FUNDAMENTAL PROPERTIES OF NON-NEGATIVE IMPULSE RESPONSE FILTERS — THEORETICAL BOUNDS I

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ABSTRACT

This paper presents several fundamental frequency-domain bounds for a non-negative impulse response (NNIR) filter. Upper-bounds on power spectral attenuation and power spectral gain in geometrically spaced frequency regions are derived, when power spectral attenuation near frequency zero is limited. By analyzing the tightnesses of these bounds, the relationship between the maximally allowable power attenuation and gain is also treated. All results hold for both continuous and discrete-time domains.

Index Terms— Nonnegative impulse response, bounds, filter, frequency response.

1. INTRODUCTION

Non-negativity of an impulse response is widely required in many applications, such as congestion control, machine tool axis control, trajectory following in robotics, etc., where local extrema of the step response are not acceptable [1]. One emerging class of applications comes from the design of so called Evidence Filters [2] based on Dempster-Shafer Theory. An evidence filter combines evidence (pieces of possibly imperfect information) from heterogeneous sources and makes inferences on various events of interest. The combining process requires the feature of an NNIR.

It is well known that the relationship between an NNIR and its corresponding frequency response is still not completely clear. However, such a relationship is important for designing NNIR systems, especially when optimization approaches based on non-linear programming techniques are employed. If a unachievable specification is specified, it may lead to significant approximation errors.

To guide the NNIR filter design, we explore some fundamental properties in the frequency domain, when the impulse response is constrained to be non-negative in the time domain.

This paper is organized as follows: Section 2 states the related work. Section 3 constitutes the main body of the paper. It presents the upper-bounds on the power spectral attenuation and gain in a certain frequency region and discusses their relationship. Section 4 presents the conclusion.

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2. RELATED WORK

The work by Papoulis [3] shows that if the impulse response is non-negative, the frequency domain attenuation is severely limited. It provides the power attenuation over subsequent octaves on the frequency axis when the attenuation over the first octave is given. The work in [4] shows that a highpass filter with more than 6dB/octave roll-off near the origin of the frequency response will have a step response that is non-negative/non-positive. The work in [5] [6] provides theorems regarding the transient response of fixed, lumped, linear, and stable networks. These theorems provide the bounds on the impulse response and step response of various classes of system functions when the system function is restricted in certain ways, as well as the bounds on the frequency response when the impulse response is restricted. The work in [7] gives the bounds on the attenuation of the real part of the transfer or immittance function $R(\omega)$ when the impulse response is restricted to be non-negative. Tighter bounds are given when $R(\omega)$ is further restricted to be non-negative and monotonically decreasing. "Dual" bounds on the impulse response are found when $R(\omega)$ is restricted to be non-negative. The work in [8] provides a set of limitations to the even-order transient response of a positive real network, when the real part of the transfer function is limited to be positive real.

The work in [9, 10] is motivated by the applications in the field of optics, where inherent requirements of non-negativity and band-limitedness of point-spread functions for luminance exist. The work in [9] provides the best possible bound on the passband of a non-negative integrable time-domain function whose frequency response vanishes for $|\omega| \geq 1$ on the continuous frequency axis. The work in [10] provides pointwise and integral frequency-domain bounds for non-negative, strictly band-limited 1-D or radially symmetric 2-D functions. However, all these results only work for *passband* analysis of *lowpass* filters, and are *not* applicable to other filter types or to frequency regions that are not in the passband, while the bounds given in this paper have no such limitations.

3. FUNDAMENTAL LIMITS OF NNIR FILTERS

Before discussing fundamental bounds on power spectral attenuation or gain it is convenient to present Lemma 3.1 first.

Lemma 3.1 *If the impulse response of a filter is non-negative, i.e., $h(n) \geq 0,\ n \in \mathbb{Z}$, then the power spectrum denoted by $|H(\omega)|^2$ satisfies:*

$$|H(\omega)|^2 \leq |H(0)|^2 .$$

where $0 \leq \omega \leq \pi$.

