

# A CLASS OF COMPREHENSIVE CONSTRAINTS FOR THE PCWLSE FILTER DESIGN: A BOOST IN PERFORMANCE

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## ABSTRACT

The complex Chebyshev error criterion is usually used as a general constraint in design of peak constraint weighted least square error (PCWLSE) filters. It applies an upper bound on the maximum magnitude of error between the desired and designed transfer functions of the filter. Therefore, it confines the corresponding maximum phase error as well. However, it imposes an over restricted constraint which reduces the feasibility region of the filter design problem. In this paper, a new and comprehensive class of constraints is proposed for design of PCWLSE filters that provides a larger feasible region than that of the complex Chebyshev constraint. Hence, the filter weights acquire larger feasible space to search and select better optimal values and consequently boost up the performance of the designed filter by achieving less weighted least square error. Simulation results show superiority of the proposed constraints over that of the complex Chebyshev criterion.

**Index Terms**— Peak Constraint weighted least square design, Complex Chebyshev error, Linear constraints

## 1. INTRODUCTION

Adams in [1] has shown that filter design based on trade-off between the least square error ( $L_2$  norm) and the Chebyshev error ( $L_\infty$  norm) has more desirable properties than using any of them alone. These are called peak-constraint weighted least-square error (PCWLSE) filters and have drawn attention of several researchers in the past decade [2-5]. Although these filter design procedures are mainly realized by the FIR filters, however in [4-7] they have been extended to the Laguerre digital filters. The Laguerre filter makes a general filter structure that comprises the FIR filter structure as its own specific sub-structure. Adams in [3] introduced the general form of the PCWLSE criterion and proposed different constraints including the complex Chebyshev error criterion for this type of filter design. The reason for using the complex Chebyshev error constraint, as it is used in several works [2-8], is to achieve linear constraints. With linear constraints efficient algorithms can be exploited to solve the corresponding semi-infinite quadratic optimization problems [4-7] and [9]. Although, the complex Chebyshev error is used to achieve linear constraints, one should be aware of the following problems:

i) with the complex Chebyshev error constraint, the maximum magnitude and phase errors are restricted by a single threshold. In other words, one can not use distinct thresholds for magnitude and phase errors independently. While, in some applications it may be required to do so.

ii) Using the complex Chebyshev error constraint along with other constraints, for example phase constraint, may dominantly over restrict (unnecessarily restrict) the feasible region and make the other constraint ineffective. On the contrary, the other constraint may be dominant and over restrict the feasible region and make the complex Chebyshev error constraint redundant. Both over restricted feasible region and redundant constraint are not desirable. Since redundant constraint unnecessarily increases the complexity of computations, and over restricted feasible region results in larger least square errors. Achieving larger error is due to providing less room for the search region and consequently preventing weights from selecting the best optimal solutions. The set of comprehensive constraints proposed in this paper which is novel and has been presented for the first time in literature, deals with all the aforementioned problems and provides optimal solution with less weighed least square error.

## 2. THE PROPOSED AND THE COMPLEX CHEBYSHEV ERROR CONSTRAINTS

The frequency response of a Laguerre filter of order  $N$  with real coefficients  $\xi_l$  is defined as [4-5]:

$$H(e^{j\omega}) = L^T(e^{j\omega})\xi \quad (1)$$

where

$$\xi = [\xi_0 \ \xi_1 \ \dots \ \xi_{N-1}]^T \in \Re^N \quad (2)$$

with  $\Re^N$  the set of real numbers and

$$L(e^{j\omega}) = [L_0(e^{j\omega}) \ L_1(e^{j\omega}) \ \dots \ L_{N-1}(e^{j\omega})]^T \quad (3)$$

Here  $L_l(e^{j\omega})$ ,  $l = 0, 1, \dots, N-1$ , are the Laguerre basis functions and form a complete orthonormal set that are given by:

$$L_l(e^{j\omega}) = \sqrt{(1-b^2)} \frac{(e^{-j\omega} - b)^l}{(1 - be^{-j\omega})^{l+1}} \quad (4)$$

where  $b$  denotes the Laguerre parameter and  $|b| < 1$  guarantees the stability of this filter and by setting  $b=0$ , Laguerre filter reduces to FIR filter. Let  $\delta_p$  and  $\delta_s$  denote the maximum allowable deviation in magnitude of the filter response from the desired frequency response in passband and stopband, respectively. Also let  $\delta_i$

denote the corresponding maximum allowable deviation in phase of the filter in passband. Hence, the problem of filter design is originally as follows:

