A VARIABLE STEP SIZE AND VARIABLE TAP LENGTH LMS ALGORITHM FOR IMPULSE RESPONSES WITH EXPONENTIAL POWER PROFILE

Kun Shi, Xiaoli Ma, and G. Tong Zhou

School of Electrical and Computer Engineering, Georgia Tech, Atlanta, GA 30332-0250, USA

ABSTRACT

Step size and tap length play critical roles in balancing the complexity and steady-state performance of an adaptive filter. For an impulse response with an exponential power decay profile, which models a wide range of practical systems, such as an acoustic echo path, this paper proposes a new variable step-size and tap-length least mean square (LMS) algorithm. In each iteration, the optimal step-size and tap-length are derived by minimizing the mean square deviation (MSD) between the true and the estimated filter coefficients. The proposed algorithm performs better in terms of both convergence rate and steady-state performance than the existing ones. Effectiveness of the proposed algorithm is demonstrated through computer simulations.

Index Terms— Exponential power decay profile, least mean square (LMS), variable step size, variable tap length, adaptive filters, acoustic echo cancellation.

1. INTRODUCTION

Adaptive filters have been successfully applied to diverse fields including digital communication, speech recognition, control systems, radar systems, seismology, and biomedical engineering. Among various types of adaptive algorithms, the least mean square (LMS) algorithm is well known and widely utilized for its simplicity and robustness [1]. The performance of the LMS algorithm, in terms of convergence rate, misadjustment, mean square error (MSE), and computational cost, is believed to be governed by both the tap length and step size of the adaptive filter [2–4].

Usually, to describe an unknown linear time-invariant system accurately, a sufficiently large filter tap length is needed, since the MSE is likely to increase if the tap length is undermodeled [4, 5]. However, the computational cost is proportional to the tap length. Moreover, an increase in filter length can slow down the convergence rate dramatically due to the step-size restrictions [5, 6]. Thus, a variable tap-length algorithm, which finds the appropriate tap-length for each iteration, is necessary to achieve both small MSE and fast convergence. Existing variable tap-length algorithms such as [7, 8] are sensitive to the parameter selection, i.e., different parameters result in different performance, according to the discussion in [8].

Recently, the impulse response envelope is suggested to be one essential factor that determines the convergence rate of a deficient-length filter [5,9]. In many applications such as acoustic echo cancellation, the unknown impulse response follows an exponential decay envelope. For this kind of systems, a theoretically optimal variable tap-length sequence is introduced in [5]. However, this algorithm entails large computational complexity as a result of trying to solve Lambert's W-function. To reduce the complexity, an adaptive solution for the optimal tap length is proposed in [9], which ensures a well-behaved transient tap-length convergence. However, to the best of our knowledge, variable tap-length algorithms have not been proposed in conjunction with a variable step size. It is well known that with the stability conditions, the efficient step-size control trade-offs fast convergence rate and tracking ability with filter misadjustment. Thus, we are motivated to develop a low complexity algorithm with both a variable tap length and step size.

In this paper, we propose a new variable tap-length and variable step-size LMS algorithm for applications where the unknown channel impulse response has an exponential decay envelope. The optimal solution of both tap-length and step-size at each iteration is obtained by minimizing the mean square deviation (MSD). As will be shown in the simulations, the proposed method outperforms those existing algorithms in terms of both convergence rate and misadjustment.

This work was supported in part by the Texas Instruments DSP Leadership University Program.