Proof: The proof is trivially based on the definition of $H(\omega)$ and readers may refer to addendum [11] for details. ■

3.1. Fundamental Bound on Power Spectral Attenuation

Theorem 3.2 *If the impulse response of a filter is non-negative, i.e., $h(k) \geq 0,\ k \in \mathbb{Z}$, and the power spectral attenuation within $[0, \omega_o]$ is bounded by:*

$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o,\ \omega_o \in (0, \pi]$, then the power spectral attenuation at $m^n\omega_o$ is bounded by:

$$|H(0)|^2 - m^{2n}\delta \leq |H(m^n\omega_o)|^2 \leq |H(0)|^2$$

where $m, n \in \mathbb{Z}^+$, $m^n\omega_o \leq \pi$.

Proof: Using induction, it is simple to show that:

$$\frac{1-\cos(m^n\omega)}{1-\cos(\omega)} \leq m^{2n}$$

(Readers may refer to addendum [11] for a detailed proof.)

$$\because\ h(k) \geq 0 \quad \therefore\ \tilde{h}(k) \triangleq h(k) * h(-k) \geq 0$$

Since $h(k)$ is real, we have:

$$h(k) * h^*(-k) = h(k) * h(-k) = \tilde{h}(k)$$

Therefore

$$|H(\omega)|^2 = H(e^{j\omega})H^*(e^{j\omega}) \overset{\mathscr{Z}}{\longleftrightarrow} h(k) * h^*(-k) = \tilde{h}(k)$$

Since $|H(\omega)|^2$ is non-negative real, we have:

$$|H(\omega)|^2 = \sum_{l=-\infty}^{\infty} \tilde{h}(l)e^{-j\omega l} = \sum_{l=-\infty}^{\infty} \tilde{h}(l)\cos(\omega l)$$

Therefore:

$$|H(0)|^2 - |H(m^n\omega_o)|^2 = \sum_{l=-\infty}^{\infty} \tilde{h}(l)(1-\cos(m^n\omega_o l))$$
$$\leq m^{2n} \sum_{l=-\infty}^{\infty} \tilde{h}(l)(1-\cos(\omega_o l))$$
$$= m^{2n}(|H(0)|^2 - |H(\omega_o)|^2) \leq m^{2n}\delta$$

Together with Lemma 3.1, rewrite the above inequality as:

$$|H(0)|^2 - m^{2n}\delta \leq |H(m^n\omega_o)|^2 \leq |H(0)|^2$$ ■

Comment 1: *Theorem 3.2 infers that the stopband attenuation of an NNIR filter is severely constrained when the attenuation near the DC frequency does not exceed a prescribed value. That is, an arbitrary stopband attenuation is unachievable regardless of the system order.*

It is also noted that the results in [3] are a special case ($m = 2$) of Theorem 3.2.

Based on Theorem 3.2, we have Theorem 3.3 as follows:

Theorem 3.3 *If the impulse response of a filter is non-negative, i.e., $h(k) \geq 0,\ k \in \mathbb{Z}$, and the power spectral attenuation within the frequency region $[0, \omega_o]$ is bounded by:*

$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2 \tag{1}$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o,\ \omega_o \in (0, \pi]$, then the end-to-end power spectral attenuation of the frequency region $[m^{n-1}\omega_o, m^n\omega_o]$ is bounded by:

$$\left|H\left(m^{n-1}\omega_o\right)\right|^2 - |H(m^n\omega_o)|^2 \leq \left(m^2-1\right)m^{2n-2}\delta$$

where $m, n \in \mathbb{Z}^+$, $m^n\omega_o \leq \pi$.

