Analog flat filter design *

H.G. Hoang†, H.D. Tuan†, and T.Q. Nguyen ‡
† School of Electrical Engineering and Telecommunication, University of New South Wales,
UNSW Sydney, NSW 2052, AUSTRALIA; Email: hung@student.unsw.edu.au, h.d.tuan@unsw.edu.au
‡Department of Electrical and Computer Engineering, UCSD. Email: nguyent@ece.ucsd.edu

***Abstract*—This paper proposes a systematic approach for the design of a general class of analog infinite-impulse-response (IIR) filters, which includes all well-known classical analog filters as a special case. All specifications including the conventional ones and also filter flatness degrees are explicitly incorporated into design process. Several numerical examples are presented to demonstrate the efficiency and flexibility of the proposed method.**

I. Introduction

Analog filters are indispensable parts in interface with the analog real world. Analog and digital circuits are often implemented together on the same integrated circuit chip [1], [3], [4], [12], [13]. Also, the most popular approach in digital infinite-impulse-response (IIR) filters design is based on transformation methods from analog counterpart [2], [9], [10]. There is no doubt that design of analog filters is a fundamental problem in signal processing, communications, and control.

In contrast to digital IIR filter design, the classical design of analog IIR filters looks rather complete, so there is not much further development. The Chebyshev min-max approximation works so well. Each classical filter is optimal in some sense. Given the filter order, the stop-band ripple and the cut-off frequency the Chebyshev filter has the least peak ripple in the pass-band among all pole filters [9]. On the other hand, given the filter order, the pass-band ripple and the cut-off frequency the inverse Chebyshev filter has the least peak ripple in the stop-band among the maximally flat (at pass-band) filters [9]. Finally, given three out of four design parameters: the filter order, the pass-band ripple, the transition band width, the stop-band ripple, the elliptic (Cauer) filter minimizes the only one remaining design parameter.

Because of the nature of the minimax optimality, all classical filters however are unable to satisfy additional regularity or flatness conditions, which are desirable in many practical applications. For instance, Chebyshev filters are maximally flat at the stop-band (like other all-pole filters) but cannot be flat for any degree at the pass-band. The inverse Chebyshev filters are maximally flat (at pass-band) but cannot be flat for any degree at the stop-band. Elliptic (Cauer) filters cannot be flat for any degree at the both pass-band and stop-band. In our previous work [6], an alternative design to Chebyshev and inverse Chebyshev filters has been proposed. The designed filters have the same structure as of Chebyshev and inverse Chebyshev ones but they possess additional flatness for any degree at either pass-band or stop-band. The design is based on a new semi-definite programming (SDP) formulation, which also includes Chebyshev and inverse Chebyshev filer designs as a special case. This paper is a further development of [6], where a complete formulation for problem of designing general IIR flat filters is proposed and tested in several examples.

The rest of the paper is structured as follows. Section 2 describes mathematical tools that will be used throughout the paper. Section 3 presents the reduced order SDP formulation for generalized elliptic filters. In section 4, various numerical examples are given to demonstrate the viability of the proposed method. Finally, concluding remarks are presented in Section 5.

The following notation is used in the paper. Vectors and matrices will be represented by italicized bold lower case and upper case letters, respectively. The superscript "T" denotes the transpose (without conjugation) whereas the superscript "H" denotes Hermitian transpose. Symbols $\mathbb{R}$ and $\mathbb{C}$ are used to denote real and complex spaces. Real part and imaginary part of a complex number w are denoted by $\Re(w)$ and $\Im(w)$, respectively. The round-down and the round-up operations to the closest integers of a number a are $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle$ respectively denoted by $\lfloor a \rfloor$ and $\lceil a \rceil$. The standard notation $\boldsymbol{X} \geq 0$ defines a positive semi-definite Hermitian matrix, while $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle$ is the inner product of two matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, i.e. $\langle \boldsymbol{X}, \boldsymbol{Y}\rangle = \text{Trace}(\boldsymbol{XY})$. For a given set $C \subset \mathbb{R}^n$ its convex hull (conic hull), denoted by conv(C) (cone(C)), is the smallest convex set (cone) in $\mathbb{R}^n$ that contains C.