2. VARIABLE TAP-LENGTH AND VARIABLE STEP-SIZE LMS ALGORITHM

Consider an unknown length N exponential decay impulse response $\boldsymbol{c}_N = [c_0, c_1, ..., c_{N-1}]^T$ modeled by

$$c_i = e^{-(i-1)\tau} r(i), \;\; i = 0, 2, ..., N-1, \tag{1}$$

where the decay rate τ is a known positive constant and $r(i)$ is a zero-mean i.i.d. Gaussian random process with variance σ_r^2. The observed signal is a linear convolution of the transmitted signal and the impulse response:

$$d(n) = \boldsymbol{x}_N^T(n)\boldsymbol{c}_N + v(n), \tag{2}$$

where $\boldsymbol{c}_N = [c_0, c_2, ..., c_{N-1}]^T$ is the channel response and $\boldsymbol{x}_N(n) = [x(n), x(n-1), ..., x(n-N+1)]^T$ is the input vector and $v(n)$ is the additive noise. Here the problem we are considering is to estimate $\{c_i\}$ given $d(n)$ and $x(n)$ using an LMS algorithm with variable tap length and step size.

In the variable tap-length and variable step-size LMS algorithm, both the tap-length and step-size are time-varying rather than fixed. We denote by $M(n)$ and $\mu(n)$, respectively, the integer tap-length and step-size for the coefficients updated at the n^{th} iteration, and assume that $M(n) \leq N$. With the LMS criterion, the filter coefficients are updated by [5]

$$\boldsymbol{w}_{M(n+1)} = \begin{bmatrix} \boldsymbol{w}_{M(n)}(n) \\ \mathbf{0}_{M(n+1)-M(n)} \end{bmatrix} + \mu(n+1)e(n)\boldsymbol{x}_{M(n+1)}(n+1), \tag{3}$$

where $e(n)$ is the estimated error defined as

$$e(n) = d(n) - \boldsymbol{x}_{M(n)}^T(n)\boldsymbol{w}_{M(n)}, \tag{4}$$

$\boldsymbol{x}_{M(n)}(n) = [x(n), x(n-1)..., x(n-M(n)+1)]^T$ and $\boldsymbol{w}_{M(n)} = \left[w_1(n), w_2(n)..., w_{M(n)}(n)\right]^T$ are the $M(n)$-tap adaptive filter input vector and the coefficients vector, respectively, and $\mathbf{0}$ denotes a vector with all-zero entries. In the following, we introduce how to update the tap-length $M(n)$ and step-size $\mu(n)$ at each iteration.

Similar to [5, 9], we partition the impulse response $\boldsymbol{c}_N$ into two parts as

$$\boldsymbol{c}_N \triangleq \begin{bmatrix} \boldsymbol{c}'_{M(n)} \\ \boldsymbol{c}''_{N-M(n)} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{c}'_{M(n)}$ can be viewed as the part modeled by $\boldsymbol{w}_{M(n)}$, and $\boldsymbol{c}''_{N-M(n)}$ is the undermodeled part. Define the estimation errors of partial and total coefficients, respectively, as

$$\boldsymbol{\delta}_{M(n)}(n) = \boldsymbol{w}_{M(n)} - \boldsymbol{c}'_{M(n)}, \tag{6}$$

and

$$\boldsymbol{\delta}_N(n) = \begin{bmatrix} \boldsymbol{w}_{M(n)} \\ \mathbf{0}_{N-M(n)} \end{bmatrix} - \boldsymbol{c}_N. \tag{7}$$

Combining (2) and (6), we rewrite the signal estimate error in (4) as

$$e(n) = -\boldsymbol{x}_N^T(n)\boldsymbol{\delta}_N(n) + v(n). \tag{8}$$

Substituting (8) into (3), we obtain

$$\boldsymbol{\delta}_N(n+1) = \mathbf{A}(n)\boldsymbol{\delta}_N(n) + \mu(n+1)v(n) \begin{bmatrix} \boldsymbol{x}_{M(n+1)}(n+1) \\ \mathbf{0}_{N-M(n+1)} \end{bmatrix}, \tag{9}$$

where

$$\mathbf{A}(n) = \boldsymbol{I}_N - \mu(n+1) \begin{bmatrix} \boldsymbol{x}_{M(n+1)}(n+1) \\ \mathbf{0}_{N-M(n+1)} \end{bmatrix} \boldsymbol{x}_N^T(n), \tag{10}$$

and $\boldsymbol{I}_N$ is the $N \times N$ identity matrix.