Proof: Since $m^n\omega_o = m \cdot \left(m^{n-1}\omega_o\right)$, from Theorem 3.2, we have:

$$|H(0)|^2 - |H(m^n\omega_o)|^2 \leq m^2\left(|H(0)|^2 - \left|H\left(m^{n-1}\omega_o\right)\right|^2\right)$$

Therefore:

$$\left|H\left(m^{n-1}\omega_o\right)\right|^2 - |H(m^n\omega_o)|^2$$
$$= \left(|H(0)|^2 - |H(m^n\omega_o)|^2\right) - \left(|H(0)|^2 - \left|H\left(m^{n-1}\omega_o\right)\right|^2\right)$$
$$\leq (m^2-1)\left(|H(0)|^2 - \left|H\left(m^{n-1}\omega_o\right)\right|^2\right)$$
$$\leq \left(m^2-1\right)m^{2n-2}\delta$$ ■

Comment 2: *Theorem 3.3 infers that the spectral attenuation in the transition band of a lowpass/bandpass NNIR filter is severely constrained with a limited spectral attenuation near the DC frequency.*

3.2. Fundamental Bound on Power Spectral Gain

Theorem 3.4 *If the impulse response of a filter is non-negative, i.e., $h(k) \geq 0,\ k \in \mathbb{Z}$, and the power spectral attenuation in the frequency region $[0, \omega_o]$ is bounded by:*

$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o,\ \omega_o \in (0, \pi]$, then the end-to-end power spectral gain of the frequency region $[m^{n-1}\omega_o, m^n\omega_o]$ is bounded as follows:

$$|H(m^n\omega_o)|^2 - \left|H\left(m^{n-1}\omega_o\right)\right|^2 \leq \begin{cases} \delta, & n = 1 \\ m^2\delta, & n = 2 \\ \left(m^2-1\right)m^{2n-4}\delta, & n \geq 3 \end{cases}$$

where $m, n \in \mathbb{Z}^+$, $m^n\omega_o \leq \pi$.

Proof: $n = 1$ and $n = 2$ are the trivial cases. This is due to the fact that the maximal power spectral attenuation at ω_o and $m\omega_o$ cannot exceed δ and $m^2\delta$ (Theorem 3.2) respectively, and the maximal power gain from δ and $m^2\delta$ cannot go beyond $0dB$ (Lemma 3.1).

When $n \geq 3$, it can be proved via induction that:

$$\frac{\cos\left(m^n\omega\right) - \cos\left(m^{n-1}\omega\right)}{1 - \cos(\omega)} \leq \left(m^2 - 1\right) \cdot m^{2n-4}$$

(Please refer to addendum [11] for a detailed proof.)
Therefore, for $n \geq 3$ we have:

$$\begin{aligned}
&|H(m^n\omega_o)|^2 - \left|H(m^{n-1}\omega_o)\right|^2 \\
&= \sum_{l=-\infty}^{\infty} \tilde{h}(l)\left(\cos(m^n\omega_o l) - \cos(m^{n-1}\omega_o l)\right) \\
&\leq \left(m^2 - 1\right) \cdot m^{2n-4} \sum_{l=-\infty}^{\infty} \tilde{h}(l)\left(1 - \cos\left(\omega_o l\right)\right) \\
&\leq \left(m^2 - 1\right) \cdot m^{2n-4}\delta
\end{aligned}$$

■

It should be noted that the bound on the power spectral gain shown in Theorem 3.4 is quite loose for arbitrary $m \in \mathbb{Z}^+$. The closed form of a much tighter bound is hard to obtain. However, for $m = 2$ and $m = 3$, the closed form of a tight bound does exists, and it is stated in Theorems 3.5 and 3.6 respectively with omitted proofs. Interested readers may refer to addendum [11] for a detailed proof.