II. Optimization tool: SDP

For $\boldsymbol{\varphi}_n(\omega) = (1, \omega, \omega^2, ..., \omega^n)^T$, a polynomial curve $C_{a,b} \subset \mathbb{R}^{n+1}$ is defined as $C_{a,b} := \{\boldsymbol{\varphi}_n(\omega) : \ \omega \in [a, b]\} \subset \mathbb{R}^{n+1}$, and its polar $C^*_{a,b}$ is given by $C^*_{a,b} = \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \langle \boldsymbol{u}, \boldsymbol{v}\rangle \geq 0 \ \forall \ v \in C_{a,b}\}$. For an integer k define the linear matrix valued functions

$$\boldsymbol{T}_k(\boldsymbol{y}) = \begin{bmatrix} y_0 & y_1 & \cdots & y_k \\ y_1 & y_2 & \cdots & y_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_k & y_{k+1} & \cdots & y_{2k} \end{bmatrix},$$

$$\boldsymbol{T}_{1k}(\boldsymbol{y}) = \begin{bmatrix} y_1 & y_2 & \cdots & y_{k+1} \\ y_2 & y_3 & \cdots & y_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k+1} & y_{k+2} & \cdots & y_{2k+1} \end{bmatrix}.$$

Theorem 1 ([16]): The conic hull $\text{cone}(C_{a,b}), b < +\infty$ of the polynomial curve $C_{a,b}$ is fully characterized by LMIs: $\boldsymbol{y} = (y_0, y_1, ..., y_n)^T \in \text{cone}(C_{a,b})$ if and only if it satisfies the linear matrix inequalities (LMIs)

$$\boldsymbol{T}_{\lfloor n/2 \rfloor}(\boldsymbol{y}) \geq 0, \ b\boldsymbol{T}_{\lfloor n/2 \rfloor}(\boldsymbol{y}) \geq \boldsymbol{T}_{1\lfloor n/2 \rfloor}(\boldsymbol{y}) \geq a\boldsymbol{T}_{\lfloor n/2 \rfloor}(\boldsymbol{y}) \quad (1)$$

while $\boldsymbol{y} = (y_0, y_1, ..., y_n)^T \in \text{conv}(C_{a,b})$ if and only if it satisfies the LMIs (1) with $y_0 = 1$.

The conic hull $\text{cone}(C_{a,+\infty})$ is fully characterized by LMIs: $\boldsymbol{y} \in \text{cone}(C_{a,+\infty})$ if and only if it satisfies the LMIs

$$\boldsymbol{T}_{\lfloor n/2 \rfloor}(\boldsymbol{y}) \geq 0, \ \boldsymbol{T}_{1\lfloor n/2 \rfloor}(\boldsymbol{y}) \geq a\boldsymbol{T}_{\lfloor n/2 \rfloor}(\boldsymbol{y}). \quad (2)$$

while $\boldsymbol{y} \in \text{conv}(C_{a,+\infty})$ if and only if it satisfies the LMIs (1) with $y_0 = 1$.

Note that for n even, by definition, $\boldsymbol{T}_{1\lfloor n/2 \rfloor}(\boldsymbol{y})$ is a matrix function of $(y_0, y_1, \ldots, y_{n+1})$ and accordingly LMIs (1) are understood for some $y_{n+1} \in \mathbb{R}$.

In the next section, we will see that any filter design problem can be easily reformulated to the following semi-infinite programming

978-1-4244-2354-5/09/$25.00 ©2009 IEEE

(SIP)

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{c}^T\boldsymbol{x} \text{ s.t. } \boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{d}_i\in C_i^*,\ i=1,2,\ldots,m, \qquad (3)$$

where matrices $\boldsymbol{A}_i$ and $\boldsymbol{Q}>0$ are given and $C_i=\text{cone}(C_{a_i,b_i})$ for some $a_i,b_i, i=1,2,...,m$.
In general, SIP (3) is intractable optimization. However, by Theorem 1, its dual

$$\max_{\boldsymbol{y}_i\in C_i}\min_{\boldsymbol{x}}\left[\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{c}^T\boldsymbol{x}-\sum_{i=1}^m(\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{d}_i)^T\boldsymbol{y}_i\right]$$

is in fact the following convex semi-definite programming (SDP) of tractable optimization:

$$\max_{\boldsymbol{y}_i,\nu}\ -\sum_{i=1}^m\boldsymbol{y}_i^T\boldsymbol{d}_i-\nu \quad : \quad \begin{bmatrix}\nu & \boldsymbol{c}^T-\sum_{i=1}^m\boldsymbol{y}_i^T\boldsymbol{A}_i\\ \boldsymbol{c}-\sum_{i=1}^m\boldsymbol{A}_i^T\boldsymbol{y}_i & 4\boldsymbol{Q}\end{bmatrix}\geq 0,$$
$$(1) \text{ for } a_i\leftarrow a,\ b_i\leftarrow b,\ i=1,\ldots,m, \qquad (4)$$