To quantitatively evaluate the misadjustment of the filter coefficients, MSD is taken as a figure of merit, which is defined as

$$\Lambda(n) \triangleq \Lambda(M(n), \mu(n)) = E\left[\|\boldsymbol{\delta}_N(n)\|_2^2\right], \tag{11}$$

where $\|\cdot\|_2$ denotes the l_2 norm. Note that at each iteration, MSD depends on both $M(n)$ and $\mu(n)$. Assume that both the signals $x(n)$ and $v(n)$ are i.i.d. zero-mean Gaussian with variances σ_x^2 and σ_v^2, respectively. According to the analysis in [5, 9], we find that

$$\begin{aligned}\Lambda(n+1) =& \beta(n+1)\Lambda(n) + (\eta(n+1) - \beta(n+1)) \\ & E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right] + \gamma(n+1),\end{aligned} \tag{12}$$

where

$$\beta(n+1) = 1 - 2\mu(n+1)\sigma_x^2 + (M(n+1)+2)\mu^2(n+1)\sigma_x^4, \tag{13}$$

$$\eta(n+1) = 1 + M(n+1)\mu^2(n+1)\sigma_x^4, \tag{14}$$

$$\gamma(n+1) = M(n+1)\mu^2(n+1)\sigma_x^2\sigma_v^2. \tag{15}$$

We propose to find the optimal tap length and step size by minimizing MSD at each iteration. Taking the first-order partial derivative of $\Lambda(n+1)$ with respect to $M(n+1)$ and $\mu(n+1)$, respectively, we obtain

$$\begin{aligned}\frac{\partial\Lambda(n+1)}{\partial M(n+1)} =& \mu^2(n+1)\sigma_x^4\Lambda(n) + \mu^2(n+1)\sigma_x^2\sigma_v^2 + 2\mu(n+1) \\ & \sigma_x^2\left(1 - \mu(n+1)\sigma_x^2\right)\frac{\mathrm{d}E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right]}{\mathrm{d}M(n+1)},\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial\Lambda(n+1)}{\partial\mu(n+1)} =& 2\sigma_x^2\left((M(n+1)+2)\mu(n+1)\sigma_x^2 - 1\right)\Lambda(n) \\ & + 2\sigma_x^2(1 - 2\mu(n+1)\sigma_x^2)E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right] \\ & + 2\mu(n+1)\sigma_x^2\sigma_v^2 M(n+1).\end{aligned} \tag{17}$$

Based on the impulse pulse model in (1), we obtain

$$E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right] = \frac{e^{-2M(n+1)\tau} - e^{-2N\tau}}{1 - e^{-2N\tau}}E\left[\|\boldsymbol{c}_N\|_2^2\right], \tag{18}$$

$$E\left[\|\boldsymbol{c}_N\|_2^2\right] = \frac{1 - e^{-2N\tau}}{1 - e^{-2\tau}}\sigma_r^2. \tag{19}$$

Substituting (18) and (19) into (16) and setting the first-order partial derivatives $\partial\Lambda(n+1)/\partial M(n+1)$ and $\partial\Lambda(n+1)/\partial\mu(n+1)$ to zero, we obtain

$$M(n+1) = -\frac{1}{2\tau}\ln\frac{\mu(n+1)\left(\sigma_x^2\Lambda(n) + \sigma_v^2\right)\left(1 - e^{-2\tau}\right)}{4\tau\left(1 - \mu(n+1)\sigma_x^2\right)\sigma_r^2}, \tag{20}$$

$$\mu(n+1) = \frac{1 - \frac{E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right]}{\Lambda(n)}}{(M(n+1)+2)\sigma_x^2 + \frac{M(n+1)\sigma_v^2}{\Lambda(n)} - \frac{2\sigma_x^2 E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right]}{\Lambda(n)}}. \tag{21}$$