Theorem 3.5 *For $m = 2$, if*

$$h(k) \geq 0,\ k \in \mathbb{Z}$$
$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o,\ \omega_o \in (0, \pi]$, then

$$|H(2^n\omega_o)|^2 - \left|H(2^{n-1}\omega_o)\right|^2 \leq \begin{cases} \delta,\ n = 1 \\ 4\delta,\ n = 2 \\ 8\delta,\ n = 3 \\ \left(2^{2n-3} - 2^{2n-6} + 2^{2n-8} - 2^{2n-10} + \cdots \pm 2^0\right)\delta,\ n \geq 4 \end{cases}$$

where $2^n\omega_o \leq \pi$. (In the last expression, the plus sign is used when n is even, and minus sign is used with n is odd.)

Theorem 3.6 *For $m = 3$, if*

$$h(k) \geq 0,\ k \in \mathbb{Z}$$
$$|H(0)|^2 - \delta \leq |H(\omega)|^2 \leq |H(0)|^2$$

where $\delta > 0,\ 0 \leq \omega \leq \omega_o,\ \omega_o \in (0, \pi]$, then

$$|H(3^n\omega_o)|^2 - \left|H(3^{n-1}\omega_o)\right|^2 \leq \begin{cases} \delta,\ n = 1 \\ 9\delta,\ n = 2 \\ \left(3^{2n-2} - 3^{2n-3} + 3^{2n-5} - 3^{2n-7} + \cdots \pm 3^1\right)\delta,\ n \geq 3 \end{cases}$$

where $3^n\omega_o \leq \pi$. (In the last expression, the plus sign is used when n is odd, and the minus sign is used with n is even.)

Comment 3: *Theorems 3.4-3.6 can be used to study how the spectral gain in the transition band of a high-pass/bandpass/multiband NNIR filter is constrained when the attenuation near frequency zero does not exceed a prescribed value.*

3.3. Relationship between Maximally Allowable Attenuation and Gain

Based on the observations in *Comment 2* and *Comment 3*, this section explores the relationship between the maximal end-to-end attenuation and the maximal end-to-end gain that are allowed for the frequency region $[m^{n-1}\omega_o, m^n\omega_o]$. To do so, it is helpful to know the tightness of the bounds first.

Definition When (1) is satisfied, the *tightness* of a bound, t, can be defined as:

$$t_a \triangleq \frac{e_a}{A_a} \triangleq \frac{|B_a - A_a|}{A_a},\ \ t_r \triangleq \frac{e_r}{A_r} \triangleq \frac{|B_r - A_r|}{A_r} \tag{2}$$

where A denotes the amount of maximally allowable power attenuation or gain, B denotes the power attenuation bound or power gain bound obtained in 3.1 and 3.2:

$$A_a = \max_{\omega\in[0,\omega_o]} \left\{\left|H\left(m^{n-1}\omega\right)\right|^2 - \left|H\left(m^n\omega\right)\right|^2\right\} \tag{3a}$$
$$A_r = \max_{\omega\in[0,\omega_o]} \left\{\left|H\left(m^n\omega\right)\right|^2 - \left|H\left(m^{n-1}\omega\right)\right|^2\right\} \tag{3b}$$
$$B_a = \left(m^2 - 1\right)m^{2n-2}\delta,\ \forall m \in \mathbb{Z}^+ \tag{3c}$$
$$B_r = \left(m^2 - 1\right)m^{2n-4}\delta,\ \ n \geq 3,\ \forall m \in \mathbb{Z}^+ \tag{3d}$$

Particularly, for $m = 2, 3$:

$$\begin{aligned}
B_r &= \left(2^{2n-3} - 2^{2n-6} + 2^{2n-8} - 2^{2n-10} + \cdots \pm 2^0\right)\delta,\ n \geq 4 \\
B_r &= \left(3^{2n-2} - 3^{2n-3} + 3^{2n-5} - 3^{2n-7} + \cdots \pm 3^1\right)\delta,\ n \geq 3
\end{aligned} \tag{4}$$

and e denotes the bounding error:

$$e_a \triangleq |B_a - A_a|,\ \ e_r \triangleq |B_r - A_r| \tag{5}$$

(Subscripts a and r denote *attenuation* and *gain* respectively.)

