FUNDAMENTAL PROPERTIES OF NON-NEGATIVE IMPULSE RESPONSE FILTERS — THEORETICAL BOUNDS II

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ABSTRACT

This paper presents fundamental limitations imposed on the frequency response of a filter by a non-negative impulse response (NNIR). Several upper-bounds on power spectral attenuation/gain in linearly spaced frequency regions are derived. These upper-bounds serve as a guide in the design of NNIR filters.

***Index Terms*—** Non-negative impulse response, bounds, filter, frequency response.

1. INTRODUCTION

This paper is a companion paper to [1] and provides new insights into the fundamental limitations on the frequency response of a filter with non-negativity constraints on the impulse response. While the paper in [1] analyzed the frequency response constraints for geometrically spaced frequency samples, this paper approaches the problem for linearly spaced frequency points. The formulation of frequency response properties in terms of equi-distantly spaced frequency points is of particular importance for the characterization of evidence filters [2]. This filter type has the capability of selectively fusing information from a variety of data and signal sources, while still weighing previously acquired information. For these types of tasks, frequency regions of interest are often not geometrically spaced. The performance boundaries of evidence filters are currently not clear and this paper sheds some light on this problem. The paper shows that the non-negativity constraint is a severe constraint and, unlike in classical filter design, increasing the filter order often does not lead to meeting given frequency specifications.

Previous results on the topic of non-negative impulse response filters and their corresponding frequency response properties have been reported in numerous publications over several decades, some of which can be found in [3–9].

This paper revisits the performance constraints of non-negative response filters in [1] for the case of equidistantly spaced frequency points. Section 2 introduces the main theorems and also illustrates the results through some simple design examples. Section 3 summarizes the obtained insights and offers conclusions.

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2. FUNDAMENTAL LIMITS OF NNIR FILTERS

Theorem 2.1 *If the impulse response of a filter is non-negative, i.e., $h(n) \geq 0,\ n \in \mathbb{Z}$, and the power spectral attenuation in the frequency region $[0, \omega_o]$ is bounded by:*

$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o$, $\omega_o \in (0, \pi]$, then the end-to-end power spectral attenuation or gain of the frequency region $[(m-1)\omega_o, m\omega_o]$ is bounded by:

$$\left| |H((m-1)\omega_o)|^2 - |H(m\omega_o)|^2 \right| \leq (2m-1)\delta$$

where $m \in \mathbb{Z}^+, m \geq 2, m\omega_o \leq \pi$.

Proof: Here only the proof for the power spectral attenuation case is given. The case of power spectral gain can be proven in a similar manner.

Let $U_k[x]$ be the k^{th}-degree Chebyshev polynomial of the second kind in variable x. Let

$$F(\omega) \triangleq \frac{\cos((m-1)\omega) - \cos(m\omega)}{1-\cos\omega} = \frac{\sin\left(\left(\frac{2m-1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$
$$= U_{2m-2}\left[\cos\left(\frac{\omega}{2}\right)\right]$$

From the property of $U_k[x]$:

$$\max_x U_k[x] = \lim_{x \to \pm 1} U_k[x]$$

we have:

$$\max_\omega F(\omega) = \lim_{\omega \to 0} U_{2m-2}\left[\cos\left(\frac{\omega}{2}\right)\right] = 2m-1$$

Since

$$h(n) \geq 0$$
$$\Rightarrow \quad \tilde{h}(n) \triangleq h(n) * h(-n) = h(n) * h^*(-n) \geq 0$$

and

$$|H(\omega)|^2 = H(e^{j\omega})H^*(e^{j\omega}) \overset{\mathscr{Z}}{\longleftrightarrow} h(n) * h^*(-n) = \tilde{h}(n)$$

we have:

$$|H(\omega)|^2 = \sum_{l=-\infty}^{\infty} \tilde{h}(l) e^{-j\omega l} = \sum_{l=-\infty}^{\infty} \tilde{h}(l)\cos(\omega l)$$

Therefore

$$\begin{aligned}
&|H((m-1)\omega_o)|^2-|H(m\omega_o)|^2\\
&=\sum_{l=-\infty}^{\infty}\tilde h(l)\left(\cos((m-1)\omega_o l)-\cos(m\omega_o l)\right)\\
&\le(2m-1)\sum_{l=-\infty}^{\infty}\tilde h(l)\left(1-\cos(\omega_o l)\right)\\
&=(2m-1)\left(|H(0)|^2-|H(\omega_o)|^2\right)
\end{aligned}\tag{1}$$