Then, at each iteration, a pair of stationary points $M(n+1)$ and $u(n+1)$ can be obtained by jointly solving Eqs. (20) and (21). Based on Eqs. (20) and (21), it is difficult to find closed-form solutions for $M(n+1)$ and $\mu(n+1)$. Moreover, the stationary points from (20) and (21) lead to the global minimum of $\Lambda(n+1)$ only if the MSD is a convex function with respect to the tap-length and step-size. However, the convexity is difficult to be verified due to the complicated Hessian matrix. In the following, we find an approximate solution of $M(n)$ and $\mu(n)$ rather than explicitly solving (20) and (21).

By assuming that $M(n)$ is close to $M(n+1)$, we replace $M(n+1)$ by $M(n)$ in (21)

$$\mu(n+1) = \frac{1 - \frac{E\left[\|\boldsymbol{c}''_{N-M(n)}\|_2^2\right]}{\Lambda(n)}}{(M(n)+2)\sigma_x^2 + \frac{M(n)\sigma_v^2}{\Lambda(n)} - \frac{2\sigma_x^2 E\left[\|\boldsymbol{c}''_{N-M(n)}\|_2^2\right]}{\Lambda(n)}}. \tag{22}$$

Thus, in each iteration $\mu(n+1)$ and $M(n+1)$ are obtained in an alternating manner by using (22) and (20). Next, we show that in this alternating manner, convergence condition is satisfied. Moreover, by removing the dependence in Eqs. (21) and (20) between each other, $\mu(n+1)$ in (22) and $M(n+1)$ in (20) are optimal solutions in terms of minimizing $\Lambda(n+1)$ given the other.

Combining (5), (7), and (11), we obtain

$$\Lambda(n) = E\left[\|\boldsymbol{\delta}_{M(n)}(n)\|_2^2\right] + E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right]. \tag{23}$$

Substituting (23) into (21), it is then straightforward to verify that $u(n+1)$ ensures the convergence of (12) according to the condition in [5]

$$0 < \mu(n+1) < \frac{2}{(M(n+1)+2)\sigma_x^2}. \tag{24}$$

Moreover, if there is no background noise $\left(\sigma_v^2 = 0\right)$ and the filter tap length is perfectly modeled $\left(\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2 = 0\right)$, the step size in (21) simplifies to

$$\mu(n+1) = \frac{1}{(M(n+1)+2)\sigma_x^2}, \tag{25}$$

which is consistent with the step-size that achieves the optimum convergence rate and adjustment in [10].

To analyze the behavior of $\mu(n+1)$ in (22), we take the second-order partial derivative of (12) with respect to $\mu(n+1)$:

$$\frac{\partial^2 \Lambda(n+1)}{\partial \mu^2(n+1)} = 2M(n+1)\sigma_x^2\left(\sigma_x^2\Lambda(n)+\sigma_v^2\right) + 4\sigma_x^4\left(\Lambda(n) - E\left[\|\boldsymbol{c}''_{N-M(n+1)}\|_2^2\right]\right). \tag{26}$$

Based on (23), we know that

$$\frac{\partial^2 \Lambda(n+1)}{\partial \mu^2(n+1)} > 0, \tag{27}$$

which indicates that for any given tap length, MSD is a convex function in the step size parameter. Therefore, with tap-length $M(n)$, the step-size in (22) minimizes the MSD at the $(n+1)^{st}$ iteration. Similarly, the second-order partial derivative of (12) with respect to $M(n+1)$ leads to

$$\frac{\partial^2 \Lambda(n+1)}{\partial M^2(n+1)} = \frac{8\mu(n)\tau^2\sigma_x^2(1-\mu(n)\sigma_x^2)e^{-2M(n+1)\tau}\sigma_r^2}{1-e^{-2\tau}}. \tag{28}$$