The optimal solution $\boldsymbol{x}^*$ of the intractable programm (3) is directly retrieved from the optimal solution $\boldsymbol{y}_i^*$ of the tractable programm (4) involving just $(n+1)m$ scalar variables by the formula $\boldsymbol{x}^*=-\frac{1}{2}\boldsymbol{Q}^{-1}(\boldsymbol{c}-\sum_{i=1}^m\boldsymbol{A}_i^T\boldsymbol{y}_i^*)$.

III. Elliptic flat filter design

Generally, the transfer function of an elliptic (Cauer) filter takes the form [11] $|T(j\omega)|^2=(1+\epsilon^2R_n^2(\omega)^{-1}$ with the n-th-order rational Chebyshev function $R_n(\omega)$. As briefly mentioned, this specialized form is enough for its optimality in any of four design parameters (order, pass-band and stop-band ripples, transition bandwidth) given the remained three others. Its drawback is non-regularity or non-flatness for any degree at both pass-band and stop-band, which are actually desirable in certain applications [14], [17]. Our main target is to resolve this drawback by considering it in more flexible class beyond the rational Chebyshev functions. Namely,

$$|T(j\omega)|^2=\frac{1}{1+H^2(\omega)}=\frac{D^2(\omega)}{D^2(\omega)+N^2(\omega)}, \qquad (5)$$

where $H(\omega)=N(\omega)/D(\omega)$ is a rational (structure-free) function, i.e. $N(\omega)$ and $D(\omega)$ are polynomials in ω. Knowing $D^2(\omega)$ and $N^2(\omega)$ the stable filter $T(j\omega)$ can be obtained through minimal phase factorization [8].
Let n be the Macmillan order of $T(j\omega)$. Since $|T(j\omega)|^2$ is a function of ω^2 with $|T(0)|^2\neq 0$ (for low pass filters), $N(\omega)$ and $D(\omega)$ must take the following form

$$(N(\omega),D(\omega)=(\omega^{\tilde{n}}\sum_{i=0}^{\lfloor n/2\rfloor}n_i\omega^{2i},\sum_{i=0}^{\lfloor n/2\rfloor}d_i\omega^{2i}), \qquad (6)$$

where $\tilde{n}=\lceil n/2\rceil-\lfloor n/2\rfloor$, which is either 1 or 0.
Typically, the following constraints are imposed on the filter:

- Peak ripples in the pass-band and stop-band

$$1-\delta\leq\frac{1}{1+H^2(\omega)},\forall\omega\in[0,\omega_p], \qquad (7)$$
$$\frac{1}{1+H^2(\omega)}\leq\epsilon,\forall\omega\in[\omega_s,+\infty), \qquad (8)$$

  with the transition bandwidth $\Delta=\omega_s-\omega_p$;
- k-th-order flatness at the pass-band

$$\frac{d^{2i}|T(j\omega)|^2}{d\omega^{2i}}|_{\omega=0}=1$$
$$\Leftrightarrow\quad\frac{d^iH(\omega)}{d\omega^i}|_{\omega=0}=0,\quad i=0,2,\ldots,2(k-1)$$

  which is equivalent to

$$n_0=n_1=\cdots=n_{k-1}=0,d_0\neq 0. \qquad (9)$$

  The inverse Chebyshev filter $|T(j\omega)|^2=1/(1+(\epsilon^2T_n^2(\omega_S/\omega))^{-1})$ with the n-th order Chebyshev polynomial $T_n(.)$ is maximally flat ($k=n$) at the pass-band.
- ℓ-th flatness at the stop-band

$$\frac{d^i|T(j\omega)|^2}{\partial\omega^i}|_{\omega=+\infty}=0,\quad i=0,2,\ldots,2(k-1),$$

  which is equivalent to

$$d_n=d_{n-1}=\cdots=d_{n-\ell+1}=0,\ n_0\neq 0 \qquad (10)$$

  The Chebyshev filter $|T(j\omega)|^2=(1+\epsilon^2T_n(\omega))^2$ with the n-th order Chebyshev polynomial $T_n(.)$ is maximally flat ($\ell=n$) at the stop-band. The Butterworth filter is maximally flat at both pass-band and stop-band, while the elliptic filter is not flat in any degree at the pass-band and stop-band.