Since

$$\begin{aligned}
&|H(0)|^2-|H(\omega)|^2\le\delta,\ w\in[0,\omega_o]\\
\Rightarrow\ &|H(0)|^2-|H(\omega_o)|^2\le\delta
\end{aligned}$$

Inequality (1) then becomes

$$|H((m-1)\omega_o)|^2-|H(m\omega_o)|^2\le(2m-1)\delta \qquad \blacksquare$$

Comments: Theorem 2.1 can be used to study the relationship between the roll-off/gain of the falling/rising edges of a variety of NNIR filter types and the power spectral attenuation near frequency zero.

Example: If an NNIR highpass filter's specifications are as follows:

- passband frequency region: $0.6\pi\sim\pi$
- passband ripple: $(-10)\,dB\sim 0dB$ $(0.3162\sim 1)$
- stopband gain: $\le -70dB$ (3.162×10^{-4})
- transition frequency region: $0.5\pi\sim0.6\pi$

then it is found that the power spectral attenuation should at least have an attenuation of $0.04dB$ at 0.1π, regardless of the attenuation at the frequencies less than 0.1π (Note that the DC gain of $0dB$ always exists regardless of filter type).

Consider $\omega_o=0.1,\ m=6$, then the transition band lies in $[(m-1)\omega_o, m\omega_o]$. Let δ be the maximal *magnitude* attenuation at ω_o. Since:

$$x=-10dB=0.3162,\ \ y=-70dB=3.162\times10^{-4}$$

and

$$|H(6\omega_o)|\ge x,\ \ |H(5\omega_o)|\le y$$

from Theorem 2.1, we have:

$$\begin{aligned}
&x^2-y^2\le|H(6\omega_o)|^2-|H(5\omega_o)|^2\le(2\times6-1)\delta\\
\Rightarrow\ &\delta\ge\frac{x^2-y^2}{11}=0.0091
\end{aligned}$$

Therefore:

$$|H(\omega_o)|=\sqrt{1-\delta}\le0.9954=-0.04dB.$$

Theorem 2.2 *If the impulse response of a filter is non-negative, i.e., $h(n)\ge0,\ n\in\mathbb{Z}$, and the power spectral attenuation within $[0,(m-1)\omega_o]$ is bounded by:*

$$|H(0)|^2-\delta\le|H(\omega)|^2\le|H(0)|^2$$

where $m\in\mathbb{Z}^+, m\ge2, \delta>0, 0\le\omega\le(m-1)\omega_o, \omega_o\in\left(0,\frac{\pi}{m}\right]$, then the end-to-end power spectral attenuation of the frequency region $[(m-1)\omega_o, m\omega_o]$ is bounded by:

$$|H((m-1)\omega_o)|^2-|H(m\omega_o)|^2\le\frac{2m-1}{(m-1)^2}\delta$$

where $m\omega_o\le\pi$.

Proof: To find the relationship between the attenuations in the two frequency regions, consider the function $f(\omega)$:

$$\begin{aligned}
&f(\omega)\triangleq\frac{\cos((m-1)\omega)-\cos(m\omega)}{1-\cos((m-1)\omega)}\\
\Rightarrow\ &\frac{d}{d\omega}f(\omega)\Big|_{\omega=\omega_o}\triangleq p=\frac{A}{\left[1-\cos((m-1)\omega_o)\right]^2}
\end{aligned}$$

where

$$\begin{aligned}
A&=4\sin\left(\frac{m\omega_o}{2}\right)\sin\left(\frac{m-1}{2}\omega_o\right)\cdot B\\
B&=\sin\left(\frac{\omega_o}{2}\right)\left[-\left(m-\frac12\right)+\frac12\frac{\sin\left(\left(m-\frac12\right)\omega_o\right)}{\sin\left(\frac{\omega_o}{2}\right)}\right]
\end{aligned}$$