For any step size that guarantees convergence (see (24)), it is straightforward to show

$$\frac{\partial^2 \Lambda(n+1)}{\partial M^2(n+1)} > 0, \tag{29}$$

which indicates that with a given step size, MSD is also a convex function in the tap length parameter. Therefore, with the step size $\mu(n+1)$, the tap length in (20) achieves the minimum MSD. So far, an optimal solution for the step size and tap length at each iteration is described by (22) and (20). However, the estimates of $\mu(n)$ and $M(n)$ still depend on $\Lambda(n)$. Next, we show how to estimate $\Lambda(n)$ in the $(n+1)^{st}$ iteration.

Based on the independence assumption between the filter input signal and the filter coefficients, the MSE of the LMS filter is expressed as (see also [1, 9])

$$E\left[e^2(n)\right] = \sigma_x^2\Lambda(n) + \sigma_v^2. \tag{30}$$

Combining (30), (22), and (20), the tap length and step size are obtained as follows:

$$\mu(n+1) = \frac{E\left[e^2(n)\right] - \sigma_v^2 - \sigma_x^2 E\left[\|\boldsymbol{c}''_{N-M(n)}\|_2^2\right]}{\sigma_x^2\Big(M(n)E\left[e^2(n)\right] - 2\sigma_x^2 E\left[\|\boldsymbol{c}''_{N-M(n)}\|_2^2\right]\Big)}, \tag{31}$$

$$M(n+1) = -\frac{1}{2\tau}\ln\frac{\mu(n+1)\left(1-e^{-2\tau}\right)E\left[e^2(n)\right]}{4\tau\left(1-\mu(n+1)\sigma_x^2\right)\sigma_r^2}. \tag{32}$$

In practice, the statistical average $E\left[e^2(n)\right]$ can be estimated recursively by its time average:

$$\overline{e^2(n)} = \rho\overline{e^2(n-1)} + (1-\rho)e^2(n), \tag{33}$$

where $0 < \rho < 1$ is the forgetting factor. Moreover, since the tap-length of a filter must be an integer, we choose to only keep the integer part of $M(n+1)$ after its update by Eq. (32). Finally, the entire adaptive algorithm is described sequentially by (4), (18), (31), (32), and (3).

3. SIMULATION RESULTS

In this section, the performance of the proposed method is assessed via computer simulations. For comparison purposes, we also implemented the fixed tap-length LMS algorithm and the variable tap-length LMS algorithm in [9]. The setup of all the simulations is similar to that in [9]: The impulse response was generated according to (1), which was a white Gaussian noise sequence with zero-mean and variance σ_r^2 of 0.01 weighted by an exponential decay profile. The impulse response length was $N = 1024$, and the envelope decay rate τ was 0.005. One realization of the unknown response is shown in Fig. 1. The filter input was a zero-mean i.i.d. Gaussian process with variance $\sigma_x^2 = 1$. The noise was another white Gaussian process with zero mean and variance σ_v^2 of 0.01. All the following results were obtained by averaging over 100 Monte Carlo trials.

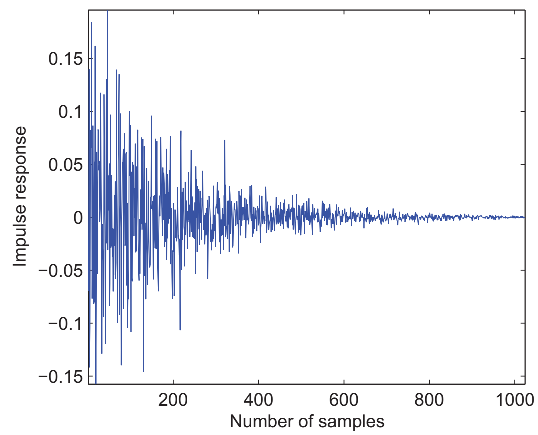

Fig. 1. One realization of impulse response.