Fig. 1 illustrates the tightness of the power attenuation bound and power gain bound stated in Theorem 3.4 for $m = 2$. As shown in Fig. 1, the attenuation bound is pretty tight — the bounding error is close to 0, while the gain bound is very loose, with the bounding error *at least* 70% of the actual maximal gain. Similar observation can be found for other $m \in \mathbb{Z}^+$.

The tightness of the two bounds can then described as:

$$t_a < t_r \tag{6}$$

Based on (6), the relationship described in Theorem 3.7 is obtained.

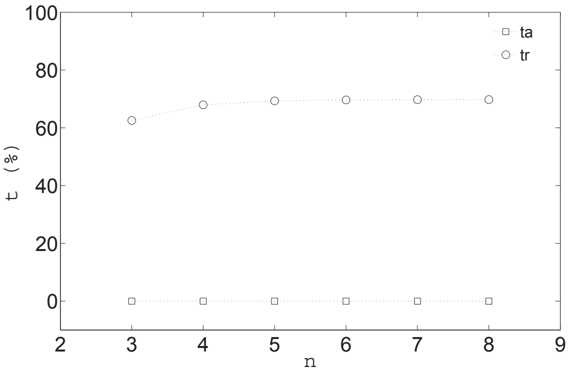

Fig. 1. Tightness of attenuation and gain bounds ($m = 2$)

Theorem 3.7 *Let* A_a *be defined as in* (3a)*,* A_r *be defined as in* (3b)*,* Δ_k *denotes the amount of power attenuation at frequency point* $m^k\omega_o$*, i.e.,* $\Delta_k \triangleq 1 - \left|H\left(m^k\omega_o\right)\right|^2$*,* $k \in \mathbb{Z}^+$*. If* Δ_{n-1} *is fixed, then:*

$$\frac{A_r}{A_a} < \frac{1}{m^2} \tag{7}$$

Proof: With a given Δ_{n-1}, there must exist some δ as defined in (1), according to Theorem 3.2. With this δ, the following result can be obtained immediately from Theorem 3.3:

$$A_a \le B_a = \left(m^2 - 1\right) m^{2n-2}\delta$$

According to (5), we can rewrite A_a as:

$$A_a = \left(m^2 - 1\right) m^{2n-2}\delta - e_a$$

Similarly, from Theorem 3.4, we have:

$$A_r \le B_r = \left(m^2 - 1\right) m^{2n-4}\delta$$

Rewrite Δ_r :

$$A_r = \left(m^2 - 1\right) m^{2n-4}\delta - e_r$$

From (2), we have:

$$\begin{aligned} \frac{A_a}{A_r} &= \frac{\left(m^2 - 1\right) m^{2n-2}\delta - e_a}{\left(m^2 - 1\right) m^{2n-4}\delta - e_r} \\ &= \frac{\left(m^2 - 1\right) m^{2n-2}\delta - A_a t_a}{\left(m^2 - 1\right) m^{2n-4}\delta - A_r t_r} \end{aligned} \tag{8}$$

From (6), we have the desired result after simplifying (8):

$$\frac{A_r}{A_a} < \frac{1}{m^2}$$ ■

Comment 4: *Theorem 3.7 infers that the maximally achievable end-to-end spectral gain is much less than the maximally achievable end-to-end spectral attenuation for the transition band of an NNIR filter when the attenuation near frequency zero does not exceed a prescribed value. This means that lowpass NNIR filters are theoretically much easier to achieve than highpass or bandpass NNIR filters.*

4. CONCLUSION

This paper presents the fundamental limitations on power spectral attenuation and gain in geometrically spaced frequency regions of an NNIR filter, when attenuation near DC frequency is given. The relationship between the maximally allowable power spectral attenuation and gain in the same frequency region is analyzed. These results provide an important tool to analyze the frequency response of an NNIR filter.

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