Since

$$\frac{\sin\left(\left(m-\frac12\right)\omega_o\right)}{\sin\left(\frac{\omega_o}{2}\right)}=\frac{\sin\left((2m-1)\frac{\omega_o}{2}\right)}{\sin\left(\frac{\omega_o}{2}\right)}=U_{2m-2}\left[\cos\left(\frac{\omega_o}{2}\right)\right]$$

and from the property of $U_l[x], l\in\mathbb{Z}^+$, we have:

$$\frac{\sin\left(\left(m-\frac12\right)\omega_o\right)}{\sin\left(\frac{\omega_o}{2}\right)}\le\lim_{\omega_o\to0}\frac{\sin\left(\left(m-\frac12\right)\omega_o\right)}{\sin\left(\frac{\omega_o}{2}\right)}=2m-1$$

Since

$$\sin\left(\frac{\omega_o}{2}\right)\ge0,\ \sin\left(\frac{m\omega_o}{2}\right)\ge0,\ \sin\left(\frac{m-1}{2}\omega_o\right)\ge0$$

we have $B\le0$, and therefore $A\le0$. So we have:

$$p\le0\tag{2}$$

Inequality (2) indicates that $f(\omega_o)$ is monotonically decreasing in $\left(0,\frac{\pi}{m}\right]$. Therefore a maximum is obtained when $\omega_o\to0$:

$$\max_{\omega_o\in\left(0,\frac{\pi}{m}\right]}f(\omega_o)=\lim_{\omega_o\to0}f(\omega_o)=\frac{2m-1}{(m-1)^2}\tag{3}$$

From (3) we have:

$$\begin{aligned}
&|H((m-1)\omega_o)|^2-|H(m\omega_o)|^2\\
&=\sum_{l=-\infty}^{\infty}\tilde h(l)\left(\cos((m-1)\omega_o l)-\cos(m\omega_o l)\right)\\
&\le\frac{2m-1}{(m-1)^2}\sum_{l=-\infty}^{\infty}\tilde h(l)\left(1-\cos((m-1)\omega_o l)\right)\\
&=\frac{2m-1}{(m-1)^2}\left(|H(0)|^2-|H((m-1)\omega_o)|^2\right)\le\frac{2m-1}{(m-1)^2}\delta
\end{aligned}$$

$\blacksquare$

Comments: Theorem 2.2 is useful for studying low-pass NNIR systems. It indicates the relationship between the passband ripple in ($[0, (m-1)\,\omega_o]$) and the maximal roll-off that can be achieved in the transition-band ($[(m-1)\,\omega_o, m\omega_o]$).

Example: An NNIR lowpass filter is specified as follows:

- passband frequency region: $0 \sim 0.4\pi$
- passband ripple: $4dB$
- stopband gain: $\leq -20dB$
- transition frequency region: $0.4\pi \sim 0.5\pi$

While this specification appears to be relatively mild, it turns out to be unachievable. From Theorem 2.2, a low-pass filter with a $4dB$ passband ripple only has a maximal attenuation to $12.25dB$ that can be achived in the transition band. (Please refer to [10] for a detailed discussion.)

The upper-bound given in Theorem 2.1, being a global upper-bound for any $\omega_o \in [0, \pi]$, can be further tightened if ω_o is known. This is stated in Theorem 2.3.

Theorem 2.3 *If the impulse response of a filter is non-negative, i.e., $h\,(n) \geq 0,\ n \in \mathbb{Z}$, and the power spectral attenuation within $[0, \omega_o]$ is bounded by:*

$$|H\,(0)|^2 - \delta \leq |H\,(\omega)|^2 \leq |H\,(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o$, and $\omega_o \in \left[\frac{4k\pi}{2m-1}, \frac{(4k+2)\pi}{2m-1}\right]$ (for power attenuation case) or $\omega_o \in \left[\frac{(4k+2)\pi}{2m-1}, \frac{4(k+1)\pi}{2m-1}\right]$ (for power gain case), $k \in \{1, 2, \ldots, \frac{m-1}{2}\}$, $m \in \mathbb{Z}^+$ is odd, and $m > 2$, then the end-to-end power spectral attenuation/gain in $[(m-1)\,\omega_o, m\omega_o]$ is bounded by:

$$\begin{aligned} &\left| |H\,((m-1)\,\omega_o)|^2 - |H\,(m\omega_o)|^2 \right| \\ &\leq \csc\left(\frac{(4k+1)\,\pi}{4m-2}\right)\left[1 + \frac{\pi}{4m-2}\cot\left(\frac{(4k+1)\,\pi}{4m-2}\right)\right]\delta \end{aligned} \tag{4}$$

where $m\omega_o \leq \pi$. (Similar results exist for m being even).