Problem 1:

$$\min_{\xi \in \mathfrak{R}^N} \frac{1}{2} \xi^T \psi_r \xi + \varphi^T \xi \quad \text{Subject to} \quad (5)$$

$$\left| |H(e^{j\omega})| - |D(e^{j\omega})| \right| \leq \delta_p, \quad \omega \in \Omega_p \quad (6)$$

$$|H(e^{j\omega})| \leq \delta_s, \quad \omega \in \Omega_s \quad (7)$$

$$|\angle H(e^{j\omega}) - \angle D(e^{j\omega})| \leq \delta_1, \quad \omega \in \Omega_p \quad (8)$$

where  $D(e^{j\omega})$  denotes the desired frequency response which is set to 0 in stopband. The passband and stopband regions of  $D(e^{j\omega})$  are stated by  $\Omega_p$  and  $\Omega_s$  that are two compact and uncountable subsets of  $[0, \pi]$  for the real filter coefficients and  $[0, 2\pi]$  for the complex filter coefficients and  $\psi_r$  is the real part of the matrix  $\psi \in \mathbb{C}^{N \times N}$   $\psi(m, n)$ , defined as:

$$\begin{aligned} \psi(m, n) = & \left[ 2w_p \int_{\Omega_p} \frac{e^{j\omega}(1-be^{j\omega})^{m-n-1}}{(e^{j\omega}-b)^{m-n+1}} d\omega \right. \\ & \left. + 2w_s \int_{\Omega_s} \frac{e^{j\omega}(1-be^{j\omega})^{m-n-1}}{(e^{j\omega}-b)^{m-n+1}} d\omega \right] (1-b^2) \end{aligned} \quad (9)$$

and  $\varphi \in \mathfrak{R}^N$  is a vector with elements  $\varphi(k)$ , where

$$\varphi(k) = -2 \operatorname{Re} \left( w_p \int_{\Omega_p} D(e^{j\omega}) L_k^*(e^{j\omega}) d\omega \right) \quad (10)$$

in which  $\operatorname{Re}\{\cdot\}$  and  $*$  are the real part and complex conjugate notations, respectively. In conventional filter design, the complex Chebyshev error criterion is used with problem 1, to combine the nonlinear constraints of this problem. Then by real rotation theorem, a set of linear constraints are produced under which the filter can be solved using the aforementioned new approaches applied to semi-infinite quadratic programming problems. Using the complex Chebyshev error criterion in problem 1, results in the following constraints:

$$\begin{aligned} |H(e^{j\omega}) - D(e^{j\omega})| & \leq \delta(\omega), \quad \omega \in \Omega_p \\ |H(e^{j\omega})| & \leq \delta_s, \quad \omega \in \Omega_s \end{aligned} \quad (11)$$

in which  $\delta(\omega) = \min(\delta_p(\omega), |D| \sin \delta_1(\omega))$ . Depending on the maximum allowable errors for the magnitude and phase, we have different situations. However, we explain the new proposed set of constraints using the case where  $\delta_p < \sin \delta_1$ , which itself is the most common case in practice. Other situations are very similar. In fact, it means that we are more sensitive to the magnitude error in the passband rather than the phase error in the same band. The shaded region in figure 1, represents a typical feasible region for the passband region of the problem, in this situation. The circle with center  $D$  shows the region covered by the complex Chebyshev error constraint for this problem. It is clear that the regions out of this circle are also feasible regions of the problem