First, we evaluated the convergence performance of the proposed method. MSD and MSE curves with respect to the number of iterations are depicted with different types of LMS algorithms. The step size for the fixed tap-length LMS algorithm is set to 1/1026, which corresponds to the optimal step-size in (25). For both the algorithm in [9] and the proposed method, the initial tap-length $M(0)$ was chosen as 20 and the forgetting factor in (30) was 0.99. The MSDs are shown in Fig. 2(a). It is seen that the algorithm in [9] converged faster than the fixed tap-length LMS algorithm due to the variable step size, and both exhibited similar steady-state MSDs. The proposed method further improved the convergence rate and achieved lower steady-state MSD, due to the fact that MSD is minimized in terms of both the tap length and step size at each iteration. The MSEs are shown in Fig. 2(b), which also validates the advantages of the proposed method in terms of both convergence rate and steady-state performance. We point out that the consistency between the MSD and MSE results is in agreement with the theoretical analysis in (30). Both of them are presented here since different applications may focus on different criteria. For instance, MSD is more suitable in channel estimation, whereas MSE is preferable in echo cancellation applications.

The values of tap length and step size of the proposed method and the method in [9] are shown in Fig. 3. The step sizes are shown in log scale for demonstration purposes. For the method in [9], it is seen that the tap length saturates at around 800. Similar variability is observed for the step size, since the step size simply follows $\mu(n) = \mu'/(M(n-1)+\delta)\sigma_x^2$, with the parameters δ and μ' being set to 5 and 0.5, respectively. Comparatively, the step size in the proposed method saturates at a smaller value, which provides finer coefficients update. Therefore, the proposed method achieves better performance (see Fig. 2).

Finally, we evaluate the performance of the proposed method with respect to the initial value. The steady-state MSDs with different initial tap-length are shown in Table 1. It can be observed that with a wide range of the initial tap-length, the MSD converges to the value that achieves an effective modeling of the significant energy within the impulse response. Thus, we claim that the proposed algorithm is robust to the selection of initial tap-length.

Table 1. Steady-state MSDs with different initial tap-length.

$M(0)$	20	50	100	200	500	800
MSD(∞) (dB)	-29.4	-29.7	28.6	-29.3	-30.1	-29.4

4. CONCLUSIONS

A new variable tap-length and variable step-size LMS algorithm is proposed in this paper. For the impulse response with an exponential decay envelope, the tap-length and step-size are obtained by minimizing the MSD at each iteration. Simulation results show that the proposed method achieves faster convergence rate as well as better steady-state performance (in terms of MSD and MSE).

5. REFERENCES

[1] A. H. Sayed, *Fundamentals of Adaptive Filtering*, New York, Wiely, 2003.

[2] H. -C. Shin, A. H. Sayed, and W. -J. Song, "Variable step-size NLMS and affine projection algorithms," *IEEE Signal Processing Letters*, vol. 11, pp. 132–135, Feb. 2004.

[3] C. Paleologu, S. Ciochină, and J. Benesty, "Variable step-size NLMS algorithm for under-modeling acoustic echo cancellation," *IEEE Signal Processing Letters*, vol. 51, pp. 5–8, Apr. 2008.

[4] K. Mayyas, "Performance analysis of the deficient length LMS adaptive algorithm," *IEEE Trans. on Signal Processing*, vol. 53, pp. 2727–2734, Aug. 2005.

[5] Y. Gu, K. Tang, H. Cui, and W. Du, "Convergence analysis of a deficient-length LMS filter and optimal-length sequence to model exponential decay impulse response," *IEEE Signal Processing Letters*, vol. 10, no. 1, pp. 4–7, Jan. 2003.

[6] T. -K. Woo, "Fast hierarchical least mean square algorithm," *IEEE Signal Processing Letters*, vol. 8, pp. 289–291, Nov. 2001.