The above design constraints determine the set of feasible filters. Among these feasible filters, the "best" one is defined by the objective function. In this paper, we use the objective function that minimizes either the aggregated deviation of $|T(j\omega)|^2$ from 1 in the pass-band:

$$\Sigma_p=\int_0^{\omega_p}\left[1-\frac{1}{1+H^2(\omega)}\right]d\omega, \qquad (11)$$

or the aggregation of $|T(j\omega)|^2$ in the stopband:

$$\Sigma_s=\int_{\omega_s}^{+\infty}\frac{1}{1+H^2(\omega)}d\omega. \qquad (12)$$

As both Σ_p and Σ_s cannot be expressed by closed form formula, we must find good approximations for Σ_p and Σ_s. With the pass-band constraint (7), equation (11) is rewritten as

$$(1-\delta)\int_0^{\omega_p}H^2(\omega)d\omega\leq\Sigma_p\leq\int_0^{\omega_p}H^2(\omega)d\omega \qquad (13)$$

Therefore, a sensible approximation of Σ_p is

$$\Sigma_p\approx\bar{\Sigma}_p=\int_0^{\omega_p},H^2(\omega)d\omega \qquad (14)$$

because minimizing (14) also minimizes the gap between lower bound and upper bound of Σ_p.
Next, as $H(\omega)$ is a rational function, there is still no analytical expression for the integral in (14). Following the strategy in [5], we can replace the minimizer of $\bar{\Sigma}_p$ by the following quadratic objective function

$$\min_{\boldsymbol{n},\boldsymbol{d}}\int_0^{\omega_p}\left[N^2(\omega)+\gamma\big[D(\omega)-1\big]^2\right]d\omega \qquad (15)$$

where the scalar γ is a predefined small weight. For simplicity, in this paper we focus only on minimizing Σ_p. The quadratic objective function for the minimization of Σ_s can be effectively replaced in a similar manner by

$$\min_{\boldsymbol{n},\boldsymbol{d}}\int_0^{1/\omega_s}\left[D^2(\Omega)+\gamma\big[N(\Omega)-1\big]^2\right]d\Omega,$$

where the variable $\Omega=1/\omega$ is used to flip the magnitude response $|T(j\omega)|^2$.

Now the design of a generalized elliptic filter becomes the following optimization problem

$$\min_{\boldsymbol{n},\boldsymbol{d}} \int_0^{\omega_p} \Big[N^2(\omega) + \gamma\big[D(\omega) - 1\big]^2\Big] d\omega : \ (9), (10), \quad (16a)$$

$$-\sqrt{\frac{\delta}{1-\delta}} \leq \frac{N(\omega)}{D(\omega)} \leq \sqrt{\frac{\delta}{1-\delta}}, \ \forall \omega \in [0, \omega_p] \quad (16b)$$

$$-\sqrt{\frac{\epsilon}{1-\epsilon}} \leq \frac{D(\omega)}{N(\omega)} \leq \sqrt{\frac{\epsilon}{1-\epsilon}}, \forall \omega \in [\omega_s, +\infty) \quad (16c)$$

By defining

$$\bar{\boldsymbol{Q}} = \int_0^{\omega_p} \boldsymbol{\varphi}_n^T(\omega)\boldsymbol{\varphi}_n(\omega) d\omega, \ \bar{\boldsymbol{c}} = \int_0^{\omega_p} \boldsymbol{\varphi}_n(\omega) d\omega \quad (17a)$$

$$\boldsymbol{x} = [\boldsymbol{n}^T, \boldsymbol{d}^T]^T, \Delta_p = \sqrt{\frac{\delta}{1-\delta}}, \Delta_s = \sqrt{\frac{\epsilon}{1-\epsilon}} \quad (17b)$$