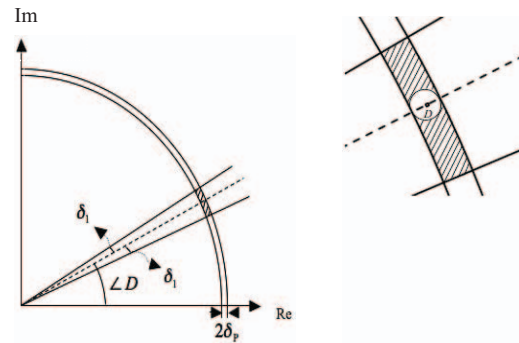
and may contain the optimal solution. Our objective is to find the closest region to the feasible region but with linear constraints. In order to accomplish this goal, first, we have proposed to multiply the frequency response of the filter  $H(e^{j\omega})$  by  $e^{-\angle D(e^{j\omega})}$ . In fact, by this multiplication, the feasible region is rotated such that the search regions for optimal solution lie around the real axis of the complex plain. In this way, a new frequency response, called  $K(e^{j\omega})$  is designed that is equal to  $H(e^{j\omega})e^{-j\angle D(e^{j\omega})}$ . When the optimal solution is obtained, then the results in frequency domain must be multiplied by  $e^{j\angle D(e^{j\omega})}$  to rotate them back to the original location. The result of this rotation is the region represented in figure 2. The linear constraints that are proposed for design of  $K(e^{j\omega})$  are formed by the approximate dashed lines plotted in right side of this figure. The proposed region has been developed as what follows:

**A-** Simplification of the passband magnitude constraint (6) by considering  $K(e^{j\omega})$  instead of  $H(e^{j\omega})$  with the assumption of

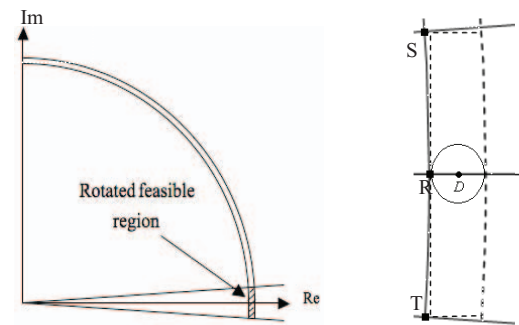
$|D| = 1$ , which leads to the following two equations:

$$|K(e^{j\omega})| \leq 1 + \delta_p, \quad \omega \in \Omega_p \quad (12)$$

$$1 - \delta_p \leq |K(e^{j\omega})|, \quad \omega \in \Omega_p \quad (13)$$



**Fig. 1.** The feasible region in original space (left). The magnified version of the feasible region (right)



**Fig. 2.** The rotated feasible region (left). The proposed constraints for designing  $K(e^{j\omega})$ -dashed lines/' (right).

By the real rotation theorem [8], a complex magnitude inequality as  $|z| \leq \delta$  can equivalently be expressed by:

$$\operatorname{Re} \{ z e^{j\lambda} \} \leq \delta, \quad \forall \lambda \in [0, 2\pi] \quad (14)$$

Thus, the inequality (12) can be expressed as a linear constraint. Therefore, this inequality can be used as a constraint in the design process and is represented as the outer circle in left side of figure 2 or the curved dashed line in right side of this figure. The real rotation theorem can not be applied to inequality (13). Consequently, it may not be simplified to a linear constraint. Instead we approximate this inequality by a line perpendicular to the real axis. This constraint is represented by the straight dashed line passing through the point R and perpendicular to the real axis in figure 2. Using this approximation for the equation (13) results the following linear inequality constraint:

$$-\operatorname{Re}(K(e^{j\omega})) \leq \delta_p - 1, \quad \omega \in \Omega_p \quad (15)$$

**B-** The constraints on the phase of  $K(e^{j\omega})$  can be approximated by two parallel lines that are drawn perpendicular to the imaginary axis. These lines are represented by the two remaining dashed lines parallel to the real axis in figure 2 passing through the points S and T. By evaluating the imaginary parts of points S and T, we can express these constraints by the following two equations:

$$\operatorname{Im}(K(e^{j\omega})) \leq (1 - \delta_p) \sin \delta_1, \quad \omega \in \Omega_p \quad (16)$$

$$-\operatorname{Im}(K(e^{j\omega})) \leq (1 - \delta_p) \sin \delta_1, \quad \omega \in \Omega_p \quad (17)$$