Proof: For the sake of brevity, we only give the proof for the power spectral attenuation case.

Since $U_{2m-2}\left[\cos\left(\frac{\omega}{2}\right)\right]$ is symmetric with respect to $\omega = \pi$ [10], and since a $(2m-2)^{th}$-degree Chebyshev polynomial of the second kind has $2m-2$ different simple Chebyshev roots, we only need to consider the first $m-1$ roots.

Let $\cos\left(\frac{\omega_r}{2}\right)$ denote these Chebyshev roots:

$$\cos\left(\frac{\omega_r}{2}\right) = \cos\left(\frac{p\pi}{2m-2+1}\right) \ \Rightarrow\ \omega_r = \frac{2p\pi}{2m-1}$$

where $p = 1, 2, \ldots, m-1$.

Denote the $m-2$ equi-length frequency regions split by these ω_r's as:

$$I_k^a:\ \left[\frac{4k\pi}{2m-1}, \frac{(4k+2)\,\pi}{2m-1}\right]$$
$$I_k^r:\ \left[\frac{(4k+2)\,\pi}{2m-1}, \frac{4\,(k+1)\,\pi}{2m-1}\right]$$

where $k = 1, 2, \ldots, \frac{m-1}{2}$. It can easily be proven that the power spectrum within frequency region $[(m-1)\,\omega_o, m\omega_o]$ must *decreases* from one end to the other when $\omega \in I_k^a$ and must *increases* when $\omega \in I_k^r$ [10].

Now let ω_* be the mid-point of a I_k^a:

$$\omega_* = \frac{1}{2}\left(\frac{4k\pi}{2m-1} + \frac{(4k+2)\,\pi}{2m-1}\right) = \frac{(4k+1)\,\pi}{2m-1}$$
$$q \triangleq U_{2m-2}\left[\cos\left(\frac{\omega_*}{2}\right)\right],\quad s \triangleq U'_{2m-2}\left[\cos\left(\frac{\omega_*}{2}\right)\right]$$

and let d be the width of each I_k^a:

$$d = \frac{(4k+2)\,\pi}{2m-1} - \frac{4k\pi}{2m-1} = \frac{2\pi}{2m-1}$$

Then an upper-bound B can be found by linear approximation, as shown in (5) and illustrated in Fig. 1 for $m = 7$. (Please refer to [10] for a step-by-step derivation of (5).) In Fig. 1, A denotes the actual maximum of the left side of (4), and B denotes the bound obtained via linear approximation.

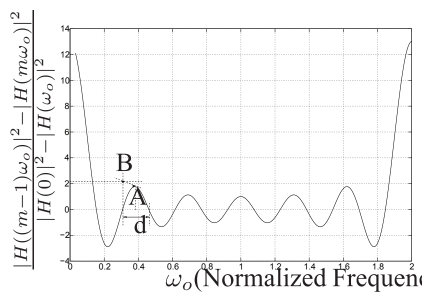

Fig. 1. An upper-bound obtained by linear approximation

$$\begin{aligned} B &= q + \left|\frac{d}{2}s\right| \\ &= U_{2m-2}\left[\cos\left(\frac{\omega_*}{2}\right)\right] + \left|\left(\frac{\pi}{2m-1}\right)U'_{2m-2}\left[\cos\left(\frac{\omega_*}{2}\right)\right]\right| \\ &= \csc\left(\frac{(4k+1)\,\pi}{4m-2}\right)\left[1 + \frac{\pi}{4m-2}\cot\left(\frac{(4k+1)\,\pi}{4m-2}\right)\right] \end{aligned} \tag{5}$$

