x_3(t)} \\ x_4(t) \\ \frac{\epsilon\cos x_3(t)(x_1(t)-\epsilon x_4^2(t)\sin x_3(t))}{1-\epsilon^2\cos^2 x_3(t)} \end{bmatrix} + \tilde{B}_1\tilde{w}(t), \tag{26}$$

where $\tilde{w}(t) \sim \mathcal{N}(0, \tilde{Q}(t))$ with $\tilde{Q}(t) = diag([\,0.04,\ 0.001\,])$. The sampled-data measurements comprise of $(\xi(t),\, \theta(t))$ available at sampling interval $T = 10\,ms$ and corrupted by noise $v_k = \mathcal{N}(0, R_k)$ with $R_k = diag([\,\sigma_\xi,\ \sigma_\theta\,])$, $\sigma_\xi = 0.1\,m$ and $\sigma_\theta = \pi/180\,rad$. The LFT model for (26) is given in [13]. Given the initial condition $x(0) = [\,0.5,\, 0,\, 0,\, 0\,]^T$ and $P(0) = diag([\,3,\, 3,\, \pi/60,\, \pi/60\,])$ and for $\epsilon = 0.2$ the true trajectories of the oscillator and actuator for $10\,s$ are shown in Fig. (1). In Fig. (2) the mean square error (MSE) in the estimates of the oscillator and actuator are shown using the proposed filtering approach and UKF. The simulation results indicate that the proposed approach gives better performance than UKF.

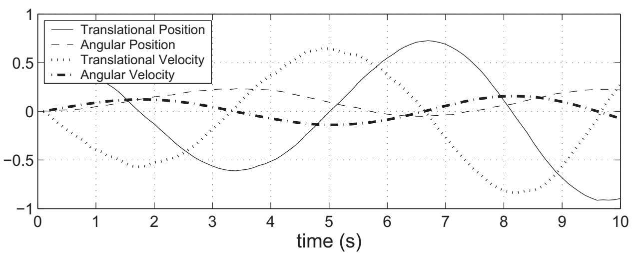

Fig. 1. Oscillator and actuator trajectories.

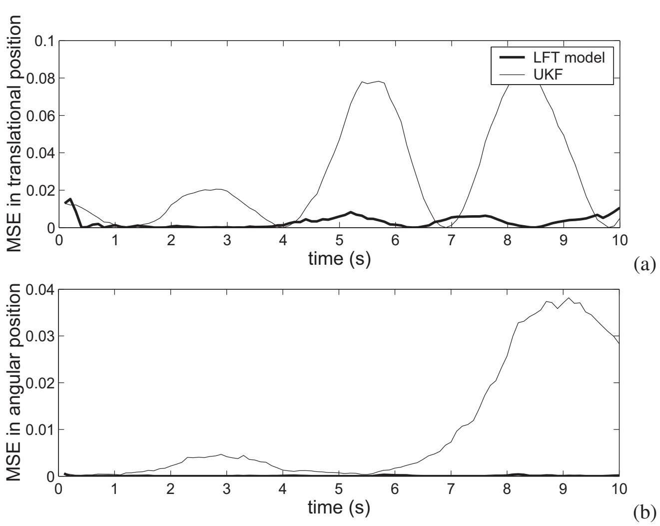

Fig. 2. MSE in oscillator/ actuator positions using LFT model and UKF.

5. CONCLUSIONS

In this paper, nonlinear Bayesian filtering using the LFT model is proposed for continuous-time problems with sampled-data measurements. By applying the unscented transformation to the feedback path of the LFT model only we derive a closed form solution to estimate the conditional mean of the state given a set of observations. Simulation results demonstrate that the proposed approximation works better than UKF.

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