**C-** Using the proposed method, it is easy to obtain the constraint for the group delay error in case it is needed. Let  $E_G$  denote the group delay error. Then

$$E_G(\omega) = \frac{d\angle H(e^{j\omega})}{d\omega} - \frac{d\angle D(e^{j\omega})}{d\omega} = \frac{d(\angle H(e^{j\omega}) - \angle D(e^{j\omega}))}{d\omega} = \frac{d\angle K(e^{j\omega})}{d\omega} \quad (18)$$

which by definition stated for  $K(e^{j\omega})$ , it is clear that the phase of  $K(e^{j\omega})$  is actually the phase error for  $H(e^{j\omega})$ . On the other hand, we have approximated the phase constraint for  $K(e^{j\omega})$  by equations (16) and (17). Hence, by substituting the approximate values, we can evaluate the group delay error constraint as follows:

$$|E_G(\omega)| = \left| \frac{d\angle K(e^{j\omega})}{d\omega} \right| = \left| \frac{d\{\operatorname{Im}[K(e^{j\omega})]\}}{d\omega} \right| \quad (19)$$

By definition of  $K(e^{j\omega})$  and considering the Laguerre basis functions introduced in (4), the group delay error can be represented as  $|E_G(\omega)| = \mathbf{h}^T \boldsymbol{\xi}$  where  $\mathbf{h}$  is a vector that for

$n = 0, \dots, N-1$ , we have:

$$h_n = \operatorname{Re} \left\{ \sqrt{(1-b^2)} \frac{(e^{-j\omega} - b)^n}{(1 - b e^{-j\omega})^{n+1}} e^{-\angle D(e^{j\omega})} \left\{ \frac{n}{1 - b e^{j\omega}} + \frac{(n+1)b}{e^{j\omega} - b} + \frac{d\angle D(e^{j\omega})}{d\omega} \right\} \right\} \quad (20)$$

Combining all the aforementioned constraints, the PCWLSE design problem is converted to the following problem:

Problem 2:

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^N} \frac{1}{2} \boldsymbol{\xi}^T \boldsymbol{\Psi}_r \boldsymbol{\xi} + \boldsymbol{\varphi}^T \boldsymbol{\xi} \quad \text{Subject to} \quad (21)$$

$$|K(e^{j\omega})| \leq 1 + \delta_p, \quad \omega \in \Omega_p \quad (22)$$

$$-\operatorname{Re}(K(e^{j\omega})) \leq \delta_p - 1, \quad \omega \in \Omega_p \quad (23)$$

$$\operatorname{Im}(K(e^{j\omega})) \leq (1 - \delta_p) \sin \delta_1, \quad \omega \in \Omega_p \quad (24)$$

$$-\operatorname{Im}(K(e^{j\omega})) \leq (1 - \delta_p) \sin \delta_1, \quad \omega \in \Omega_p \quad (25)$$

$$\mathbf{h}^T \boldsymbol{\xi} \leq \delta_{gd}, \quad \omega \in \Omega_p \quad (26)$$

$$|K(e^{j\omega})| \leq \delta_s, \quad \omega \in \Omega_s \quad (27)$$

where  $\boldsymbol{\varphi} \in \mathbb{R}^N$  is a vector with elements  $\varphi'(k)$ :

$$\varphi'(k) = -2 \operatorname{Re}(w_p \int_{\Omega_p} L_k^*(e^{j\omega}) e^{-j\angle D(e^{j\omega})} d\omega) \quad (28)$$

and the group delay limit is denoted as  $\delta_{gd}$ . Using real rotation theorem stated earlier, problem 2 is converted to a semi-infinite quadratic programming with linear constraints. Although any of the methods in [4-7] and [9] can be used to solve the resultant problem, but in this paper the discretization method is exploited to solve the problem. The reason for using this simple choice is that our main concern in this paper is to introduce the most appropriate feasible region for the optimum filter design and study its effect in the final solution, rather than selecting the best solution approach.

#### 4. NUMERICAL EXAMPLE

Design a lowpass Laguerre filter with following specifications:

$$D(e^{j\omega}) = \begin{cases} e^{-j20\omega}, & \omega \in [0, 0.04\pi] \\ 0, & \omega \in [0.13\pi, \pi] \end{cases}, \quad w_s / w_p = 1000$$

$$\delta_p = \delta_s = 0.01, \quad \delta_1 = 0.1.$$

Using the same procedure stated in [4], the suboptimal value of the Laguerre parameter is found to be 0.77. Here, the Laguerre filters are of order 12 while the FIR filters are of order 46.