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{0}_{(\tilde{n}+2\lceil\frac{k-\tilde{n}}{2}\rceil)\times(\lfloor n/2\rfloor+1-\lceil\frac{k-\tilde{n}}{2}\rceil)} \\ \mathbf{diag}\big([1,0,1,\ldots,0,1]\big) \end{bmatrix} \quad (17c)$$

$$\subset \mathbb{R}^{(n+1)\times(\lfloor n/2\rfloor+1-\lceil\frac{k-\tilde{n}}{2}\rceil)} \quad (17d)$$

$$\boldsymbol{B}_2 = \begin{bmatrix} \mathbf{diag}\big([1,0,1,\ldots,0,1]\big) \\ \boldsymbol{0}_{(\tilde{n}+2\lceil\frac{\ell-\tilde{n}}{2}\rceil)\times(\lfloor n/2\rfloor+1-\lceil\frac{\ell-\tilde{n}}{2}\rceil)} \end{bmatrix} \quad (17e)$$

$$\subset \mathbb{R}^{(n+1)\times(\lfloor n/2\rfloor+1-\lceil\frac{\ell-\tilde{n}}{2}\rceil)} \quad (17f)$$

$$\boldsymbol{Q} = [\boldsymbol{B}_1 \quad \boldsymbol{0}]^T\bar{\boldsymbol{Q}}[\boldsymbol{B}_1 \quad -\boldsymbol{0}] + \gamma[\boldsymbol{0} \quad \boldsymbol{B}_2]^T\bar{\boldsymbol{Q}}[\boldsymbol{0} \quad \boldsymbol{B}_2] \quad (17g)$$

$$\boldsymbol{c} = -2\gamma[\boldsymbol{0} \quad \boldsymbol{B}_2]^T\bar{\boldsymbol{c}} \quad (17h)$$

$$\boldsymbol{A}_1 = \big[\boldsymbol{B}_1 \quad \Delta_p\boldsymbol{B}_2\big], \ \boldsymbol{A}_2 = \big[-\boldsymbol{B}_1 \quad \Delta_p\boldsymbol{B}_2\big], \quad (17i)$$

$$\boldsymbol{A}_3 = \big[\Delta_s\boldsymbol{B}_1 \quad \boldsymbol{B}_2\big], \ \boldsymbol{A}_4 = \big[\Delta_s\boldsymbol{B}_1 \quad -\boldsymbol{B}_2\big], \quad (17j)$$

$$C_{a_1,b_1} = C_{a_2,b_2} = C_{0,\omega_p}, \ C_{a_3,b_3} = C_{a_4,b_4} = C_{\omega_s,+\infty} \quad (17k)$$

the optimization problem (16) is rewritten in the form of (3), which is solved via SDP (4).

In contrast to classical design methods, this optimization approach offers several advantages. First, we have direct control on design parameters δ, ϵ, ω_p, ω_s. Second, as $N(\omega)$ and $D(\omega)$ are structure-free, the proposed approach is more general than the classical one so that other requirements such as flatness can be easily incorporated. Furthermore, the classical elliptic filter can also be easily designed by the proposed approach.

According to (5) and (6), it is obvious that flatness of even order is automatically satisfied for n even. More precisely, a $(2k-1)$-th order flatness filter of even order will automatically satisfy the $2k$-order flatness condition. Similarly, a $2k$-th order flat filter of odd order is also a $(2k+1)$-th order flat.

In addition, if we set the maximal flatness at $\omega = 0$ or $\omega = +\infty$, we will obtain generalizations of classical Chebyshev filter, inverse Chebyshev filter, and Butterworth filter. The classical Elliptic filter is optimal in terms of transition bandwidth, i.e. given δ_p, δ_s, and ω_p, the stop-band edge ω_s of the classical Elliptic filter is the smallest among all analog filters of the same order. Therefore, with n, δ_p, δ_s, and ω_p given, we can use the proposed method to derive the classical Elliptic filter if we set ω_s in (16) to the optimum value. The optimum value of ω_s can be obtained by bisection method on ω_s satisfying (16).

Likewise, the classical Chebyshev filter is maximally flat in the pass-band and optimum in terms of stop-band attenuation. Therefore, with n, δ_p, ω_p, and ω_s given, we can use the proposed method to derive the classical Elliptic filter if we set δ_s in (16) to the smallest possible value. This optimum value can also be obtained by bisection method on δ_s satisfying (16). Derivation of classical inverse Chebyshev and Butterworth filters using proposed method is similar.