Therefore, the end-to-end spectral attenuation is obtained as:

$$\begin{aligned} &|H\,((m-1)\,\omega_o)|^2 - |H\,(m\omega_o)|^2 \\ &= \sum_{l=-\infty}^{\infty} \tilde{h}\,(l)\,(\cos\,((m-1)\,\omega_o l) - \cos\,(m\omega_o l)) \\ &\leq B \sum_{l=-\infty}^{\infty} \tilde{h}\,(l)\,(1 - \cos\,(\omega_o l)) \\ &= B\left(|H(0)|^2 - |H\,(\omega_o)|^2\right) \leq B\delta \end{aligned}$$

where B is as described in (5). ■

Theorem 2.4 *If the impulse response of a filter is non-negative, i.e.,* $h(n) \geq 0,\ n \in \mathbb{Z}$*, and the power spectral attenuation within the frequency region* $[0, \omega_o]$ *is bounded by:*

$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o,\ \omega_o \in \left(0, \frac{\pi}{m+1}\right], m \in \mathbb{Z}^+, m \geq 2$*, let* $\Delta^a_{m-1,m}$ $(\Delta^r_{m-1,m})$ *and* $\Delta^a_{m,m+1}$ $(\Delta^r_{m,m+1})$ *denote the* end-to-end power spectral attenuation (gain) of the frequency region $[(m-1)\omega_o, m\omega_o]$ *and* $[m\omega_o, (m+1)\omega_o]$ *respectively, then the variance between the two attenuations (gains) is bounded by a* constant*:*

$$\Delta^r_{m,m+1} - \Delta^r_{m-1,m} \leq 2\delta$$
$$\Delta^a_{m-1,m} - \Delta^a_{m,m+1} \leq 2\delta$$

where $m \in \mathcal{Z}^+, m \geq 2$*,* $m\omega_o \leq \pi$*, and*

$$\Delta^r_{k-1,k} = |H(k\omega_o)|^2 - |H((k-1)\omega_o)|^2 \geq 0,\ k \in \mathcal{Z}^+$$
$$\Delta^a_{k-1,k} = |H((k-1)\omega_o)|^2 - |H(k\omega_o)|^2 \geq 0,\ k \in \mathcal{Z}^+.$$

Proof: Based on the recurrence relation of the Chebyshev polynomial of the first kind in variable x:

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x)$$

we have:

$$T_{m+1}(x) - T_m(x) = (2x-1)T_m(x) - T_{m-1}(x)$$
$$= T_m(x) - T_{m-1}(x) + 2(x-1)T_m(x)$$

Therefore:

$$\frac{T_{m+1}(x) - T_m(x)}{1-x} - \frac{T_m(x) - T_{m-1}(x)}{1-x} = -2T_m(x)$$

Let $x = \cos(\omega)$, simplifying the above equation, we have:

$$\frac{\cos((m+1)\omega) - \cos(m\omega)}{1-\cos(\omega)} - \frac{\cos(m\omega) - \cos((m-1)\omega)}{1-\cos(\omega)}$$
$$= -2\cos(m\omega) \leq 2$$

Therefore

$$\Delta^r_{m,m+1} - \Delta^r_{m-1,m}$$
$$= \left\{ \sum_{l=-\infty}^{\infty} \tilde{h}(l) [\cos((m+1)\omega_o l) - \cos(m\omega_o l)] \right\}$$
$$- \left\{ \sum_{l=-\infty}^{\infty} \tilde{h}(l) [\cos(m\omega_o l) - \cos((m-1)\omega_o l)] \right\}$$
$$\leq 2 \sum_{l=-\infty}^{\infty} \tilde{h}(l)(1 - \cos(\omega_o l)) = 2\left(|H(0)|^2 - |H(\omega_o)|^2\right) \leq 2\delta$$

Similarly, we can prove: $\Delta^a_{m-1,m} - \Delta^a_{m,m+1} \leq 2\delta$ ■

Comments: Theorem 2.4 describes the behavior of power spectral attenuation/gain in terms of higher granular frequency regions: There is a *linear increase* in the achievable attenuation/gain in subsequent equi-length frequency regions along the frequency axis. Obviously, this result is consistent with the result in Theorem 2.1. Similarly, it is consistent with the results in [1], i.e., a *geometric increase* in the achievable attenuation/gain in subsequent geometrically-spaced frequency regions.

3. CONCLUSION

This paper revisits the performance constraints in the frequency domain of NNIR filters in [1] for the case of equidistantly spaced frequency points. The obtained results are of particular importance to some NNIR systems. Frequency-domain performance boundaries of various types of NNIR filters can be studied based on these results.

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