[7] Y. Gu, K. Tang, H. Cui, and W. Du, "LMS algorithm with gradient descent filter length," *IEEE Signal Processing Letters*, vol. 11, pp. 305–307, Mar. 2004.

[8] Y. Gong and C. F. N. Cowan, "An LMS style variable tap-length algorithm for structure adaptation," *IEEE Trans. on Signal Processing*, vol. 53, pp. 2400–2407, Jul. 2005.

[9] Y. Zhang, J. A. Chambers, S. Sanei, P. Kendrick, and T. J. Cox, "A new variable tap-length LMS algorithm to model an exponential decay impulse response," *IEEE Signal Processing Letters*, vol. 14, pp. 263–276, Apr. 2007.

[10] T. Hsia, "Convergence analysis of LMS and NLMS adaptive algorithms," *Proc. IEEE ICASSP*, Boston, MA, vol. 8, pp. 667–670, Apr. 1983.

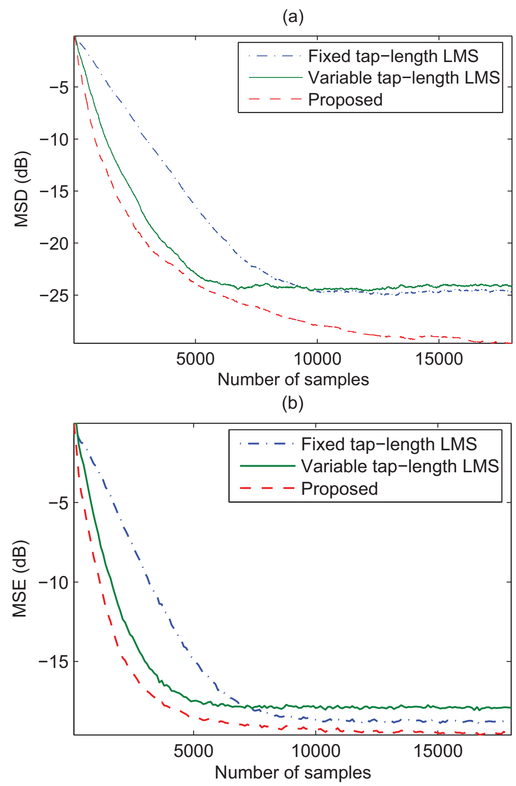

Fig. 2. Comparison of convergence performance with different LMS algorithms: (a) MSD; (b) MSE.

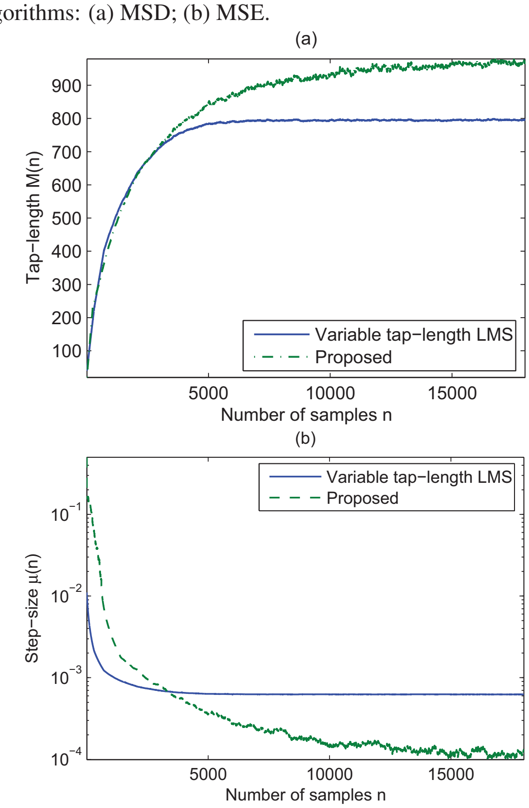

Fig. 3. Comparison of tap-length and step-size with different LMS algorithms: (a) tap-length $M(n)$; (b) step-size $\mu(n)$.