IV. Numerical examples

In this section, we demonstrate the viability of the proposed method in designing of several analog filters. All filters are required to meet specifications given as

$$\omega_p = 1[rad/s], \ \omega_s = 1.25[rad/s], \ \delta_p = 3\times10^{-3}, \ \delta_s = 4\times10^{-3}. \quad (18)$$

The resultant SDPs are solved on a standard personal computer using SeDuMi, a Matlab-based general purpose SDP solver [15], and Yalmip, a Matlab toolbox for rapid prototyping of optimization problems [7].

In order to fulfill the predefined specifications, it requires a Butterworth filter of order 26 or a Chebyshev filter of order 10. The necessary order for an elliptic filter is only 6, which is a significant improvement. Recall that the classical elliptic filter is

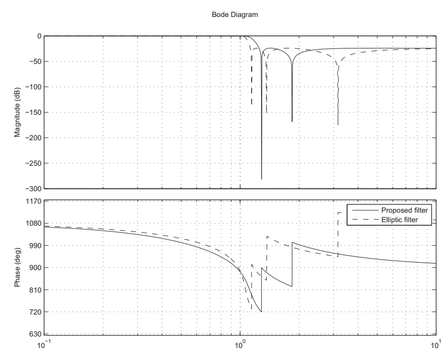

Fig. 1. Bode plot of the 6th-order generalized elliptic filter

optimum in the sense that given filter order and the peak ripples the transition bandwidth is minimum. Thus, the elliptic filter is usually over-designed in terms of transition bandwidth, i.e. the bandwidth is usually less than the required value as can be seen clearly in Fig. 1. The proposed method is more flexible in that it allows appropriate regularity requirements to be imposed on the elliptic filter while other specifications are still satisfied. In this generalized elliptic filter design example, we can set the regularity of the filter to 3 at either 0 or $+\infty$, as shown in Fig. 2 and Fig. 3.

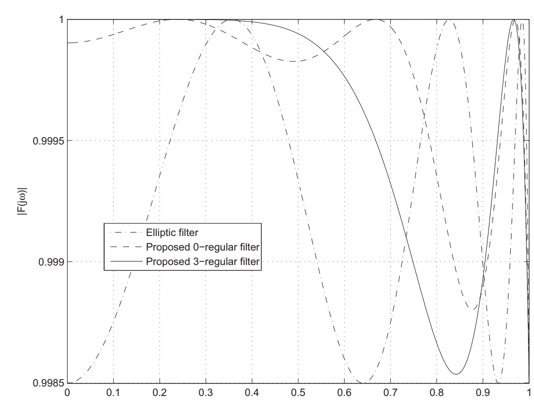

Fig. 2. Passband details comparison

On the other hand, if we take regularity as a new design specification, it must be traded-off with the conventional ones in (18). For example, higher regularity requires higher peak pass-band ripple

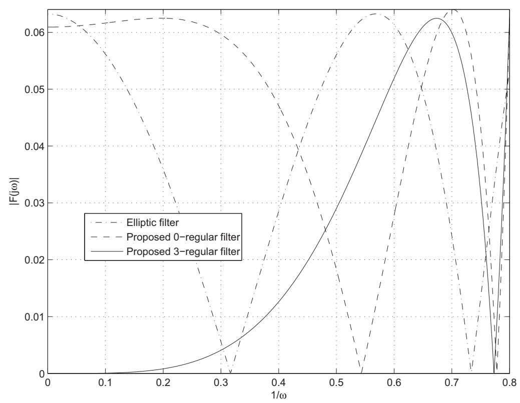

Fig. 3. Stopband details comparison

provided that other specifications remain the same. Figure 4 shows the trade-off between regularity and peak pass-band ripple for the 6-order generalized elliptic filter.

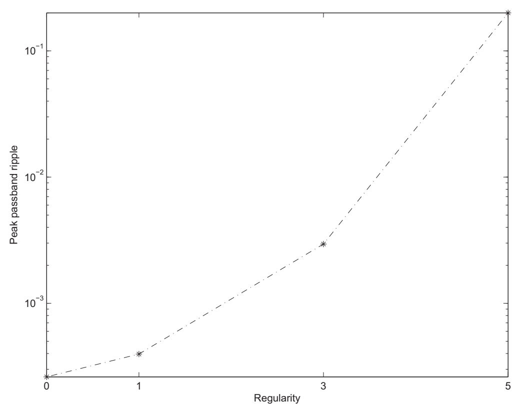

Fig. 4. Peak passband ripple versus regularity

The flexibility of the proposed method and the importance of regularity setting is further demonstrated in the design of the generalizations for the remaining classical analog filters. By setting the regularity in the pass-band, in the stop-band, and in both bands to maximum values, the proposed method subsequently yields the generalizations of Chebyshev filter, inverse Chebyshev filter, and Butterworth filter in Fig. 5-Fig. 6.