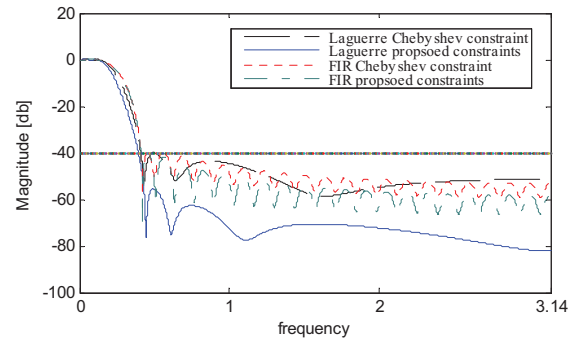
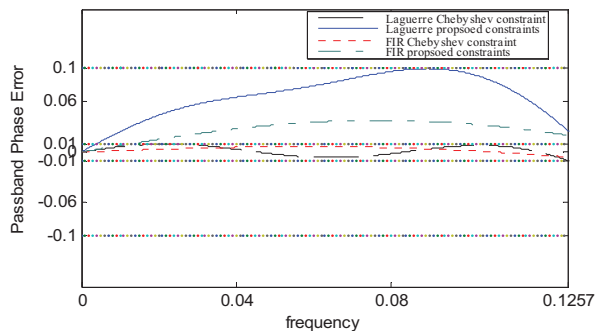


Fig. 3. Magnitude of the Laguerre filters of order 12 and FIR filters with 46 taps.

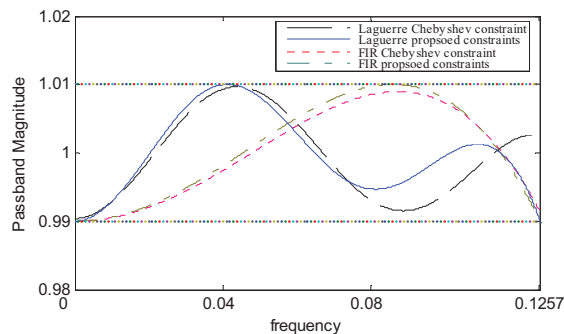
From figure 3, it is obviously seen that the PCWLSE design filters with the proposed constraints have much lower least square error in the stopband compared to those designed by the complex Chebyshev error. Here, Figure 4 illustrates the passband phase errors for the above filters. The constraints of the problem are the straight lines on 0.1 and -0.1. It is clearly seen that using the complex Chebyshev error restricts the phase error to 0.01 and -0.01 which is the same as the magnitude error, while the maximum

allowable value for the passband phase error is 0.1. The proposed method allows the problem to use approximately all its feasible region. Therefore, we can reach the LSE for the designed filters.



**Fig. 4.** Passband Phase error of the Laguerre filter of order 12 and FIR filter with 46 stages for  $[0, \pi]$ .

Figure 5, shows the passband magnitude of the designed filters. They satisfy the magnitude constraint in the passband. Table 1 shows the weighted least square error of the designed filters. Obviously, filters designed by the proposed method have better performance (less WLS) than those designed by the complex Chebyshev error constraint.



**Fig. 5.** The passband magnitude of the filters.

**Table 1.** Weighted Least square error for designed filters with Laguerre filters of 12 stages and FIR filters of order 46.

Filter Type	WLSE (db)
Laguerre filter (complex Cheb. Error)	-28.4732
Laguerre filter (Proposed Constraints)	-57.7211
FIR filter (complex Cheb. Error)	-29.4413
FIR filter (Proposed Constraints)	-35.5116

#### 4. CONCLUSION

In this paper, instead of dealing with different solution approaches for the quadratic problem of the PCWLSE filter design, we have proposed a new and comprehensive set of constraints which results in significant improvement in the performance of the designed filter in terms of overall least square error. This is due to the fact that in conventional filter design, the complex Chebyshev error criterion is used that restricts the phase error to the same amount as the magnitude constraint. By the new proposed constraints, not

only we are able to define distinct independent values for the maximum magnitude and phase errors, but also a larger feasible region is introduced to the problem, and consequently more freedom is given to the filter coefficients to search for the smallest WLSE solution in this region.

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