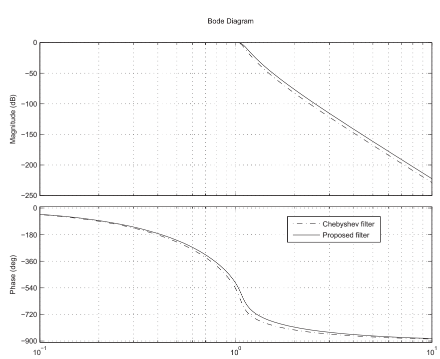

Fig. 5. Bode plot of the 10th-order generalized Chebyshev filter

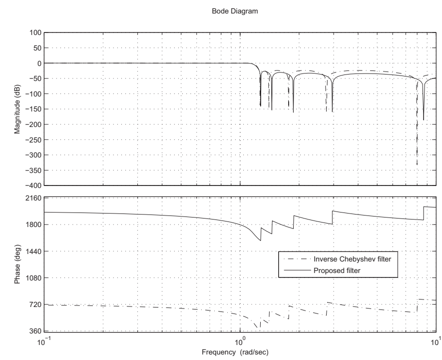

Fig. 6. Bode plot of the 10th-order generalized inverse Chebyshev filter

V. Conclusion

We proposed systematic approach to design generalized Elliptic filters. The attractive feature of the method is that regularity condition is easily incorporated. Thus, the proposed method is more general and more flexibility than classical methods as all classical filters can be obtained as special cases. Several numerical examples are presented to demonstrate the viability of the proposed method.

References

[1] A. Aktas and M. Ismail, *CMOS PLLs and VCOs for 4G wireless*. Kluwer Academic, 2004.

[2] S. M. Bozic and R. J. Chance, *Digital filters and signal processing in electronic engineering: theory, applications, architecture, and code*. Horwood, 1998.

[3] R. Gregorian and G. C. Temes, *Analog MOS integrated for signal processing*. Prentice-Hall, 1986.

[4] M. R. Haskard and I. C. May, *Analog VLSI design nMOS and CMOS*. Prentice-Hall, 1988.

[5] H. G. Hoang, H. D. Tuan, and T. Q. Nguyen, "Frequency selective KYP lemma and its applications to IIR filter bank design," in *Proc. of the International Conference on Acoustic, Speech, and Signal Processing (ICASSP 2007)*, vol. III, 2007, pp. 1457–1460.

[6] N.T. Hoang, H.D. Tuan, T.Q. Nguyen, H.G. Hoang, Optimized analog filter designs with flat responses by semi-definite programming, to appear in *IEEE Trans. Signal Processing*

[7] J. Löfberg, "Yalmip : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: http://control.ee.ethz.ch/~joloef/yalmip.php

[8] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, Third Edition, MC Graw-Hill.

[9] H. W. Park and C. S. Burrus, *Digital Filter Design*. John Wiley & Sons, 1987.

[10] M. E. V. V. R. Schaumann, *Design of analog filters*. Oxford University Press, 2001.

[11] R. Schaumann and M. E. V. Valkenburg, *Design of Analog Filters*. Oxford University Press, 2001.

[12] S. Soclof, *Analog integrated circuits*. Prentice-Hall, 1985.

[13] W. D. Stanley and J. M. Jeffords, *Electronic communications: principles and systems*. Thompson Delmar Learning, 2006.

[14] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Wellesley College, 1996.

[15] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimiz. Methods Softw.*, pp. 625–653, 1999.

[16] H. D. Tuan, N. T. Hoang, H. Q. Ngo, T. Hoang, and B.-N. Vo, "A frequency-selective positive real lemma and its applications to IIR filter design," in *Proc. of the 45th IEEE Conference on Decision and Control (CDC 2006)*, 2006, pp. 1478–1485.

[17] P. P. Vaidyanathan, *Multirate systems and filter banks*. Prentice-Hall, 1993.